Analytic interpolation and spectral analysis: Advances and Applications *

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The relevance of analytic interpolation in the analysis of time-series became increasingly apparent with the pioneering work of Levinson, Burg and others in the second half of the 20th century. By exploiting this link, a wide spectrum of techniques were devised during the 80's and 1990's—now known as “modern nonlinear methods” [15]. The connection between the two stems from the fact that a partial autocorrelation sequence

\[ R_k := E\{u_\ell u_{\ell+k}\}, \ k = 0, 1, 2, \ldots, n, \]

of a stationary (zero-mean) stochastic process \{u_\ell\} represents the Fourier coefficients of a positive measure \(d\mu\) on the unit circle—the power spectral distribution, which is the object of interest in time-series analysis. These coefficients can be viewed either as moments of the measure with respect to the kernel functions \(e^{ik\theta}\), or as interpolation constraints at the origin for the function

\[ F(z) := \int \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} d\mu \]

which is defined by the measure \(d\mu\) via Herglotz’ formula. This function is analytic in the unit disc and has positive real part. Thus, the problem of identifying the class of spectra consistent with a given partial autocorrelation sequence amounts to the problem of characterizing analytic functions with positive-real parts which satisfy the relevant interpolation constraints at the origin. As a result the rich theory of Szegő orthogonal polynomials, of the moment problem and of Carathéodory interpolation provided a set of mathematical tools and ideas to time-series analysts. This paradigm witnessed a new phase of development when, in recent years, output/state second-order statistics of a linear dynamical system

\[ x_\ell = Ax_{\ell-1} + Bu_\ell, \ \text{and} \ y_\ell = Cx_\ell + Du_\ell, \]

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were treated as generalized interpolation constraints of the same “generating” function $F(z)$ [2, 3, 11, 12]. The dynamical system $(A, B, C, D)$ under consideration may represent a physical or, even, a virtual (algorithmic) device designed to filter out “noise” and improve “resolution.”

In the talk we will first present recent theoretical results in analytic interpolation which have been motivated by engineering issues of time-series analysis, and then we will overview a number of techniques for spectral analysis that utilize generalize statistics, assess their resolution capabilities, and conclude with highlighting their performance on an application study in non-invasive temperature sensing via ultrasound. More specifically, in:

**Part I** of the talk, we will first explain the connection between output/state covariances and their rôles as data for a Sarason-Carathéodory-Fejer type interpolation problem [1, 13]. We will then present a characterization of all interpolants of degree bounded by the “size” of the interpolation problem (where, more specifically, $n + 1$ is the size of the corresponding Pick matrix, or equivalently, the size of the dynamical system $(A, B, C, D)$) [7], [8], [9], [4], [5], [10], [2], [6].

Characterization of such interpolants is non-classical and has not been considered in earlier mathematical literature. Interpolants of degree $\leq n$ give rise to “parsimonious” autoregressive moving-average power spectra for the relevant time-series. Their study was motivated by engineering issues and the problem of characterizing such interpolants was first formulated in [7]. The parametrization of interpolants was developed in stages: first in [7–9] it was shown that for each selection of $\leq n$ spectral zeros in the unit disc there exist an interpolant of degree $\leq n$ such that the corresponding power spectrum has the specified spectral zeros. That such an interpolant is uniquely defined and, hence, that the spectral zeros provide a complete parametrization remained for several years as a conjecture [7] until it was settled in the affirmative in [4] (see also [10], where a technical condition in [4] was removed to include spectral zeros on the boundary). The talk will touch upon a new approach which originates in [5] and was developed further in [2, 3, 6, 14]. The main new ingredient is a (Kullback-Leibler-von Neumann) convex functional

$$\mathbb{S}(\nu||\mu) = \int \log\left(\frac{d\mu}{d\nu}\right) d\nu$$

where $d\nu$ and $d\mu$ are nonnegative measures on the unit circle and can be thought of as a distance measure between the distributions $d\nu$ and $d\mu$. It turns out that, if $d\nu$ ranges over absolutely continuous nonnegative measures with a certain structure (poles fixed both in location and number by the underlying dynamics and arbitrary zeros—bounded only in number by the degree constraint), then the functional $\mathbb{S}(\nu||\mu)$ with $d\mu$ constrained to satisfy
the required moments, achieves a minimum at $d\mu$ with the desired spectral zeros and degree bound. I.e., the optimal $d\mu$ gives via Herglotz’ formula a positive real function $F(z)$ of degree $\leq n$ and the spectral zeros of $d\mu$ coincide with those selected for $d\nu$. This part is largely based on joint work with C.I. Byrnes, A. Lindquist and A. Megretski [2,3,6,14].

In Part II of the talk, we will focus on the effect of selecting second order statistics (and designing the corresponding filtering device) for improved resolution and robustness of the spectral estimate in a specified frequency range. We will demonstrate a dramatic improvement as compared with traditional methods and highlight the advantages of this new paradigm on an application study. The purpose of the (experimental) study was to detect and map changes in the temperature field inside tissue by processing the echo of a probing ultrasound signal. This part is based on joint work with A. Nasiri-Amini and E. Ebbini [16].

References


