Extended diffusive representations and application to non-standard oscillators

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Keywords: diffusive representations, pseudo-differential operators, fractional differential equations, fractional ARMA models, Lyapunov functionals

1 Context and Motivation

Pseudo-differential operators (PDO, see [1]) $\mathcal{D}$ of diffusive type have been first introduced in [2] for fractional derivatives and integrals (see [3]), and their theory has been elaborated in [4], the use of the diffusive representation (DR) $\mu_\mathcal{D}$ in a diffusive realisation – in the sense of systems theory – helps transforming a non-local in time pseudo-differential equation into a first order differential equation on an infinite-dimensional state-space, endowed with a Hilbert structure, which allows stability analysis (see [5]) and straightforward finite-dimensional approximations (see [6]). This approach reveals useful for both theoretical and numerical treatment of pseudo-differential equations (not only fractional differential equations), even time-varying and non-linear ones (see [7], [8]).

In this paper, several extensions of diffusive representations (DR) are presented.

- Extensions for continuous-time systems are first addressed in section 2: in the study of some fractional differential equations, the fractional integrals and derivatives can be extended to diffusive pseudo-differential operators (see [9]) that share a dissipativity property, which can easily be read on $\mu_\mathcal{D}$; this makes an interesting qualitative jump in the analysis.

Moreover, following ideas stemming from complex variable theory already introduced in [10, 5], diffusive representations of the second kind are presented in a general way: they can be used to synthesize, analyze and approximate such special functions as Bessel functions, and to model non-standard oscillators.

- In a second stage (in section 3), the leading ideas of the diffusive formulation in continuous time can be extended to discrete-time systems, as first introduced in [11]: a new framework is first settled; the now classical integrating filter of fractional order (ARFIMA, see [12]) is recast in the new framework, then extended to pseudo-difference equations of diffusive type, that share a dissipativity property.

As for the continuous-time case, taking advantage of some considerations on complex variable theory already made in [10], diffusive representations of the second kind are then introduced in order to model non-standard discrete-time oscillators (like Gegenbauer filters) that combine oscillations with an anomalous decay.

Throughout the paper, some worked out examples will be treated both from theoretical and numerical point of views; illustrative simulation results will be provided.

2 Extensions in continuous time

This section is divided in four parts. In §2.1, we recall the main definitions and properties of the so-called diffusive pseudo-differential operators of the first kind introduced in [4]; in §2.2, a family of oscillators with damping of diffusive type is fully analyzed and numerical simulations are presented in §2.3; finally, in §2.4, we turn to DR of the second kind introduced in [5] to synthesize Bessel functions and to model other types of oscillators.

2.1 DR of the first kind

The key idea to diffusive representations of the first kind is to decompose long-range functions on a continuous family of purely damped exponentials, with weight $\mu_\mathcal{D}$: thus, the convolution with the diffusive impulse response $h_\mathcal{D}(t) = \int_0^\infty \mu_\mathcal{D}(\xi) e^{-\xi t} d\xi$ can be realised by the system:

$$\begin{cases}
\partial_t X(\xi, t) = -\xi X(\xi, t) + u(t); & X(\xi, 0) = 0 \\
y(t) = \int_0^\infty \mu_\mathcal{D}(\xi) X(\xi, t) d\xi
\end{cases}$$

(1)

where the state $X(., t)$ belongs to a convenient Hilbert space $V$ and $\mu_\mathcal{D}$ is the so-called diffusive representation of the causal pseudo-differential operator $\mathcal{D}: u \mapsto y = \mathcal{D} u$. This infinite-dimensional realisation makes sense in a classical $V \subset H \subset V'$ framework which can be specified: let $H^k = \{X(., t) \in L_1^2(\mathbb{R}^+; |\mu_\mathcal{D}(\xi)|) /$
\[
\int_0^{+\infty} (1 + \xi)^k |X|^2 \left| \mu_D(\xi) \right| d\xi < +\infty \quad \text{then} \quad V = H^1, \quad H = H^0 \text{ and } V' = H^{-1}. \text{ In the theory elaborated in [4], the DR } \mu_D \text{ is a causal tempered distribution belonging to an appropriate space } \Delta \subset \mathcal{S}'(\mathbb{R}) : \text{ we shall here be concerned only with } \text{DR } \mu \in \Delta \text{ such that } \mu_{\mathcal{D}}(\xi) \in L^1(\mathbb{R}^+) \text{, which ensures that the transfer function}
\]
\[
H_D(s) \triangleq \mathcal{L}[h_D(t)] = \int_0^{+\infty} \frac{\mu_D(\xi)}{s + \xi} d\xi \quad \Re(s) > 0 \tag{2}
\]
satisfies
\[
\lim_{|4| \to +\infty} H_D(s) = 0 \quad \lim_{|4| \to 0} s H_D(s) = 0 \tag{3}
\]
(see [4, §4.2] for the proof). These conditions express the high- and low-frequency behaviors of the operator \( \mathcal{D} \) and allow straightforward analysis and numerical simulation. Fractional integral of order \( \beta < 1 \) is the basic example of PDO of diffusive type with DR \( \mu_D(\xi) = \mu_{\beta}(\xi) \triangleq \frac{m!}{\Gamma(\beta)} \xi^{-\beta} \) (see [2], [6], [4]) and transfer function \( H_D(s) = s^{-\beta} \) for \( \Re(s) > 0 \).

Extending the ideas of Matignon ([5, theorem 2.27]):

**Proposition 1** The DR \( \mu_D \) can be computed from the discontinuity of the transfer function \( H_D \) on the cut along \( \mathbb{R}^- \), by:

\[
\mu_D(\xi) = \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \left[ H_D(-\xi - i\varepsilon) - H_D(-\xi + i\varepsilon) \right] \tag{4}
\]

The convergence is to be understood in the sense of distributions.

**Sketch of the proof.** From (2), we get
\[
2i\pi (\delta_\varepsilon * \mu_D)(\xi) = H_D(-\xi - i\varepsilon) - H_D(-\xi + i\varepsilon), \quad \text{where } \delta_\varepsilon(\xi) = \frac{\pi}{\varepsilon} (\varepsilon^2 + \xi^2)^{-1} \text{ converges towards } \delta \text{ as } \varepsilon \to 0.
\]
Some more involved regularity results can be investigated under appropriate additional assumptions on \( \mu_D \).

This link proves useful to establish the external stability of coupled systems like the oscillators presented in §2.2.

Next, note that due to condition (3), the PDO \( \mathcal{D} \) defined by (1) is strictly proper, which means that its order \( \sigma(\mathcal{D}) \) defined by (see [1, 4]):

\[
\sigma(\mathcal{D}) = \inf \{ k \in \mathbb{R} \mid \frac{H_D(s)}{s^k} \to 0 \text{ when } |s| \to +\infty \} \tag{5}
\]
is strictly negative. Rational systems without direct transfer are classical examples of strictly proper operators with order \( \sigma = -n < 0, \quad n \in \mathbb{N} \). For PDO \( \mathcal{D} \), we have \( \sigma(\mathcal{D}) \in ] -1, 0 [ \). As pointed out in [4], the framework of DR can also be extended to proper (i.e. \( \sigma < 1 \)) operators by derivation as follows:

\[
\begin{align*}
\{ \partial_\varepsilon X(\xi, t) &= -\xi X(\xi, t) + u(t); \quad X(\xi, 0) = 0 \\
y(t) &= \int_0^{+\infty} \mu_D(\xi) \partial_\varepsilon X(\xi, t) d\xi
\end{align*}
\tag{6}
\]
is a realisation of the PDO \( \frac{d}{dt} \mathcal{D} \). Rational systems with a non-null direct transfer are classical examples of proper operators with order \( \sigma = 0 \). For PDO \( \frac{d}{dt} \mathcal{D} \), we have \( \sigma(\frac{d}{dt} \mathcal{D}) \in ] 0, 1 [ \). Fractional derivator of order \( \alpha < 1 \) is the basic example of proper PDO of diffusive type \( (\mathcal{D}^\alpha = \mathcal{D}^1 \circ \mathcal{D}^{1-\alpha}) \) with DR \( \mu_D(\xi) = \mu_{-\alpha}(\xi) \) (see [4]).

**Proposition 2** The transfer function of system (6)
\[
H_D(s) = s H_D(s)
\]
satisfies
\[
\lim_{|4| \to +\infty} \frac{H_D(s)}{s} = 0 \quad \lim_{|4| \to 0} s^2 H_D(s) = 0 \tag{7}
\]

**Remark 1** When \( \mu_D \in L^1(\mathbb{R}^+) \), which is not the case for fractional derivator \( \mathcal{D}^\alpha \), the integral term \( \partial_\varepsilon X(\xi, t) \) in (6) can be split into two parts; it yields that (6) exhibits the direct transfer \( \int_0^{+\infty} \mu_D(\xi) d\xi \).

Finally, we turn to **positivity** property of PDOs \( \mathcal{D} \) and \( \frac{d}{dt} \mathcal{D} \), which proves crucial when studying coupled systems like the oscillators presented in §2.2.

**Proposition 3** When \( \mu_D \geq 0, \forall T > 0 \), the quadratic forms
\[
L^2(0, T) \to \mathbb{R} \quad H^1(0, T) \to \mathbb{R}
\]
\[
X(\xi, 0) = 0 \quad u \longmapsto \int_0^T u \mathcal{D} u dt \quad u \longmapsto \int_0^T u \frac{d}{dt} \mathcal{D} u dt
\]
éven positive.

**Proof.** From Fubini theorem and since \( X(\xi, 0) = 0 \), we get
\[
\int_0^T u \mathcal{D} u dt = \frac{1}{2} \int_0^{+\infty} \mu_D(\xi) X^2(\xi, T) d\xi + \int_0^{+\infty} \int_0^T \xi \mu_D(\xi) X^2(\xi, T) d\xi dt \geq 0
\]
Similarly,
\[
\int_0^T u \frac{d}{dt} \mathcal{D} u dt = \int_0^{+\infty} \int_0^T \mu_D(\xi) (\partial_\varepsilon X(\xi, t))^2 d\xi dt + \frac{1}{2} \int_0^{+\infty} \xi \mu_D(\xi) X^2(\xi, T) d\xi \geq 0
\]
Note that in both cases, all the integrals make sense in the convenient functional framework.

**2.2 Application to oscillators with damping of diffusive type**

We consider the following family of second order oscillators
\[
\ddot{x} + \varepsilon \dot{x} + \mathcal{D}(\dot{x}) + \omega^2 x = u; \quad \varepsilon \geq 0 \tag{8}
\]
where \( \mathcal{D} = \mathcal{D}_\mu + \frac{d}{dt} \mathcal{D}_v \) is a positive PDO with \( \mathcal{D}_\mu \) and \( \mathcal{D}_v \) respectively of the form (1) and (6). By combining (3) and (7), it comes that the transfer function \( H_D \) satisfies:
\[
\lim_{|4| \to +\infty} \frac{H_D(s)}{s} = 0 \quad \lim_{|4| \to 0} s H_D(s) = 0 \tag{9}
\]
\footnote{in the sense defined by proposition 3.}
As a first step, we can notice that positivity of $D$ yields boundedness of the mechanical energy of the oscillator $E(t) \triangleq \frac{1}{2} \dot{x}^2(t) + \frac{1}{2} \omega^2 x(t)$. Indeed, when $u = 0$
\[
\forall t \in [0, T] \quad \frac{d E}{d t} = \dot{x} \dot{x} + \omega^2 x \dot{x} = \dot{x} (-D(x) - \varepsilon \dot{x})
\]
which gives by integration
\[
E(T) - E(0) = -\int_0^T \dot{x} D(x) dt - \varepsilon \int_0^T \dot{x}^2 dt \leq 0 \quad (10)
\]
and therefore $\forall T > 0$, $E(T) \leq E(0)$. However, in view of the analysis of the asymptotic stability of the system, such a result is of little help. In fact, positivity is not sufficient and a dissipativity property is here required for $D$. This is achieved by means of a diffusive realisation of $D$ which gives rise to an explicit dissipative semigroup, and hence to a Lyapunov functional.

Using DR, we rewrite (8) under two standard abstract forms
\[
W(t) = A W(t) + B u(t), W(0) = 0, x(t) = C \left( W(t), \dot{W}(t) \right)
\]
where the state $W$ belongs to a convenient Hilbert space and $C$ is a continuous linear form. These different realisations will help us in a theoretical analysis to prove both internal and external stabilities of the system. They can be approximated by truncation in finite dimension (see e.g. [2], [6]) from which some numerical simulations are derived.

**Coupled realisation** It is based upon a diffusive realisation of $D$ and writes out:
\[
\begin{cases}
\dot{x} + \varepsilon \dot{x} + y + \omega^2 x = u \\
v(t) = \dot{x}(t) \\
\partial_X X(\xi, t) = -\xi X(\xi, t) + v(t): \ X(\xi, 0) = 0 \\
y(t) = \int_0^\infty [\mu_D(\xi) X(\xi, t) + \nu_D(\xi) \partial_X X(\xi, t)] d\xi
\end{cases}
\]
(11)
with $DR$ of the system. This gives rise to an explicit dissipative semigroup, and hence to a Lyapunov functional.

Direct realisation It is obtained from the transfer function
\[
H(s) = \frac{1}{s^2 + \varepsilon s + s H_D(s) + \omega^2} \quad \left\{ \begin{array}{ll}
\Re(s) \geq 0 \\
\Re(s) \not= 0
\end{array} \right. \quad (14)
\]
First, let us justify the condition $\Re(s) \geq 0, s \not= 0$ by

**Proposition 4** System (11) has no unstable poles.

**Proof.** We prove the result in two steps. Firstly, observe that since $\varepsilon, \mu_D, \nu_D \geq 0$ and $\omega^2 > 0$, the system has no real unstable pole. Indeed,
\[
s^2 + \varepsilon s + s \int_0^\infty \frac{\mu_D(\xi) + \nu_D(\xi)}{s + \xi} d\xi + \omega^2 = 0 \quad (15)
\]
cannot hold for $s \in \mathbb{R}^+$. Secondly, suppose there exists two complex conjugate unstable poles $p$ and $\bar{p}$ with $p = \rho e^{i\theta}$, $\rho > 0$ and $\theta \in [0, \pi/2]$. Then $p$ satisfies (15) which cannot hold (take the imaginary part of the expression) because $\varepsilon, \mu_D, \nu_D \geq 0$ and $\omega^2 > 0.$

By inversion of (14) and following the process developed in [5, theorem 2.27] for fractional differential systems, we get the following decomposition for the impulse response:
\[
h(t) = \sum_{k=1}^K r_k e^{s_k t} + \int_0^\infty \mu(\xi) e^{-\xi t} \ d\xi \quad (16)
\]
where $s_k$ are simple complex poles in $\{ s \in \mathbb{C}, \ \Re(s) < \arg(s) \ < \pi \}, r_k$ their residues and $\mu$ the DR of the system. If a pole is to be found on $\mathbb{R}^-$, then the integral term in (16) must be understood in the principal value sense of Cauchy and should be read $\mu, e^{-\xi t} > s'. s$ with $\mu$ a distribution (for more details see [4, §3.2 table 1]). Using (4), the DR has the analytic expression:
\[
\begin{align*}
\mu(\xi) &= \lim_{\varepsilon \to 0} \frac{1}{2i \pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left[ H(-\xi - i\varepsilon) - H(-\xi + i\varepsilon) \right] \ d\xi \\
&= \frac{\xi \mu_D(\xi) - \xi \nu_D(\xi)}{\xi^2 + m(\xi)} + \omega^2
\end{align*}
\]
where $m$ is such that $\lim_{\xi \to +\infty} m(\xi) = 0$ and $\lim_{\xi \to 0} m(\xi) = 0$ thanks to (9). It follows that $x = h \ast u$ can be realised by:
\[
\begin{cases}
\dot{X}_f = A_f X_f + B_f u, \ X_f(0) = 0 \\
\partial_X X(\xi, t) = -\xi X(\xi, t) + u(t): \ X(\xi, 0) = 0 \\
x(t) = \int_0^\infty \mu(\xi) X(\xi, t) \ d\xi + C_f X_f
\end{cases} \quad (18)
\]
where $(A_f, B_f, C_f)$ realises the finite-dimensional transfer $\sum_{k=1}^K r_k e^{s_k t} \ast v$. By mimicking the ideas developed by Matignon in the fractional case ([5, theorem 2.24] proved in [15]), it comes

**Proposition 5** System (8) is externally stable (BIBO stability).

---

1 straightforward extension of this decomposition holds in the case of multiple poles.
Sketch of the proof. Since the system has no unstable poles, we get from decomposition (16) that the impulse response of (8) is in $L^1(\mathbb{R}^+)$ if and only if the integral term belongs to $L^1(\mathbb{R}^+)$, which is the case because $\frac{\mu(\xi)}{\xi} \in L^1(\mathbb{R}^+)$ (analogously to proposition 6). Indeed,

- at infinity: since $m(\xi) = o(\xi^{-4})$, $\frac{\mu(\xi)}{\xi} \sim \frac{\mu(\xi)}{\xi} \nu(\xi) = o(\xi^{-3})$ because $\sigma(D_\mu + \frac{\nu}{\xi} D_\nu) < 1$.
- near $\xi = 0$: since $m(\xi) = o(1)$, $\frac{\mu(\xi)}{\xi} \sim \mu(\xi) - \xi \nu(D(\xi)) \in L^1_{\text{loc}}(\mathbb{R}^+)$ by definition. \qed

Finally note that the direct realisation combined with the Watson lemma (see e.g. [16, chap. II, §2.2]) gives the asymptotics of the impulse response; namely, up to $\omega^4$ and with a slight abuse of notation,

$$\mu(\xi) \sim 0 \quad \xi \left( \mu(D(\xi)) - 2 \nu(D(\xi)) \right) \Rightarrow h(t) \xrightarrow{t \to \infty} \frac{h_D(t)}{t} \quad (19)$$

2.3 Examples and numerical simulations

When

$$D = p D^\alpha + q I^\beta \quad p, q \geq 0, \quad p \neq 0 \quad (20)$$

with $\alpha, \beta \in [0, 1]$, the system (8) is a so-called fractional differential equation which has already been studied in depth in [17] and [18]. Such model has been used as constitutive equation of viscoelastic behavior of materials and members (see e.g. [19] and [20, §2] for a closely related model) but it also arises in the so-called fractional modal decomposition of the wave equation with fractional damping studied in [21]. In this case like in most fractional differential equations, both the coupled (11) and the direct (18) realisations may be used for numerical simulations but special attention should be paid to the direct realisation which requires prior knowledge of the poles and residues of the system. Except for the case of systems of commensurate orders (see e.g. [21], [5]), the computation of the poles has to be performed by means of numerical methods. When the poles are simple, the computation of the residues then follows, by use of classical results on partial fractions. In addition, note that the DR (17) can be computed exactly (see [5, theorem 2.27]).

As an example, consider now the system of incommensurate orders:

$$x + 1/2 D^{3/4} x + 1/4 x + 2 I^{1/3} x + x = u \quad (21)$$

Numerical computations give the following poles for the system: $-0.9830 \pm 1.0365i$. Therefore, as shown from (16) and illustrated in figure 1, the impulse response (dash-dot) can be decomposed under direct approximation in an exponential part due to the poles (dash) and a long-range (diffusive) function (plain). Figure 2 represents the root locus of the coupled approximation of the system (+), which can be divided in two parts: one over $\mathbb{R}^-$ to perform the approximation of the diffusive term and another one which coincides with the computed poles (○).

In order to illustrate the non-fractional case and to exhibit even slower asymptotics, we consider a family of strictly proper diffusive dampings, characterized by the following DR:

$$\mu_D^{p,q}(\xi) = \xi^p \ln \left( 1 + \frac{1}{\xi} \right) \quad (22)$$

with $p, q \in \mathbb{R}$ such that (3) holds (namely $p > q > -1$ or $p < -1$ when $q = -1$). By application of a generalized version of the Watson lemma, we get that the corresponding impulse response has the asymptotic expansion ($q \neq -1$):

$$h_D^{p,q}(t) \xrightarrow{t \to \infty} \frac{\Gamma(q + 1)}{\Gamma(p + 1)} \ln^p(t) \quad (23)$$
Thus, the impulse response of system (8) has an asymptotic expansion (see (19)) which is neither of exponential type (class of linear Ordinary Differential Equations) nor of power-law type (class of linear Fractional Differential Equations). Contrarily to the fractional case, the use of the direct realisation seems not possible to simulate the solutions of system (8) in so far as characteristic equation (15) is not explicit.

Turning to numerical simulations, let $\mu_D^1(\xi) = 5 \ln(1+\xi^{-2})$ and $\mu_D^2(\xi) = 5(\xi \ln(1+\xi^{-1})(1+\xi^2))^{-1}$ be two examples of DR close to family (22); indeed, the corresponding impulse responses have the asymptotic expansions $h_D^1(t) \underset{t \to +\infty}{\sim} \frac{\ln(t)}{t}$ and $h_D^2(t) \underset{t \to +\infty}{\sim} \frac{1}{\ln^{1/2}(t)}$. The step responses of the system in figure 4 are in perfect accordance with the behaviors of the DRs $\mu_D^1$ and $\mu_D^2$. Indeed, since $\mu_D^1(\xi) \propto \mu_D^2(\xi)$ when $\xi \to 0$, the convergence (towards 1) of the $\mu^2$ step response is much slower than that of $\mu^1$. Note in both graphs the transient due to the presence of oscillating poles.

2.4 DR of the second kind

The key idea to diffusive representation of the second kind is to decompose long-range oscillating functions on a continuous family of complex-valued damped exponentials, with complex-valued weight $\mu$, namely:

$$\begin{align*}
    \begin{cases}
        \frac{\partial}{\partial t} X(\xi, t) = (-\xi + i \omega_0) X(\xi, t) + u(t); & \xi > 0 \\
        X(\xi, 0) = 0 \\
        y(t) = \Re \left\{ J_0^+ \mu(\xi) X(\xi, t) d\xi \right\}
    \end{cases}
\end{align*}
$$

(24)

where $\omega_0 > 0$. The impulse response of (24) can then be written under the following form:

$$h(t) = \Re \left( e^{i\omega_0 t} \int_0^{+\infty} \mu(\xi) e^{-\xi t} d\xi \right)
$$

(25)

Such transfers have already been used in absorbing feedback problems for propagative 2-D systems (see [22]) but can also be used to synthesize, analyze and approximate such special functions as Bessel functions and to model non-standard oscillators.

As studied in [5, §3.3], we recall that the causal Bessel function of the first kind $J_0$, the Laplace transform of which is:

$$J_0(s) = \frac{1}{\sqrt{\pi^2 + s^2}} \Re(s) > 0
$$

(26)

can be realised by means of system (24) with $\mu(\xi) = \mu_{J_0}(\xi) = 2 \pi^{-1} \xi^{-1/2} (-\xi + 2 i)^{-1/2}$. Numerical simulation of $J_0$ is presented in figure 5 under finite-dimensional approximation of (24). Note the long-range oscillating behavior with envelope $\sqrt{\frac{2}{\pi}} t$.

As far as modelling is concerned, both systems (8) and (24) behave as non-standard oscillators. But, we shall insist on the fact that due to decomposition (16), the oscillating part of system (8) is of exponential type which is not the case for system (24): in that sense, only system (24) generates non-standard oscillations. We get here a structural difference between the two oscillators enlightened by the richness of diffusive representations.

3 Extensions to discrete time

This section deals with diffusive filters (i.e. discrete-time filters that share the properties of the so-called diffusive
pseudo-differential operators). After presenting a new framework for diffusive filters in §3.1, two examples are examined in §3.2 (both are numerical approximations of fractional integrators). Then in §3.3 dissipativity is addressed and enables to prove internal stability of a discrete oscillator with diffusive damping as shown in §3.4. Complex analysis performed on the transfer function of this non-standard oscillator proves its external stability in §3.5. Besides, inspired by the ideas of [5], the framework has been extended to diffusive filters of the second kind.

### 3.1 Framework for diffusive filters

As for diffusive pseudo-differential operators in continuous time, impulse responses of diffusive filters in discrete time entail long-memory behaviors and can be decomposed on a continuous family of purely damped geometric sequences with weight \( \mu \).

\[
\forall n \geq 1, \quad h_n = \int_0^1 \mu(\rho) \rho^{n-1} \, d\rho \tag{27}
\]

\( \mu \) is called the diffusive representation (DR) of the filter. In the present paper and for simplicity sake, \( \mu \) is always an \( L^1 \)-function and \( h_0 \) is not determined by \( \mu \). This can be extended to an appropriate distributional framework. Yet this idea of aggregating is not new: in [12] it is shown that long memory stems from aggregation of AR of order 1. Using classical notations in control (\( u_n \) for input and \( y_n \) for output), the filtering relation \( y_n = (h \ast u)_n \) can then be realised by:

\[
\begin{align*}
\varphi_{n+1}(\rho) &= \mu(\rho)\varphi_n(\rho) + u_n \quad \text{with} \quad \varphi_0(\rho) = 0 \\
y_n &= \int_0^1 \mu(\rho)\varphi_n(\rho) \, d\rho + h_0 u_n 
\end{align*}
\tag{28}
\]

where \( \rho \in [0, 1] \) and the state \( \varphi_n \) belongs to the Hilbert space defined by \( ||\varphi||^2 = \int_0^1 |\mu(\rho)|\varphi^2(\rho) \, d\rho \). This realisation is of infinite dimension and it has a Markovian structure: \( \varphi_{n+1} \) depends, apart from the input, only on \( \varphi_n \). This choice remains consistent with the use of a non-null initial condition \( \varphi_0 \) (for instance \( \varphi_0 \) may stand for past inputs) as long as \( \varphi_0 \) belongs to the Hilbert space. The realisation has the following stability properties:

**Proposition 6** System (28) is always internally stable. It is BIBO-stable (bounded input–bounded output) if

\[
\int_0^1 \frac{\mu(\rho)}{1-\rho} \, d\rho < +\infty \.
\]

**Sketch of the proof.** The use of the dominated convergence theorem proves internal stability. Namely if the input becomes null, then \( \varphi_n \to 0 \) (with an almost everywhere convergence) and \( y_n \to 0 \). BIBO-stability comes from \( ||h||_1 \leq \int_0^1 \frac{|\mu(\rho)|}{1-\rho} \, d\rho \).

The transfer function \( H \) is defined outside the unit circle (since \( h_n \to 0 \)) and it becomes, once sum and integral have been exchanged:

\[
H(z) = h_0 + \frac{1}{z^{n-1}} \int_0^1 \frac{\mu(\rho)}{1-\rho z^{-1}} \, d\rho \tag{29}
\]

This expression (analogous to that in continuous time in [4]) also shows that the transfer function of a diffusive filter can always be extended to a holomorphic function defined over \( \mathbb{C}\setminus[0, 1] \). Since \( \mu \in L^1 \), \( \lim_{z \to 0} z H(z) = 0 \) and \( \lim_{z \to 1} (1-z^{-1})H(z) = 0 \).

As for proposition 1, we have:

**Proposition 7** The DR \( \mu \) can be computed from the discontinuity of the transfer function \( H \) on the cut along \([0, 1] \), by:

\[
\mu(\rho) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} [H(\rho - i\varepsilon) - H(\rho + i\varepsilon)] \tag{30}
\]

The convergence is to be understood in the sense of distributions. Now, if \( \mu \) is \( C^1 \) in the vicinity of \( \rho \), then (30) holds pointwise; moreover, \( H(\rho + i\varepsilon) \) converges when \( \varepsilon \to 0^+ \) and \( \varepsilon \to 0^- \) separately.

### 3.2 Examples of diffusive filters

**Notations:** \( \Delta = 1-z^{-1} \) stands for the backward difference operator.

In this section we present two examples of diffusive filters which are both discrete-time approximations of fractional integral or derivative operators. Their impulse responses have a slow decay compared to classical geometric decays, which occur for finite-dimensional ARMA filters. Such filters can therefore be used to model long-memory behavior in time series analysis (see [12]).

Our first example of a diffusive filter is \( \Delta^{-\alpha} \), the transfer function of which is \( H_1(z) = (1-z^{-1})^{-\alpha} \) where \( \alpha \) belongs to \((-1, 1)\setminus\{0\} \). When \( \alpha < 0 \) this filter is referred to as fractional differences in so far as it generalizes \( n \)-fold differentiation (i.e. \( (1-z^{-1})^n \)). When \( \alpha > 0 \) it is a discretisation with step 1 of fractional integration, using the scheme of backward difference. In [12], its filter is called a pure integrating filter of fractional order. The impulse response of \( \Delta^{-\alpha} \) is \( h_n^1 = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \) (Mac Laurin series decomposition of \( H_1 \) in the neighborhood of infinity) with asymptotic behavior \( h_n^1 \sim \frac{1}{\Gamma(\alpha)} n^{1-\alpha} \).

**Proposition 8** \( \Delta^{-\alpha} \) is a diffusive filter with DR:

\[
\mu_1(\rho) = \frac{\sin(\alpha\pi)}{\pi} \rho^\alpha (1-\rho)^{-\alpha} \tag{31}
\]

**Proof.** Since (28) and (29) are derived from (27) it suffices to prove that (27) holds for \( \mu_1 \). However this requires a deeper analysis of \( H_1 \). There are two singular points \( z = 0 \) and \( z = 1 \). A cut in the complex plane joining these points has to be made so as to extend \( H_1 \) inside the unit circle. Thus by analytical continuation, \( H_1 \) can now be defined over \( \mathbb{C}\setminus[0, 1] \):

\[
H_1(z) = [1-z^{-1}]^{-\alpha} e^{-i\alpha \arg(1-z^{-1})} \quad \text{where} \quad |\arg(z)| < \pi.
\]

Since \( H_1 \) is holomorphic on \( \mathbb{C}\setminus[0, 1] \), \( h_n^1 = \frac{\Gamma(n+1)}{2\pi i} \int_C H_1(z) z^{n-1} \, dz \) and \( h_n^1 \) is unchanged when the
technique of impulse response invariance (i.e. substitution
then leads to (31). This proof shows that the DR of a diffusive filter can be derived from the transfer function extended over $\mathbb{C}\setminus[0,1]$. This is true for many other diffusive filters and is in accordance with proposition 7. Because the impulse response is real-valued the discontinuity is purely imaginary. This is why it has been chosen to simulate the imaginary part of the transfer function (figure 6). $\mu_1$ can be read out on this graph:

$$\mu_1(\rho) = \frac{1}{2} \lim_{\rho \to 0^+} \Im m(H_1(\rho + i\varepsilon)).$$

Our second example is a filter introduced in [23]. The impulse response of this filter is $h_n^2 = \frac{1}{n} \frac{1}{1 - \rho^2}$, which happens to be the asymptotics of $h_n^2$. It is also a discretization of the impulse response of fractional integration $t^{\alpha - 1}$ using the technique of impulse response invariance (i.e. substitution of $t$ by $n$). Because the continuous-time impulse response is singular at $t = 0$, it was necessary to set the direct transfer arbitrarily: $h_0^2 = 1$ has been chosen, as in the first example.

**Proposition 9** $H_2$ is the transfer function of a diffusive filter with DR

$$\mu_2(\rho) = \frac{\sin(\alpha \pi)}{\pi} \left( \ln \left( \frac{1}{\rho} \right) \right)^{-\alpha} \quad (32)$$

**Proof.** It suffices to prove that (27) holds for $\mu_2$. Since $t^{\alpha - 1} = \int_0^\infty \frac{\sin(\alpha \pi)}{\pi} \xi^{-\alpha} e^{-\xi t} d\xi$, substituting $\rho = e^{-\xi}$ then leads to (32) \hfill \Box

In both examples, the diffusive representations have common asymptotic behaviors in $\rho = 1$ and the impulse responses have also common asymptotics. This is illustrated on figure 7. Such a relation has already been noticed for the so-called Laplace integrals and has lead to the Watson lemma [16, chap. II, § 2.2]. Let us present an analogous lemma in a discrete-time context.

**Discrete-time Watson lemma** When the asymptotic behavior of the DR $\mu$ near $\rho = 1$ is described by

$$\mu(\rho) = \sum_{m=0}^M c_m \frac{(\ln(\frac{1}{\rho}))^{\gamma_m}}{\Gamma(1 + \gamma_m)} + o \left( \left( \ln \left( \frac{1}{\rho} \right) \right)^{\gamma_m} \right) \quad (33)$$

and when $\mu \in L^1(0,1)$, then the impulse response has the asymptotic behavior:

$$h_n = \sum_{m=0}^M \frac{c_m}{n^{1 + \gamma_m}} + o \left( \frac{1}{n^{1 + \gamma_m}} \right) \quad (34)$$

Applying this lemma leads for instance to:

$$h_n^2 = h_n^2 = \frac{\alpha(1-\alpha)}{2(1-\alpha)} n^{-(2-\alpha)} + o(n^{-(2-\alpha)}).$$

Similar ideas were used to give asymptotics of impulse responses in [10].

**3.3 Positivity and dissipativity in discrete time**

The aim is to extend proposition 3 to discrete time. The following definitions extend those exposed in [24] to a discrete-time context.

**Definition 1** For discrete filters with null initial condition positivity means that $\sum_{n \geq 0} u_n y_n$ is always positive.

Note that positivity implies a non-null direct transfer (i.e. $h_0 \neq 0$).

**Definition 2** A discrete filter with state $\Phi_n$ is said to be dissipative if there exists a Lyapunov functional $V$ such that $V(\Phi_{n+1}) - V(\Phi_n) \leq w(u, y)$, where $w$ is a function depending of a finite (N) number of most recent inputs and outputs: $u_n, \ldots, u_{n-N+1}$ and $y_n, \ldots, y_{n-N+1}$. This function is called a supply rate in [24].

When $w(u, y) = u_n y_n$ dissipativity yields positivity. Indeed $\sum_{n \geq 0} u_n y_n \geq V(\Phi_{N+1}) - V(0) \geq 0$. Since the Lyapunov functional depends on the state, note that dissipativity strongly depends on the choice of the realisation.

Displaying dissipative realisations proves useful in §2.2 to show the internal stability of an oscillator with damping of diffusive type. Applying such ideas to discrete time is not straightforward, (some remarks concerning these difficulties
can be found in [25]). However it has been possible to carry out an example. The following proposition gives sufficient conditions on a diffusive filter to be dissipative. For $a > 0$, $H_1$ and $H_2$ fulfill these conditions and are therefore dissipative, (dissipativity of $H_1$ is used in §3.4).

**Proposition 10** When the diffusive filter (28) fulfills the following conditions

$$\mu(\rho) \geq 0 \quad \text{and} \quad e_0 = h_0 - \int_0^1 \frac{\mu(\rho)}{1 + \rho} \, d\rho \geq 0 \quad (35)$$

dissipativity is ensured by the following Lyapunov functional

$$V(\varphi_n) = \frac{1}{2} \int_0^1 \mu(\rho) \varphi_n^2(\rho) \, d\rho \quad (36)$$

with supply rate $w(u, y) = u_n y_n$.

**Sketch of the proof.** Substituting $h_0$ for the integral expression (35) in (29) yields a new integral expression for the transfer function: $H(z) = e_0 + \int_0^1 \frac{\mu(\rho)}{1 + \rho} \frac{1+z^{-1}}{1-\rho z^{-1}}$. For each $\rho$, $\frac{1+z^{-1}}{1-\rho z^{-1}}$ is dissipative for elementary Lyapunov functions $V_\rho(\varphi_n) = \frac{1}{2}(1 + \rho)\varphi_n^2(\rho)$. The expected result then follows. □

### 3.4 Internal stability of a discrete oscillator with diffusive damping

Discretization of $x + p \, F \, \dot{x} + \omega^2 x = u$, ($p \geq 0$), using a backward difference scheme, leads to:

$$\Delta^2 x_n + p \, \Delta^{-\alpha} \circ \Delta x_n + \omega^2 x_n = u_n \quad (37)$$

with output $x_n$, and $\Delta^{-\alpha}$ the diffusive filter studied in §3.2. For $p = 0$ this filter is an exponentially damped oscillator. Indeed, the technique of discretization moves the poles of (37) inside the unit circle. For $p > 0$ the filter has a non-standard behavior that comes from its diffusive part. This is illustrated on figure 8, where the impulse response is simulated for two different values of $p$.

![Figure 8](image)

**Figure 8:** Impulse response of (37) for two different values of $p$: on the left $(p = 0.15)$, the long-memory behavior is obvious, whereas on the right and for a smaller value of $p = 0.02$, the oscillatory behavior prevails.

We are now trying to find a global Lyapunov functional $E$ to prove internal stability using energy methods. As for continuous time, from the diffusive realisation of $\Delta^{-\alpha}$ (28), we obtain a state-space realisation of (37), for $n \geq 2$:

$$\begin{align*}
\Delta^2 x_n + \omega^2 x_n + p y_n &= u_n \\
\varphi_n + 1(\rho) &= \rho \varphi_n(\rho) + u_n \\
y_n &= \int_0^1 \mu(\rho) \varphi_n(\rho) \, d\rho + h_0 \varphi_n
\end{align*} \quad (38)$$

Introducing the augmented state $\Phi_n = (x_n, x_{n-1}, \varphi_{n-1})$, the realisation can be put in compact form:

$$\begin{align*}
\Phi_{n+1} &= A \Phi_n + B u_n \\
x_n &= C \Phi_n + D u_n
\end{align*} \quad (39)$$

$\Phi_n$ belongs to the Hilbert space $H$ defined by $\|\Phi_n\|^2 = \int_0^1 \mu(\rho) \varphi_n^2(\rho) \, d\rho + x_n^2 + x_{n-1}^2$. This topology ensures the continuity of the operators $(A, B, C, D)$.

**Proposition 11** A global Lyapunov functional for (38) is given by:

$$E(\Phi_n) = \frac{1}{2} (\Delta x_n)^2 + \frac{\omega^2}{2} x_n^2 + p V(\varphi_{n+1}) \quad (40)$$

**Proof.** (38) is the coupling of two subsystems: an exponentially damped oscillator and a diffusive system. Each subsystem can be respectively stabilized by Lyapunov functionals: $E(x_n) = \frac{1}{2} (\Delta x_n)^2 + \frac{\omega^2}{2} x_n^2$ and $V(\varphi_{n+1}) = \frac{1}{2} \int_0^1 \mu(\rho) \varphi_{n+1}(\rho) \, d\rho$. (Note that $E$ is the discretization of the mechanical energy.) These two systems are dissipative in the sense of definition 2 with different supply rates. Indeed, since $\Delta (x_n)^2 \leq 2 x_n \Delta x_n$ (very reminiscent of $\frac{d(x^2)}{dt} = 2 x \, dx$), it comes $E(x_{n+1}) - E(x_n) \leq (\Delta x_{n+1} + \omega^2 x_{n+1}) \Delta x_{n+1}$, and we have $V(\varphi_{n+1}) - V(\varphi_{n+1}) \leq y_{n+1} \Delta x_{n+1}$ from proposition 10.

Hence, as soon as the input has stopped, the two supply rates become comparable: $E(\Phi_{n+1}) - E(\Phi_n) \leq (\Delta^2 x_{n+1} + \omega^2 x_{n+1}) \Delta x_{n+1} + p y_{n+1} \Delta x_{n+1} = 0$ □

Let us now look for stability results. LaSalle theorem [26] should enable to prove the asymptotic internal stability of (37) since the hypotheses are fulfilled. One of them is that the greatest set positively invariant by $A$ and included in $E^{-1}(0)$, is $\{0\}$. Another one is that $E$ tends to infinity when $\| \Phi \|_H$ tends to infinity. The asymptotic internal stability would have meant that $\varphi_n$ and $x_n$ converge to 0 when the initial condition is in $H$ and when the input has stopped. However in [26], this theorem is only established for finite-dimensional systems; and whether or not this theorem applies in the case of diffusive filters is still an open question. Yet, knowing that $E$ is not increasing is already sufficient to prove the internal stability of (37): if the initial condition $\Phi_0$ is sufficiently close to 0, and as long as the input is null, the state $\Phi_n$ remains as close to 0 as wished (i.e. for the $H$-topology).
### 3.5 External stability of a discrete oscillator with diffusive damping

Actually the method exposed in §3.4 can easily be extended to equations where $\Delta^{-\alpha}$ is replaced in (37) by a diffusive filter $D$ (i.e. of the form (28)). Namely

$$\Delta^2 x_n + p D \circ \Delta x_n + \omega^2 x_n = u_n \quad (41)$$

The DR $\mu_D$ is required to satisfy $\mu_D(\rho) > 0$ (at least for $\rho > 0$) and assumptions of propositions 7 and 10.

The purpose is here to expose how the ideas developed in §2.2 concerning the direct realisation can be extended to discrete time. Namely this method proves external stability and is based on the study of transfer functions rather than on systems analysis (as in §3.4). The transfer function of (41) is, for $|z| \geq 1$:

$$[H(z)]^{-1} = (1 - z^{-1})^2 + \omega^2 + p H_D(z) (1 - z^{-1}) \quad (42)$$

Similarly to continuous time, we have the following decomposition for the impulse response $h_n$:

$$h_n = \int_0^1 \mu(\rho) \rho^{n-1} d\rho + \sum_{i=0}^{n-1} \sum_{j=0}^I c_{ij} n^j \rho_i^n \quad (43)$$

where $\mu$ is the DR of $H$ and $\rho_i$ are poles with multiplicity $\nu_i$. $\mu$ is derived from (30) applied to (43). This decomposition into a diffusive part and a finite-dimensional part proves useful to show external stability:

**Proposition 12** System (41) is BIBO-stable.

**Sketch of the proof.** This proof can be done in two steps. Just like in continuous time, (41) has no unstable poles because $H^{-1}$ is not null on and outside the unit circle. Hence, the finite-dimensional part is externally stable. The second step is to prove the external stability of the diffusive part. From proposition 6, it suffices to prove that $\mu(\rho)\frac{1}{1-\rho}$ is $L^1$. An asymptotic analysis on $\mu$ in the vicinity of $\rho = 1$ ends the proof. \(\square\)

### 3.6 Diffusive filters of the second kind

The extension encompassed in the framework for diffusive filters of the second kind is that the impulse response is an aggregation of complex-valued geometrical sequences continuously aggregated with a complex-valued weight $\mu$. Yet the ratio of these geometrical sequences are all on a radius of the unit circle. Namely a realisation is given by:

$$\begin{align*}
\varphi_{n+1}(\rho) &= \rho e^{i\theta_0} \varphi_n(\rho) + u_n \quad \text{with} \quad \varphi_0(\rho) = 0 \\
y_n &= \Re \left( \int_0^1 \mu(\rho) \varphi_n(\rho) d\rho \right) + h_0 u_n
\end{align*} \quad (44)$$

where $\theta_0 > 0$ and $\mu$ is the DR of the filter. This framework allows for long-range oscillatory behaviors with impulse responses that can be written under the following form:

$$h_{n+1} = \Re \left( e^{i\theta_0} \int_0^1 \mu(\rho) \rho^n d\rho \right) \quad (45)$$

where $|h_n|$ can be a slowly decreasing sequence.

Let us give two examples of second kind diffusive filters with DR $\mu_1(\rho) = \frac{\sin(\alpha\theta_0)}{\pi} \rho^\alpha (1 - \rho)^{-\alpha} e^{i\theta_0}$ and $\mu_2(\rho) = 2^{1+\alpha} \frac{\sin(\alpha\theta_0)}{\pi} \rho^{2\alpha} (1 - \rho)^{-\alpha} e^{i\theta_0}$. Note that both functions are singular in $\rho = 1$. We will show that both have long-range oscillatory behaviors. Indeed the second filter has been used in time series analysis to model seasonal effects (for example in [27] to produce Gegenbauer processes and to estimate sunspots).

The transfer functions of the two examples are:

$$H_1(z) = \frac{1}{2} (1 - e^{i\theta_0} z^{-1})^{-\alpha} + \frac{1}{2} (1 - e^{-i\theta_0} z^{-1})^{-\alpha} \quad (46)$$

$$H_2(z) = \frac{1}{2} (1 - 2\cos(\theta_0) z^{-1} + z^{-2})^{-\alpha} \quad (47)$$

Both filters have analytical continuations with branching points in $z = 0$, $z = e^{i\theta_0}$, and $z = e^{-i\theta_0}$. As it has been pointed out in [10], these branching points entail the non-standard behavior. Indeed their impulse responses are oscillating at the frequency determined by $\theta_0$ with a slowly decreasing amplitude. These impulse responses are drawn on figure 9.

![Figure 9: Impulse responses of $H_1$ and $H_2$ with $\theta_0 = 0.5$ and $\alpha = 0.4$. The graph shows that the amplitude of the oscillations are slowly decreasing.](image)

As far as modelling is concerned, diffusive filters of the second kind (44) generate non-standard oscillations. Note that such behavior can not be obtained, neither with ARMA filters nor with diffusive filters of the first kind. Indeed the oscillating part of system (37) is of exponential type (see (43)).

### References


G. Montseny, J. Audounet and G. Montseny.


