Strong controllability of continuous multidimensional behaviors

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MTNS 2004

In [5, 6], two concepts of controllability of two-dimensional discrete behaviors were introduced and characterized: the first one amounts to the possibility of concatenating (by another system trajectory) any two restrictions of trajectories, given on any two (sufficiently distant) subsets of the domain $\mathbb{Z}^2$. This is the classical behavioral controllability paradigm, and it has been successfully generalized, by various authors, e.g. [1, 3], to higher dimensions, and also to continuous behaviors. The second, stronger, controllability notion resulted from dropping the assumption that the two pieces of information to be concatenated should be restrictions of trajectories. Instead, they were admitted to be arbitrary sequences that do not contradict the system laws locally. The characterization of the second definition failed to generalize in a straightforward manner to higher dimensions, and to continuous behaviors. Only recently, the study of extendibility of behaviors [4, 9] has lead to a deeper understanding of the underlying problem. This made it possible to eventually generalize “strong controllability” to continuous behaviors of arbitrary dimension, and this is the topic of the present paper. The discrete case requires more attention and is treated in [7].

Let $\mathcal{A} = C^\infty(\mathbb{R}^n, \mathbb{R})$ denote the space of real-valued smooth functions on $\mathbb{R}^n$, and let $\mathcal{P} = \mathbb{R}[s_1, \ldots, s_n]$ be the ring of polynomials in $n$ indeterminates, with real coefficients. Let $R \in \mathcal{P}^{q \times q}$ be a polynomial matrix. Then

$$B = \{w \in \mathcal{A}^q \mid R(\partial)w := R(\partial_1, \ldots, \partial_n)w = 0\}$$

is the solution space of a linear, constant-coefficient system of partial differential equations. One calls $B$ a continuous $n$-dimensional behavior.
Definition 1 $\mathcal{B}$ is controllable if for all open sets $U_1, U_2 \subset \mathbb{R}^n$ with $U_1 \cap U_2 = \emptyset$ and for all $w_1, w_2 \in \mathcal{B}$ there exists $w \in \mathcal{B}$ such that $w = w_i$ on $U_i$ for $i = 1, 2$.

Lemma 1 [1, 3] The following are equivalent:

1. $\mathcal{B}$ is controllable;
2. any kernel representation matrix $R$ of $\mathcal{B}$ is a left syzygy matrix (or: a minimal left annihilator (MLA)), that is, there exists a polynomial matrix $M$ such that $\text{im}(R^T) = \ker(M^T)$;
3. there exists a polynomial matrix $M$ such that $\mathcal{B} = \{w \in \mathcal{A}^q \mid \exists l \in \mathcal{A}^r : w = M(\partial)l\}$;
4. for all polynomial row vectors $p \in \mathcal{P}^{1 \times q}$ and all non-zero polynomials $d \in \mathcal{P}$, we have that $d(\partial)p(\partial)w = 0$ for all $w \in \mathcal{B}$ $\Rightarrow$ $p(\partial)w = 0$ for all $w \in \mathcal{B}$;
5. for all real numbers $0 < r_1 < r_2$ and for all $w_1, w_2 \in \mathcal{B}$ there exists $w \in \mathcal{B}$ such that $w = w_i$ on $U_i$ for $i = 1, 2$, where

$$U_1 = \{x \in \mathbb{R}^n \mid \|x\| < r_1\} \quad \text{and} \quad U_2 = \{x \in \mathbb{R}^n \mid \|x\| > r_2\}$$

and $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$.

Proof: Since the equivalence of the first four assertions is known [1, 3], and since the last condition is, by definition, implied by controllability, it suffices to show that if $\mathcal{B}$ is not controllable, then there exist trajectories $w_1$ and $w_2$ and sets $U_1, U_2$ as in assertion 5 such that $w_1$ and $w_2$ cannot be concatenated. Actually, this is implicitly contained in the proof given in [3]: If $\mathcal{B}$ is not controllable, then there exists a polynomial row vector $p \in \mathcal{P}^{1 \times q}$ and a non-zero polynomial $d \in \mathcal{P}$ such that $d(\partial)p(\partial)w = 0$ for all $w \in \mathcal{B}$, but for some $\bar{w} \in \mathcal{B}$, we have $p(\partial)\bar{w} \neq 0$. Let $\bar{x} \in \mathbb{R}^n$ be such that $p(\partial)\bar{w}(\bar{x}) \neq 0$, and let $r_1$ be such that $U_1$ contains $\bar{x}$. Choose $r_2 > r_1$, and set $w_1 := \bar{w}$ and $w_2 := 0$. If these two trajectories were concatenable, say by some trajectory $w$, then $\alpha := p(\partial)w$ would be identically zero in $U_2$, but not in $U_1$. This means that $\alpha \neq 0$ has compact support. On the other hand, it is contained in the autonomous behavior $\{\alpha \in \mathcal{A} \mid d(\partial)\alpha = 0\}$. This is a contradiction [3]. □
Definition 2 \( \mathcal{B} \) is extendable if for any open, bounded, and convex set \( \Omega \), and for any \( v \) defined on an open neighborhood of \( \overline{\Omega} = \mathbb{R}^n \setminus \Omega \) that satisfies the system law, there exists \( w \in \mathcal{B} \) such that \( w \) coincides with \( v \) on some open neighborhood of \( \overline{\mathbb{C}\Omega} \).

In the following, the term neighborhood will always mean an open neighborhood.

Lemma 2 [2, 9] \( \mathcal{B} \) is extendable if and only if any kernel representation matrix \( R \) of \( \mathcal{B} \) is a right syzygy matrix (or: a minimal right annihilator (MRA)), that is, there exists a polynomial matrix \( M \) such that \( \text{im}(R) = \ker(M) \).

Definition 3 \( \mathcal{B} \) is strongly controllable if for any open, bounded, and convex sets \( \Omega_1 \subset \Omega_2 \), with \( \overline{\Omega_1} \cap \mathbb{C}\Omega_2 = \emptyset \), and any \( v_{\text{out}} \), defined on a neighborhood of \( \mathbb{C}\Omega_2 \), and \( v_{\text{in}} \), defined on a neighborhood of \( \overline{\Omega_1} \), both satisfying the system law in their respective domain, there exists \( w \in \mathcal{B} \) such that \( w = v_{\text{in}} \) in a neighborhood of \( \overline{\Omega_1} \) and \( w = v_{\text{out}} \) in a neighborhood of \( \mathbb{C}\Omega_2 \).

Note that for any sets \( \Omega_1, \Omega_2 \) as above, the sets \( \overline{\Omega_1} \) and \( \mathbb{C}\Omega_2 \) have a positive distance. Therefore we may assume that their neighborhoods which appear in Definition 3 and the following proof are always such that their closures are disjoint.

Theorem 1 \( \mathcal{B} \) is strongly controllable if and only if it is both controllable and extendable.

Proof: If \( \mathcal{B} \) is strongly controllable, then it is clearly extendable (take \( \Omega_1 = \emptyset \) and \( \Omega_2 = \Omega \)) and it is also controllable (take \( \Omega_i = \{ x \in \mathbb{R}^n \mid \|x\| < r_i \} \) for \( i = 1, 2 \) and apply Lemma 1).

Now let \( \mathcal{B} \) be both controllable and extendable. Let \( \Omega_1 \) and \( \Omega_2 \) as in Definition 3 be given. By controllability, \( R \) is an MLA, say of \( M \). Then

\[
\mathcal{B} = \{ w \in \mathcal{A}^r \mid \exists l \in \mathcal{A}^r : w = M(\partial)l \}.
\]

The fundamental principle [1, 2] holds for \( \mathcal{C}^\infty(\overline{\Omega_1}, \mathbb{R}) \) [9], that is, \( v_{\text{in}} = M(\partial)l_{\text{in}} \) for some smooth function \( l_{\text{in}} \), defined on some neighborhood of \( \overline{\Omega_1} \). Thus there exists \( l \in \mathcal{A}^r \) such that \( l = l_{\text{in}} \) on some neighborhood of \( \overline{\Omega_1} \). Then \( w_1 := M(\partial)l \) is in \( \mathcal{B} \) and it coincides with \( v_{\text{in}} \) in a neighborhood of \( \overline{\Omega_1} \). By extendibility, there exists \( w_2 \in \mathcal{B} \) such that \( w_2 \) coincides with \( v_{\text{out}} \) in some neighborhood of \( \mathbb{C}\Omega_2 \). Finally, we may concatenate \( w_1 \) and \( w_2 \) due to controllability, and thus we have the desired result. \( \square \)
Corollary 1 \( B \) is strongly controllable if and only if any kernel representation matrix \( R \) of \( B \) is both a right and a left syzygy matrix.

Corollary 2 \([5, 6]\) If \( n \leq 2 \), then \( B \) is strongly controllable if and only if it admits a kernel representation with a zero left prime matrix \( R \), that is, \( R \in \mathcal{P}^{g \times q} \) is such that the mapping \( R : \mathcal{P}^q \to \mathcal{P}^g \), \( x \mapsto Rx \) is surjective.

Proof: A zero left prime matrix is minor left prime, and thus a left syzygy matrix \([8]\), and it is also a minimal right annihilator (namely, of the zero matrix), because \( R \) is surjective.

Conversely, let \( R \) be both an MLA and an MRA. The MLA property together with \( n \leq 2 \) implies that we may assume, without loss of generality, that \( R \) has full row rank. A full row rank MRA can only be an MRA of the zero matrix. This means that \( R \) must be surjective, hence, zero left prime. \( \square \)

For one-dimensional systems, controllability and strong controllability are equivalent. In the two-dimensional case, Corollary 2 coincides with the characterization of strong controllability given in \([5, 6]\). In general, the existence of a zero left prime representation matrix still implies strong controllability, but the converse does not hold in dimensions \( n \geq 3 \), as can be seen from the example

\[
R = \begin{bmatrix}
0 & -s_3 & s_2 \\
s_3 & 0 & -s_1 \\
-s_2 & s_1 & 0
\end{bmatrix}
\]

which is both an MRA and an MLA, but whose behavior

\( B = \{ w \in \mathcal{A}^3 \mid \text{curl}(w) = 0 \} \)

cannot be represented by a zero left prime matrix (or any matrix of full row rank), although it is strongly controllable.

We may characterize the existence of a zero left prime representation matrix as follows.

Corollary 3 The following are equivalent:

1. \( B \) possesses a zero left prime representation matrix;
2. \( B \) possesses a full row rank representation matrix and it is strongly controllable.
References


