A comparison between classes of perturbations allowed by some robust stability conditions

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Abstract

Some classical robust stability conditions for continuous-time linear time-invariant systems are analyzed by comparing the families of perturbed systems whose stabilization through a given compensator is guaranteed by each of them. Some properties of such families are formally derived, and some very simple examples are presented in order to clarify such properties and comparisons.

1 Introduction

Stability robustness in front of plant uncertainties is one of the main purposes in control systems design. Since the actual plant to be stabilized is not exactly known, a feasible approach to the problem is to assume that it belongs to a set of plants defining the so-called uncertainty set, and to design a compensator guaranteeing that stability is preserved for all the plants in this set. To this aim it can be helpful to make use of several robust stability conditions that are available in the literature (see [1]-[8] and the references therein) and are based on different representations of the mismatch between the nominal model and the actual plant. Notice that this approach may not achieve optimal performance, especially if the design is too conservative, i.e., when the covering of the actual plants through the uncertainty set is too large, e.g., if it takes into account perturbations in the plant parameters that rarely or never occur.

In this paper we will focus our analysis on the robust stability conditions derived by Doyle and Stein [9] and Chen and Desoer [10], based on a generalized version of the Nyquist criterion, and on some robust stability conditions based on the small gain theorem [7]. Both these types of conditions refer to the representation of the uncertainties affecting the plant to be controlled in the form of unstructured matrix perturbations, either additive or multiplicative, relating its actual transfer matrix $P(s)$ to its nominal one, namely $P_0(s)$. As it is recalled in Section 3, such representations are quite general, since they allow to take into account any kind of inaccuracies that may occur in modeling or identifying the plant, or uncertainties in the plant parameters, within the hypothesis of linear behaviour of the plant.

The purpose of this paper is to make a comparison between the families of perturbed plants whose stabilization is guaranteed by pairs of the mentioned robust stability conditions, for the same choice of the feedback compensator that stabilizes the nominal plant.

After introducing some notations and preliminaries in Section 2, a detailed comparison will be carried out in Section 3 between the families of perturbed plants whose stabilization through the given compensator is guaranteed by those of the mentioned conditions that refer to multiplicative perturbations of $P_0(s)$ (which will be recalled in Section 2). First, a formal proof will be given of the invariance of the locations and the algebraic multiplicities of the unstable eigenvalues of the perturbed plants whose stabilization is guaranteed by the stability conditions based on the small gain theorem; this seems to be a stronger assessment than the statements reported in informal reasonings in [11, 12], where multiplicative (or additive) perturbations free of unstable poles are considered. On the contrary, for the perturbed plants whose stabilization through the given compensator is guaranteed by the stability conditions in [9, 10] (that do not require the stability of the multiplicative perturbations), a formal proof will be given of the invariance of the locations and the algebraic multiplicities of the only eigenvalues lying on the Nyquist contour (i.e., on the boundary of the half-plane prescribed for the closed-loop eigenvalues), showing through an example that the other unstable eigenvalues can actually move. This will allow to infer that, in the case of $P_0(s)$ having unstable poles, the former family is less significant than the latter, since it contains only perturbed plants having just the same unstable eigenvalues (in locations and multiplicities) as the nominal one. For the sake of completeness, a similar comparison will be sketched in Section 4 also for the case of additive perturbations, although some of the related reasonings are nearly obvious in this case. Lastly, in the same
section, also the families of perturbed plants whose stabilization through the given compensator is guaranteed by the two different robust stability conditions in [9, 10] (the one referred to multiplicative perturbations, the other to additive perturbations), will be compared in some significant cases.

Up to now we implicitly referred to the mere asymptotic stability of the closed-loop control system; however, in the following, after introducing proper notations, a possibly stronger requirement will be considered, i.e., it will be assumed that the eigenvalues of the closed-loop control system are possibly required to lie in the half-plane $\text{Re}(s) < -\alpha$, for a given $\alpha \geq 0$; as it will be discussed in Section 3, a suitable choice of $\alpha > 0$ may be useful not only in order to obtain a faster convergence to zero of the free responses of the closed-loop control system, but also in order to overcome some difficulties arising in the classical use of the robust stability conditions in [9, 10] with $\alpha = 0$.

All the above mentioned discussion on the four types of conditions will be clarified by means of some very simple examples.

2 Notations and preliminaries

Consider the finite dimensional continuous-time linear time-invariant dynamical system $S$, described by the following equations:

$$
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t), \\
y(t) &= C x(t) + D u(t),
\end{align*}
$$

(1)

where $t \in \mathbb{R}$ is time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the measured output and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times p}$ are real constant matrices. In the following, for the sake of brevity, the eigenvalues of the dynamic matrix $A$ of $S$ will be also called eigenvalues of $S$; the same location will be used for any other system. Moreover, $P(s)$ will denote the input/output transfer matrix of $S$, i.e.:

$$
P(s) = C(sI - A)^{-1}B + D.
$$

The actual values of $n$ and of matrices $A$, $B$, $C$ and $D$ are uncertain, and only a nominal description of order $n_0$ of system $S$ is supposed to be available, characterized by matrices $A_0 \in \mathbb{R}^{n_0 \times n_0}$, $B_0 \in \mathbb{R}^{n_0 \times p}$, $C_0 \in \mathbb{R}^{q \times n_0}$ and $D_0 \in \mathbb{R}^{q \times p}$, and described by equations wholly similar to (1); such equations model the so-called nominal system $\Sigma_0$, whose transfer matrix will be denoted by $P_0(s)$. With abuse of location, in the following $P_0(s)$ and $P(s)$ will be also called the nominal and the perturbed plant, respectively.

If $K$ is a finite dimensional linear time-invariant dynamic compensator (whose state space description is wholly similar to (1) by means of matrices $A_K \in \mathbb{R}^{n \times n}$, $B_K \in \mathbb{R}^{n \times p}$, $C_K \in \mathbb{R}^{q \times n}$ and $D_K \in \mathbb{R}^{q \times q}$), the closed-loop system obtained by connecting systems $S$ and $K$ as in Fig.1 will be called $\Sigma$, whereas the nominal closed-loop system, obtained by replacing $S$ with $S_0$ in such a connection, will be called $\Sigma_0$. The dynamic matrices of $\Sigma$ and $\Sigma_0$ will be denoted by $A_{\Sigma}$ and $A_{\Sigma_0}$, respectively, and the transfer matrix of $K$ will be denoted by $K(s)$. It is recalled that, if $\Sigma$ is well-posed, i.e.:

$$
\det(I - DD_K) \neq 0,
$$

then the dimension of the state of $\Sigma$ is $n + n_K$, and the following relation holds:

$$
\frac{\det(sI - A_{\Sigma})}{\det(sI - A)\det(sI - A_K)} = \frac{\det(I - P(s)K(s))}{\det(I - DD_K)}. \quad (2)
$$

Asymptotic stability is an obvious requirement for $\Sigma$, but possibly too weak, if a prescribed rate of exponential decay of the free responses is to be guaranteed. Therefore, for a fixed $\alpha \geq 0$, define $C_g := \{s \in \mathbb{C} : \text{Re}(s) < -\alpha\}$; if all the eigenvalues of the closed-loop system $\Sigma$ belong to $C_g$, $\Sigma$ will be said to be $C_g$-stable; in addition, system $S$ will be said to be $C_g$-stabilizable [or $C_g$-detectable] if there exists $F \in \mathbb{R}^{p \times n}$ [or $V \in \mathbb{R}^{q \times q}$] such that all the eigenvalues of $A + BF$ [or $A + VC$] belong to $C_g$. It is clear that, if $\alpha > 0$, $C_g$-stability is a stronger requirement than the mere asymptotic stability (obtained for $\alpha = 0$), so it will be referred to also as strengthened stability. Further reasons for using $\alpha > 0$ will be discussed in the subsequent Section 3.

Remark 1 If system $S$ is $C_g$-stabilizable and $C_g$-detectable, then any eigenvalue $\lambda$ of $S$ that is not in $C_g$ is also a pole of $P(s)$ - so that, under the mentioned hypotheses, these two terms will be considered to be equivalent - and the algebraic multiplicity of $\lambda$ (i.e., its multiplicity in the characteristic polynomial of $A$) coincides with its multiplicity as a pole of $P(s)$, counted according to the McMillan degree of $P(s)$ (i.e., it coincides with its multiplicity in the least common denominator of all non zero minors of $P(s)$) [13].

The topic of this paper are well-known sufficient conditions, to be checked on the nominal closed-loop system $\Sigma_0$, guaranteeing that the property of $C_g$-stability, that is assumed for $\Sigma_0$, is maintained also in the perturbed closed-loop system $\Sigma$. Such conditions refer to the representation of the uncertainties affecting the transfer matrix $P(s)$ of system $S$ in the form of unstructured perturbations, either additive or multiplicative.

The representation of uncertainties in the form of unstructured additive perturbations, i.e.:

$$
P(s) = P_0(s) + \delta P(s), \quad (3)
$$

Figure 1: The closed-loop system $\Sigma$. 
is always possible and allows to take into account any inaccuracy in identification and all errors associated with modeling, including parameter changes and neglected dynamics (thus allowing the order \( n \) of \( S \) to be different and even much larger than the order \( n_0 \) of \( S_0 \)), within the hypothesis that \( S \) is linear. It will be assumed that the uncertainties about \( P(s) \) can be represented also in the form of unstructured output multiplicative perturbations, i.e., it will be assumed the existence for \( P(s) \) of a \( q \times q \) rational matrix \( \delta^P(s) \) such that:

\[
P(s) = \left( I + \delta^P(s) \right) P_0(s),
\]

or, equivalently, such that:

\[
\delta^P(s) = \delta^P(s)P_0(s).
\]

**Remark 2** It is stressed that, unlike what is suggested by (4), system \( S \) (whose robust \( \mathbb{C}_g \)-stabilization through the use of compensator \( K \) can be guaranteed by the subsequent Lemmas 1 and 2) is not (in general) the series connection of \( S_0 \) and a realization of \( I + \delta^P(s) \), but it is rather the actual and complete state-space representation of the plant under consideration (with the actual values of all parameters), whose nominal (and, possibly, incomplete and/or inaccurate) state-space representation is \( S_0 \). This implies that unstable cancellations between \( I + \delta^P(s) \) and \( P_0(s) \) are allowed in (4), in principle, since they do not imply any loss of stabilizability or detectability in \( S \) with respect to \( S_0 \).

Notice that the existence of \( \delta^P(s) \) such that (4) holds is guaranteed for all \( P(s) \) if \( P_0(s) \) has full column rank in the rational field.

Denoting with \( T_0(s) \) the so-called output complementary sensitivity matrix of the nominal closed-loop system \( \Sigma_0 \) (under the assumption that \( \Sigma_0 \) is well-defined), defined by:

\[
T_0(s) = P_0(s)K(s)\left( I - P_0(s)K(s) \right)^{-1},
\]

the following useful identity holds:

\[
I - P(s)K(s) = \left( I - \delta^P(s)T_0(s) \right)\left( I - P_0(s)K(s) \right).
\]

Denote with \( \sigma_{\text{max}} \) the maximum singular value of the argument matrix. The following lemma gives conditions for the robust \( \mathbb{C}_g \)-stability of system \( \Sigma \) [9, 10], and can be proven by means of a suitable extension of the Nyquist criterion.

**Lemma 1** Assume compensator \( K \) to have no uncertainties. If, for a fixed \( \alpha \geq 0 \) characterizing \( \mathbb{C}_g \):

(i) \( \Sigma_0 \) is well-posed;

(ii) \( \Sigma_0 \) is \( \mathbb{C}_g \)-stable;

(iii) \( \sigma_{\text{max}}|T_0(\alpha + j\omega)| < \frac{1}{l_m(\omega)}, \quad \forall \omega \in \mathbb{R}, \)

where \( l_m(\omega) \) is a positive and continuous function of \( \omega \), then system \( \Sigma \) is \( \mathbb{C}_g \)-stable for all the perturbed systems \( S \) such that:

(a) systems \( S \) and \( S_0 \) have the same number of eigenvalues not in \( \mathbb{C}_g \) (including algebraic multiplicities);

(b) \( \Sigma \) is still well-posed;

(c) \( P(s) \) can be expressed in the form (4) (i.e., there exists \( \delta^P(s) \) such that (4) holds), with \( \delta^P(s) \) satisfying the relation:

\[
|\delta^P(-\alpha + j\omega)| \leq l_m(\omega), \quad \forall \omega \in \mathbb{R}.
\]

In view of Remark 2, in the following, the family of perturbed systems \( S \) satisfying the conditions (a), (b) and (c) of Lemma 1 will be denoted by \( \mathcal{M}_{\mathbb{B}G}(\alpha, l_m) \). The subscript refers to the fact that Lemma 2 can be proven by means of a suitable extension of the small gain theorem, thus recalling that the multiplicative perturbations relating the transfer matrices \( P(s) \) of all the perturbed systems in this family to the nominal one \( P_0(s) \), are elements of \( \mathcal{R}\mathcal{H}_{-\alpha, \infty} \).

**Remark 3** The strict inequality in hypothesis (iii) of Lemma 1 [or 2] can be relaxed to \( \leq \) if the inequality in condition (c) [respectively, (b)] is strengthened to \( < \).
Remark 4 When the uncertainties about the description of system $S$ affect its transfer matrix in the form of unstructured input multiplicative perturbations, i.e., through a $p \times p$ rational matrix $\delta^P(s)$ such that:

$$P(s) = P_0(s) \left( I + \delta^P(s) \right)$$

(whose existence is guaranteed for all $P(s)$ by $P_0(s)$ having full row rank in the rational field), both Lemmas 1 and 2 still hold by substituting (4) with (7), and $T_0(s)$ with the so-called input complementary sensitivity matrix $R_0(s)$ of the well-posed nominal closed-loop system $\Sigma_0$, defined by:

$$R_0(s) = K(s)P_0(s) \left( I - K(s)P_0(s) \right)^{-1}.$$

\[\square\]

3 Main results

In this section the families $\mathcal{M}_{BU}(\alpha, l_m)$ and $\mathcal{M}_{SC}(\alpha, W_1, W_2, \rho)$ will be considered in detail. Firstly note that the hypotheses (i)-(iii) and the conditions (a), (b) and (c) of Lemma 1 guarantee the $\mathbb{C}_g$-stability of $\Sigma$, in addition to the one of $\Sigma_0$, so that they guarantee the $\mathbb{C}_g$-stabilizability and the $\mathbb{C}_g$-detectability of systems $S$ and $S_0$, as well as the hypotheses (i)-(iii) and the conditions (a) and (b) of Lemma 2 do; thus, the implication stressed by Remark 1 follows about the use of the terms “eigenvalue of $S$ not in $\mathbb{C}_g$” and “pole of $P(s)$ not in $\mathbb{C}_g$”.

Under the hypotheses and the conditions of Lemma 1 it is obvious that $P(s)$ cannot have poles on the $-\alpha + j\omega$-axis different from those of $P_0(s)$, since such poles should be introduced by the factor $I + \delta^P(s)$, in contradiction with condition (c). On the other hand, still under the hypotheses (i)-(iii) of Lemma 1, if $P_0(s)$ has poles on the $-\alpha + j\omega$-axis, one could ask whether there exist perturbed systems $S$ in the family $\mathcal{M}_{BU}(\alpha, l_m)$ such that these poles disappear in $P(s)$ or reduce their algebraic multiplicities (this might be compatible with conditions (a) and (c) of Lemma 1 if such poles of $P_0(s)$ are shifted to poles of $P(s)$ in the right-hand side of the $-\alpha + j\omega$-axis). The answer is negative, as it is stated by the following proposition.

Proposition 1 Under the hypotheses (i), (ii) and (iii) of Lemma 1, the nominal system $S_0$ and all the perturbed systems $S$ in the family $\mathcal{M}_{BU}(\alpha, l_m)$ have the same eigenvalues on the $-\alpha + j\omega$-axis, with the same algebraic multiplicities.

Proof. From (ii) it follows that $T_0(s)$ has no poles on the $-\alpha + j\omega$-axis, whereas from (c) it follows that $\delta^P(s)$ has no poles on the $-\alpha + j\omega$-axis. Hence $\det \left( I - \delta^P(s)T_0(s) \right)$ has no poles on the $-\alpha + j\omega$-axis, either. Moreover:

$$\sigma[\delta^P(-\alpha + j\omega)T_0(-\alpha + j\omega)] \leq \sigma[\delta^P(-\alpha + j\omega)]|T_0(-\alpha + j\omega)| < l_m(\omega) \frac{1}{l_m(\omega)} = 1, \quad \forall \omega \in \mathbb{R}. \quad (8)$$

Hence,

$$\sigma[I - \delta^P(-\alpha + j\omega)T_0(-\alpha + j\omega)] > 0, \quad \forall \omega \in \mathbb{R},$$

and therefore $\det \left( I - \delta^P(s)T_0(s) \right)$ has not even zeros on the $-\alpha + j\omega$-axis.

On the other hand, identities (2) and (6) (the former written both for the nominal and the perturbed system) and the hypothesis that $K$ has no uncertainties, yield:

$$\frac{\det(I - DD_K)}{\det(I - D_0D_K)} \frac{\det(sI - A_\Sigma)}{\det(sI - A)} \frac{\det(sI - A_0)}{\det(sI - A_{\Sigma_0})} = \frac{\det \left( I - \delta^P(s)T_0(s) \right)}{\det \left( I - \delta^P(s)T_0(s) \right)}.$$

(9)

Since $\Sigma_0$ and $\Sigma$ are $\mathbb{C}_g$-stable, and hence free of eigenvalues on the $-\alpha + j\omega$-axis, the proven lack of zeros and poles of $\det \left( I - \delta^P(s)T_0(s) \right)$ on the $-\alpha + j\omega$-axis implies that all the eigenvalues of $A_0$ on the $-\alpha + j\omega$-axis are also eigenvalues of $A$ with the same algebraic multiplicities (and vice-versa). \[\square\]

Now, under the hypotheses and the conditions of Lemma 2, it is obvious that $P(s)$ cannot have poles not in $\mathbb{C}_g$ different from those of $P_0(s)$, since such poles should be introduced by the factor $I + \delta^P(s)$, in contradiction with condition (b), implying $\delta^P(s) \in \mathcal{RH}_{-\alpha, \infty}$. On the other hand, still under the hypotheses (i)-(iii) of Lemma 2, if $P_0(s)$ has poles not in $\mathbb{C}_g$, one could ask whether there exist perturbed systems $S$ in the family $\mathcal{M}_{SC}(\alpha, W_1, W_2, \rho)$ such that these poles disappear in $P(s)$ or reduce their algebraic multiplicities (this might be compatible with the conditions of Lemma 2 if such poles of $P_0(s)$ are shifted to poles of $P(s)$ in the left-hand side of the $-\alpha + j\omega$-axis, so as to violate condition (a) of Lemma 1). Also in this case the answer is negative, as it is stated by the following proposition.

Proposition 2 Under the hypotheses (i), (ii) and (iii) of Lemma 2, the nominal system $S_0$ and all the perturbed systems $S$ in the family $\mathcal{M}_{SC}(\alpha, W_1, W_2, \rho)$ have the same eigenvalues not in $\mathbb{C}_g$, with the same algebraic multiplicities.

Proof. Let $S_{W_1}$, $S_{W_2}$ and $S_\Delta$ be minimal realizations of $W_1(s)$, $W_2(s)$ and $\Delta(s)$, respectively. Since such matrices are elements of $\mathcal{RH}_{-\alpha, \infty}$, then $S_{W_1}$, $S_{W_2}$ and $S_\Delta$ are $\mathbb{C}_g$-stable.

Consider the closed-loop system $\hat{\Sigma}$ in Fig. 2, where the subsystem $\hat{S}$ is defined in Fig.3. Notice that $\hat{S}$ is well-posed.

![Figure 2: The closed-loop system $\hat{S}$](image-url)
whereas for perturbed systems longs to eigenvalues of that eigenvalues of the nominal system it must be simply free of poles on the $-\alpha + j\omega$-axis. Moreover, under the hypotheses of Lemma 1, only to remain unchanged in number (see [9] and the subsequent Example 1).

From this reasoning it follows that the family $\mathcal{M}_{BU}(\alpha, l_m)$ is more significant, under the hypotheses of Lemma 1, than the family $\mathcal{M}_{SG}(\alpha, W_1, W_2, \rho)$ under the hypotheses of Lemma 2, at least in the case of $P_0(s)$ having poles not in $\mathbb{C}_g$, since the latter family requires, in particular, that the locations and the multiplicities of all those poles are exactly known. Consider also that it is possible to choose $\alpha$ so that the nominal system $S_0$ has no eigenvalues on the $-\alpha + j\omega$-axis.

In a particular (but significant in practice) case, it can be easily shown that, under the hypotheses of Lemma 2, the family $\mathcal{M}_{BU}(\alpha, l_m)$ contains the family $\mathcal{M}_{SG}(\alpha, W_1, W_2, \rho)$, so that the former is larger than the latter (see again the subsequent Example 1). In fact, for a fixed scalar proper rational function $r(s)$ free of poles not in $\mathbb{C}_g$ and of zeros on the $-\alpha + j\omega$-axis, and for $W_1(s) = I$ and $W_2(s) = \text{diag} \{ r(s), ..., r(s) \}$, suppose to have found a compensator $K$ such that all the hypotheses (i)-(iii) of Lemma 2 are satisfied, and in particular $||W_2(s)T_0(s)||_{-\alpha, \infty} < \rho^{-1}$, $\rho > 0$. By letting $l_m(\omega) = |r(-\alpha + j\omega)|\rho$, we have:

$$\mathcal{S}[T_0(-\alpha + j\omega)] =$$

$$= |r(-\alpha + j\omega)| W_2(-\alpha + j\omega) ||T_0(-\alpha + j\omega)||_{-\alpha, \infty} <$$

$$< |r(-\alpha + j\omega)|^{-1} \rho^{-1}, \quad \forall \omega \in \mathbb{R},$$

so that also the hypotheses of Lemma 1 are satisfied. Now, it is easy to verify that a perturbed system $S$ satisfying the conditions (a) and (b) of Lemma 2 satisfies also the conditions (a), (b) and (c) of Lemma 1. In particular, condition (a) of Lemma 1 is guaranteed by Proposition 2, condition (b) is guaranteed by Lemma 2 and condition (c) is satisfied in view of the following relations:

$$|T_0(-\alpha + j\omega)| =$$

$$\leq |r(-\alpha + j\omega)| ||T_0(-\alpha + j\omega)||_{-\alpha, \infty} \leq$$

$$\leq |r(-\alpha + j\omega)| \rho, \quad \forall \omega \in \mathbb{R}.$$
conditions (a), (b) and (c) of Lemma 1 for this $\alpha > 0$ (if it excludes such a stable pole from $\mathbb{C}_g$), but violate the condition (a) of the same lemma for $\alpha = 0$ (see the subsequent Example 2). In particular, if the nominal system $S_0$ is already asymptotically stable, by considering $\alpha = 0$, only perturbed systems $S$ maintaining the asymptotic stability of $S_0$ are allowed in the family $\mathcal{M}_{BV}(0,l_m)$, in view of condition (a), so that Lemma 1 would guarantee the robust stability for already robustly stable plants only. The choice of a suitable $\alpha > -\mbox{Re}(\lambda_i)$, where $\lambda_i$ are the stable eigenvalues of $A_0$ that move “more rapidly” across the $\omega$-axis, may allow to overcome this difficulty (see again the subsequent Example 2).

Lastly, a careful choice of $\alpha > 0$ might be useful in order to avoid that some pole or transmission zero of $P_0(s)$ lies on the $\mathbb{C}_g$-boundary, since this would restrict severely the family $\mathcal{M}_{BV}(\alpha,l_m)$, in view of Proposition 1 and the subsequent Remark 5, respectively.

Notice that there are less advantages in the application of Lemma 2 if $\alpha > 0$, even if $C_0$ is asymptotically stable, because of the severe limitations implied by Proposition 2. In addition, also in the case when $P_0(s)$ has a transmission zero with nonnegative real part, for any choice of $\alpha \geq 0$ the family $\mathcal{M}_{SC}(\alpha,W_1,W_2,\rho)$ is severely restricted, since, in view of the subsequent Remark 5, the actual perturbed systems $S$ are very likely to be characterized by corresponding $\delta^P(s)$ having such a transmission zero as a pole, so that $\delta^P(s)$ is not element of $\mathcal{R} \mathcal{H}_{\infty}$, and therefore neither of $\mathcal{R} \mathcal{H}_{-\alpha,\infty}$ for any $\alpha > 0$, thus violating the condition (b) of Lemma 2.

Remark 5 If $P_0(s)$ has a transmission zero on the $-\alpha+\omega$-axis [or not in $\mathbb{C}_g$], then it is very likely that, for any perturbed system $S$ for which there exists $\delta^P(s)$ satisfying (4), such a transmission zero appears as a pole of $\delta^P(s)$, thus preventing $S$ to belong to family $\mathcal{M}_{BV}(\alpha,l_m)$ [or $\mathcal{M}_{SC}(\alpha,W_1,W_2,\rho)$]. This assertion can be easily justified under the hypothesis that

$$\text{rank}[P_0(s)] = \min(q,p)$$

(10)
in the rational field. Then assume that (10) holds.

If $q < p$, so that $P_0(s)$ has full row rank in the rational field, denote by $P_0^{-R}(s)$ a minimal right inverse of $P_0(s)$. If, for a perturbed system $S$ and for the corresponding $P(s)$, there exists $\delta^P(s)$ such that (4) holds, from (5) it follows that

$$\delta^P(s) = \delta_P(s)P_0^{-R}(s).$$

(11)

If $q \geq p$, so that $P_0(s)$ has full column rank in the rational field, denote by $P_0^{-L}(s)$ a minimal left inverse of $P_0(s)$; then, for any perturbed system $S$ and for the corresponding $P(s)$ there exists $\delta^P(s)$ such that (4) holds, and all such $\delta^P(s)$ are expressed by the relation:

$$\delta^P(s) = \delta_P(s)P_0^{-L}(s) + \delta^P(s),$$

(12)

by considering all rational $\delta^P(s)$ such that $\delta^P(s)P_0(s) = 0$.

Since all the transmission zeros of $R_0(s)$ are poles of $P_0^{-R}(s)$, or $P_0^{-L}(s)$, respectively, then, in view of (11) and (12), if $P_0(s)$ has a transmission zero on the $-\alpha+\omega$-axis [or not in $\mathbb{C}_g$], $\delta^P(s)$ is likely to have a pole in the same location, even if $\delta_P(s) := P(s) - R_0(s)$ has no poles on the $-\alpha+\omega$-axis [or not in $\mathbb{C}_g$], thus justifying the assertion. \qed

The following very simple example shows that, under the hypotheses of Lemma 1, the family $\mathcal{M}_{BV}(\alpha,l_m)$ actually includes perturbed systems $S$ having eigenvalues lying in the right-hand side of the $-\alpha+\omega$-axis different from the ones of the nominal system $S_0$. It also confirms that the family $\mathcal{M}_{BV}(\alpha,l_m)$ may be much more significant than the family $\mathcal{M}_{SC}(\alpha,W_1,W_2,\rho)$.

**Example 1** Consider the nominal unstable plant:

$$P_0(s) = \frac{1}{s-1}$$

and the stabilizing compensator:

$$K(s) = \frac{-12}{s+6}.$$  

Then let the perturbed plant be:

$$P(s) = \frac{\beta}{s-p}, \quad \beta,p \in \mathbb{R}.$$  

By applying the Routh criterion, it is very simple to compute the whole set of pairs $(\beta,p)$ characterizing the perturbed plants that are stabilized by the compensator $K$. A portion of this set is represented by the gray region in Fig.4.

Now, referring to $\alpha = 0$, the set of pairs $(\beta,p)$ that characterize the perturbed plants satisfying the conditions (a), (b) and (c) of Lemma 1 (amended as specified in Remark 3), with the choice of the best-shaping $l_m(\omega)$, i.e.:

$$l_m(\omega) = \frac{1}{\sigma[I_0(\omega)]},$$

is represented by the dark gray region in Fig.4, whereas the set of pairs $(\beta,p)$ that characterize the perturbed plants satisfying the conditions (a) and (b) of Lemma 2 is always included in the vertical segment drawn in the same figure, for
any choice of the weighting functions $W_1(s)$ and $W_2(s)$ in $\mathcal{RH}_{\infty}$ (it is not surprising, since in view of Proposition 2 the unstable pole of $P_0(s)$ cannot change its location), and reduces to the nominal point, if at least one of such weighting functions is chosen strictly proper.

It is stressed that if the perturbed plant is:

$$P(s) = \frac{\gamma s + \beta}{s - p}, \quad \gamma, \beta, p \in \mathbb{R},$$

(thus introducing an input/output direct link), then there is no system $S$ satisfying the conditions (a) and (b) of Lemma 2, and corresponding to perturbed plants with $\gamma \neq 0$, whereas systems $S$ actually exist, satisfying conditions (a), (b) and (c) of Lemma 1, and corresponding to perturbed plants with $\gamma \neq 0$.

The following very simple example shows that the use of Lemma 1 with a proper choice of $\alpha > 0$ (thus requiring the corresponding robust strengthened stability) allows to take into account perturbed systems $S$ violating condition (a) of Lemma 1 for $\alpha = 0$ in consequence of possibly small variations of the parameters of the plant from their nominal values (such a situation might be quite frequent in practice).

**Example 2** Consider the nominal plant:

$$P_0(s) = \frac{1}{s^2 + 0.2s + 1},$$

whose poles are stable but very close to the $\omega$-axis (their real part is $\approx 0.1$), and the stabilizing compensator:

$$K(s) = \frac{-90(s + 2)}{s + 16}.$$

Then, let the perturbed plant be:

$$P(s) = \frac{\beta}{s^2 + 2\zeta s + 1}, \quad \beta, \zeta \in \mathbb{R}.$$

The set of pairs $(\beta, \zeta)$ that characterize the perturbed plants satisfying the conditions (a), (b) and (c) of Lemma 1 (amended as specified in Remark 3), with the choice of the best-shaping $l_m(\omega)$, i.e.:

$$l_m(\omega) = \frac{1}{\sigma[T_0(-\alpha + j\omega)]},$$

is represented in Fig.5 for $\alpha = 0$ and for $\alpha = 1$ (the latter choice is allowed, since the real part of the nominal closed-loop poles is less than $-1$) by the dark gray region and by the light gray region, respectively.

The analysis of these regions confirms that the most convenient choice of $\alpha \geq 0$ depends on the expected range of values of the parameter $\zeta$: e.g., for $\zeta$ ranging from 0.2 to $-1$ the choice of $\alpha = 1$ seems to be appropriate, whereas for $\zeta$ ranging from 0 to 1.5 (i.e., if it is known that the poles of $P(s)$ remain stable) the choice of $\alpha = 0$ is still the more suitable.

It is stressed that in this example the union of the two regions represented in Fig.5 can be used as an estimate of the whole set of pairs $(\beta, \zeta)$ characterizing the perturbed plants that are asymptotically stabilized by the chosen compensator $K$.

4 Additive perturbations

In this section well-known sufficient conditions for the robust $C_y$-stability of system $\Sigma$, referred to the unstructured additive perturbations $\delta_P(s)$ (defined by (3)) of the nominal transfer matrix $P_0(s)$ of the plant to be stabilized, will be analyzed.

Denoting with $V_0(s)$ the so-called control sensitivity matrix of the nominal closed-loop system $\Sigma_0$ (under the assumption that $\Sigma_0$ is well-posed), defined by:

$$V_0(s) = K(s)\left(I - P_0(s)K(s)\right)^{-1},$$

the following useful identity holds:

$$I - P(s)K(s) = \left(I - \delta_P(s)V_0(s)\right)\left(I - P_0(s)K(s)\right).$$

The following lemma gives conditions for the robust $C_y$-stability of system $\Sigma$ [10, 14], and can be proven by means of a suitable extension of the Nyquist criterion.

**Lemma 3** Assume compensator $K$ to have no uncertainties. If, for a fixed $\alpha \geq 0$ characterizing $C_y$:

(i) $\Sigma_0$ is well-posed;

(ii) $\Sigma_0$ is $C_y$-stable;

(iii) $\sigma[V_0(-\alpha + j\omega)] < \frac{1}{l_y(\omega)}, \quad \forall \omega \in \mathbb{R},$ where $l_y(\omega)$ is a positive and continuous function of $\omega$, then system $\Sigma$ is $C_y$-stable for all the perturbed systems $S$ such that:

(a) systems $S$ and $S_0$ have the same number of eigenvalues not in $C_y$ (including algebraic multiplicities);

(b) $\Sigma$ is still well-posed;

(c) $\delta_P(s)$ in (3) satisfies the relation $\sigma[\delta_P(-\alpha + j\omega)] \leq l_y(\omega), \forall \omega \in \mathbb{R}.$

![Figure 5: Robust stability regions considered in Example 2.](image-url)
In view of Remark 2, in the following, the family of perturbed systems $S$ satisfying the conditions (a), (b) and (c) of Lemma 3 will be denoted by $A_{BU}(\alpha, I_a)$. The subscript recalls that the transfer matrices $P(s)$ of all the perturbed systems in this family could be viewed as belonging to a “ball of uncertainty” around the nominal transfer matrix $P_0(s)$ [10].

Also the following lemma gives conditions for the robust $\mathbb{C}_p$-stability of $\Sigma$. It generalizes well-known results (see e.g. [7, 8]) and can be proven by means of a suitable extension of the small gain theorem.

**Lemma 4** Assume compensator $K$ to have no uncertainties. If, for a fixed $\alpha \geq 0$ characterizing $\mathbb{C}_p$,:

(i) $\Sigma_0$ is well-posed;

(ii) $\Sigma_0$ is $\mathbb{C}_p$-stable;

(iii) $\|W_2(s)V_0(s)W_1(s)\|_{-\alpha, \infty} < \rho^{-1}$, $\rho > 0$, where $W_1(s), W_2(s) \in \mathcal{R}H_{-\alpha, \infty}$, then system $\Sigma$ is well-posed and $\mathbb{C}_p$-stable for all the perturbed systems $S$ such that:

(a) system $S$ is still $\mathbb{C}_p$-stabilizable and $\mathbb{C}_p$-detectable;

(b) $\delta _P(s)$ in (3) satisfies the relation $\delta _P(s) = W_1(s)\Delta (s)W_2(s)$, where $\Delta (s) \in \mathcal{R}H_{-\alpha, \infty}$ is such that $\|\Delta (s)\|_{-\alpha, \infty} \leq \rho$.

In view of Remark 2, in the following, the family of perturbed systems $S$ satisfying the conditions (a) and (b) of Lemma 4 will be denoted by $A_{SG}(\alpha, W_1, W_2, \rho)$. The subscript refers to the fact that Lemma 4 can be proven by means of (a suitable extension of) the small gain theorem, thus recalling that the additive perturbations relating the transfer matrices $P(s)$ of all the perturbed systems in this family to the nominal one $P_0(s)$, are elements of $\mathcal{R}H_{-\alpha, \infty}$.

**Remark 6** The strict inequality in hypothesis (iii) of Lemma 3 (or 4) can be relaxed to $\leq$ if the inequality in condition (c) [respectively, (b)] is strengthened to $<$. □

An analysis of the families $A_{BU}(\alpha, I_a)$ and $A_{SG}(\alpha, W_1, W_2, \rho)$ can be carried out in a wholly similar way to what has been made in the previous section about the families $M_{BU}(\alpha, I_m)$ and $M_{SG}(\alpha, W_1, W_2, \rho)$. In particular, the following propositions:

**Proposition 3** Under the hypotheses (i), (ii) and (iii) of Lemma 3, the nominal system $S_0$ and all the perturbed systems $S$ in the family $A_{BU}(\alpha, I_a)$ have the same eigenvalues on the $-\alpha + j\omega$-axis, with the same algebraic multiplicities.

**Proposition 4** Under the hypotheses (i), (ii) and (iii) of Lemma 4, the nominal system $S_0$ and all the perturbed systems $S$ in the family $A_{SG}(\alpha, W_1, W_2, \rho)$ have the same eigenvalues not in $\mathbb{C}_p$, with the same algebraic multiplicities.

(that are nearly obvious, and can be proven similarly to Propositions 1 and 2, respectively, or by means of appropriate tools of the theory of realization) lead us to infer that the family $A_{BU}(\alpha, I_a)$ is more significant, under the hypotheses of Lemma 3, than the family $A_{SG}(\alpha, W_1, W_2, \rho)$ under the hypotheses of Lemma 4, at least in the case of $P_0(s)$ having poles not in $\mathbb{C}_p$, since the latter family requires that the locations and the multiplicities of all those poles are exactly known (what is implied by condition (b) of Lemma 4, by virtue of which $\delta _P(s)$ corresponding to a perturbed system $S$ in the family $A_{SG}(\alpha, W_1, W_2, \rho)$, must be an element of $\mathcal{R}H_{-\alpha, \infty}$). This can be stressed by stating that, for simple but reasonable choices of $I_a(\omega)$, $W_1(s)$ and $W_2(s)$, under the hypotheses (i)-(iii) of Lemma 4, the family $A_{BU}(\alpha, I_a)$ contains the family $A_{SG}(\alpha, W_1, W_2, \rho)$ (it can be derived in the same way as the similar statement about families $M_{BU}(\alpha, I_m)$ and $M_{SG}(\alpha, W_1, W_2, \rho)$ has been).

Notice that also the choice of $\alpha$ can be discussed through arguments similar to the ones discussed in the previous section, when multiplicative perturbations were considered.

Then, it seems useful to carry out the further comparison between the families $M_{BU}(\alpha, I_m)$ and $A_{BU}(\alpha, I_a)$, under the hypotheses of Lemmas 1 and 3; a compensator $K$ such that the nominal closed-loop system $\Sigma_0$ satisfies the hypotheses (i) and (ii) of Lemmas 1 and 3, will be considered. For simplicity, we shall refer to the case of $\alpha = 0$, i.e., to the mere asymptotic stability, but the subsequent conclusions hold for any $\alpha \geq 0$.

Then, suppose $P_0(s)$ is free of poles on the Nyquist contour, and consider a bound $I_a(\omega)$ satisfying the hypothesis (iii) of Lemma 3. A sufficient condition guaranteeing the bound $I_m(\omega)$ to satisfy the hypothesis (iii) of Lemma 1 is the following one:

$$I_m(\omega)\|P_0(j\omega)\| \leq I_a(\omega), \quad \forall \omega \in \mathbb{R}, \quad (13)$$

since, in this case,

$$\|P_0(j\omega)\| \leq \frac{\|P_0(j\omega)\| \|V_0(j\omega)\|}{\|I_a(j\omega)\|} \leq \frac{1}{I_m(\omega)}, \quad \forall \omega \in \mathbb{R}$$

With these choices of the bounds $I_a(\omega)$ and $I_m(\omega)$, the perturbed systems $S$ in the family $M_{BU}(0, I_m)$ also belong to the family $A_{BU}(0, I_a)$; in particular, we note that they satisfy the condition (c) of Lemma 3, since:

$$\|\delta _P(j\omega)\| \leq \|\delta _P(j\omega)\| \|P_0(j\omega)\| \leq I_m(\omega)\|P_0(j\omega)\| \leq I_a(\omega), \quad \forall \omega \in \mathbb{R}$$

Therefore, the family $M_{BU}(0, I_m)$ is contained in the family $A_{BU}(0, I_a)$.

Now, suppose $P_0(s)$ is free of transmissions zeros on the Nyquist contour and has full column rank in the rational field, so that a left inverse $P_0^{-L}(s)$ of $P_0(s)$ exists, and for all additive perturbations $\delta _P(s)$ satisfying (3) there exists $\delta _P^*(s)$ such that $\delta _P(s) = \delta _P^*(s)P_0(s)$, so as to satisfy (4) (in particular, $\delta _P(s)$ can be chosen as $\delta _P(s) = \delta _P(s)P_0^{-L}(s)$). If $P_0^{-L}(s)$ is chosen to be minimal, its poles are all the transmission zeros of $P_0(s)$, so that $P_0^{-L}(s)$ has no poles on the Nyquist contour. Then, consider a bound $I_m(\omega)$ satisfying the hypothesis (iii) of Lemma 1. A sufficient condition guaranteeing the
bound $l_a(\omega)$ to satisfy the hypothesis (iii) of Lemma 3 is the following one:

$$l_a(\omega)\sigma[P_0^{L}(\omega)] \leq l_m(\omega), \quad \forall \omega \in \mathbb{R}, \quad (14)$$

since, in this case,

$$\sigma[V_0(\omega)] \leq \sigma[P_0^{L}(\omega)]\sigma[T_0(\omega)] < \frac{1}{l_m(\omega)}, \quad \forall \omega \in \mathbb{R}.$$

With these choices of the bounds $l_m(\omega)$ and $l_a(\omega)$, the perturbed systems $S$ in the family $A_{BU}(0,l_a)$ also belong to the family $M_{BU}(0,l_m)$; in particular, we note that they satisfy the condition (c) of Lemma 1, since:

$$\sigma[\delta^P(\omega)] \leq \sigma[\delta_P(\omega)]\sigma[P_0^{L}(\omega)] \leq l_a(\omega)\sigma[P_0^{L}(\omega)] \leq l_m(\omega), \quad \forall \omega \in \mathbb{R}. $$

Therefore, the family $A_{BU}(0,l_a)$ is contained in the family $M_{BU}(0,l_m)$.

If $P_0(s)$ is free both of poles and of transmissions zeros on the Nyquist contour and has full column rank in the rational field, then both previous discussions can be applied, and one could argue that the equality of the families $M_{BU}(0,l_m)$ and $A_{BU}(0,l_a)$, if both (13) and (14) hold. However, although this is trivially true for SISO systems by choosing:

$$l_m(\omega)|P_0(\omega)| = l_a(\omega), \quad \forall \omega \in \mathbb{R},$$

nevertheless a similar choice for MIMO systems is unlikely to be available, since it is readily seen that there exists a pair of positive and continuous functions $l_a(\omega)$ and $l_m(\omega)$ satisfying (13) and (14) if and only if

$$\sigma[P_0(\omega)]\sigma[P_0^{L}(\omega)] \leq 1, \quad \forall \omega \in \mathbb{R},$$

and this, for $p = q$ (i.e., for square nonsingular $P_0(s)$), means that the maximum and the minimum singular values of $P_0(\omega)$ coincide for all $\omega \in \mathbb{R}$.

It is stressed that if, in the above analysis, (13) and (14) fail to hold, then the inclusions between the two families may fail to hold. However, even when the bounds $l_a(\omega)$ and $l_m(\omega)$ are chosen independently, some further comment may be helpful.

It was shown in Remark 5 that, if $P_0(s)$ has transmission zeros on the Nyquist contour and, for a given $P(s)$, there exists $\delta^P(s)$ such that (4) holds, then $\delta^P(s)$ is likely to have poles on the Nyquist contour, even if the corresponding $\delta_P(s)$ has not. Therefore, if $P_0(s)$ has transmission zeros, but is free of poles, on the Nyquist contour, perturbed systems $S$ might exist, that are in $A_{BU}(0,l_a)$, but not in $M_{BU}(0,l_m)$. This does not mean, in general, that, under the mentioned hypotheses, the family $A_{BU}(0,l_a)$ includes the family $M_{BU}(0,l_m)$, but we can argue that in this case the former family is more significant than the latter.

By converse, if $P_0(s)$ has poles on the Nyquist contour, then from (3) it follows that the additive perturbation $\delta_P(s)$ corresponding to a perturbed system $S$ is likely to have poles on the Nyquist contour, whereas the corresponding $\delta^P(s)$ satisfying (4) for the same $P(s)$ (if it exists) may have no such poles, since they may be introduced in (4) by $P_0(s)$. Therefore, if $P_0(s)$ has poles, but is free of transmissions zeros, on the Nyquist contour, perturbed systems $S$ might exist, that are in $M_{BU}(0,l_m)$, but not in $A_{BU}(0,l_a)$. Also in this case, this does not mean, in general, that, under the mentioned hypotheses, the family $M_{BU}(0,l_m)$ includes the family $A_{BU}(0,l_a)$, but we can argue that the former family may be more significant than the latter, for example when $\delta^P(s)$ satisfying (4) exists for all the perturbed plants $P(s)$ of interest, as it is shown by the following very simple example.

**Example 3** Consider the nominal unstable plant:

$$P_0(s) = \frac{1}{s(s-1)}$$

and the stabilizing compensator:

$$K(s) = \frac{-10(s+1)}{s+4}. $$

Then let the perturbed plant be:

$$P(s) = \frac{\beta}{s(s-p)}, \quad \beta, p \in \mathbb{R}. $$

By applying the Routh criterion, it is very simple to compute the whole set of pairs $(\beta, p)$ characterizing the perturbed plants that are stabilized by the compensator $K$. A portion of this set is represented by the gray region in Fig.6. It was shown in Remark 5 that, if $P_0(s)$ has transmission zeros on the Nyquist contour and, for a given $P(s)$, there exists $\delta^P(s)$ such that (4) holds, then $\delta^P(s)$ is likely to have poles on the Nyquist contour, even if the corresponding $\delta_P(s)$ has not. Therefore, if $P_0(s)$ has transmission zeros, but is free of poles, on the Nyquist contour, perturbed systems $S$ might exist, that are in $A_{BU}(0,l_a)$, but not in $M_{BU}(0,l_m)$. This does not mean, in general, that, under the mentioned hypotheses, the family $A_{BU}(0,l_a)$ includes the family $M_{BU}(0,l_m)$, but we can argue that in this case the former family is more significant than the latter.

![Figure 6: Robust stability regions considered in Example 3.](image-url)
Acknowledgments

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References


