Solution of inverse scattering problems by means of parametric identification techniques

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Abstract

In this paper we consider a rather general inverse scattering problem of determining the coefficients in a system of second order hyperbolic partial differential equations. By using Galerkin-type approximation we formulate this problem as a system identification problem where the unknown parameters of a linear system are to be estimated. This can be done by standard system identification algorithms without resorting to the classical gradient-type optimization routines which are complicated and time-consuming.

1 Introduction

We consider wave propagation in media of unknown and possibly space-varying propagation parameters. Recovering the unknown parameters of the media from various kinds of external measurements is a standard problem in various fields of physics and engineering, ranging from geophysical data processing to solid mechanics, electromagnetism, medical imaging, non-destructive testing, remote sensing, radar etc., which is often referred to as an inverse scattering problem in the literature.

The evolution of the state\(^1\) of such physical systems, is in general described by a system of linear second order hyperbolic partial differential equations (PDEs). The physical parameters characterizing the medium (such as density, elasticity, resistivity, magnetic permeability, permittivity..etc.) enter multiplicatively in these PDE models as unknown functions of the space variable. These functions are not known, and the inverse scattering problem can be set mathematically as the problem of recovering them

\(^1\)By state we intend quantities such as, for example, particles displacement, electromagnetic potentials and so on.
from the available sensor measurements (typically values of a finite number of linear functionals of the state) and perhaps from the known waveform of the sources generating the waves.

This problem is of paramount importance and has been intensively studied for many decades. In practice it is traditionally tackled by means of asymptotic approximations (such as the Born or Kirchhoff theories, which are valid for large and small wavelengths respectively) and by linearization of the original non-linear relation between state and parameters, about some background values of the unknown parameters [1].

The resulting algorithms can in general be viewed as regularized optimization schemes which attempt to fit iteratively the measured data by solving partial differential or integral equations involving the unknown parameters. The parameter estimates are refined iteratively, using gradient-type methods [2]. The main drawback of these approaches is a high computational cost because the forward problem must be solved, typically hundred of times before reasonable estimates are obtained.

In this paper we propose instead to tackle inverse scattering problems directly by means of parametric identification techniques. The procedure we propose involves a first step where one builds a (linear) parametric model of the system by using suitable PDEs approximation and discretization tools, and a second step where the unknown parameters of the finite-dimensional approximation are estimated. The estimation is done by combining Kalman filtering and linear least-squares parameter estimation methods. The estimation procedure turns out to be of the Expectation-Maximization (EM) type [3, 4]. However, due to linearity in the parameters the convergence turns out to be rather fast and the method quite effective. We expect to be able to solve inverse scattering problems in one spatial dimension, essentially in real time, which is a great improvement with respect to the state of the art iterative gradient minimization mentioned above. For previous work in this direction consult [5, 6].

2 Problem statement

Let $q(t, x)$ be a vector valued function representing the state of the physical system. Here $t \in \mathcal{I} = [0, \infty)$ is the time variable while $x \in \bar{D}$ is the spatial coordinate ranging on the closure of some smooth open set $\mathcal{D} \subseteq \mathbb{R}^d$ (usually $d \leq 3$). In the most general formulation, one introduces differential
operators as in the following:

\[ P = \frac{\partial^2}{\partial t^2} + \mathcal{H} \frac{\partial}{\partial t} - \mathcal{A} \]  

(1a)

\[ \mathcal{H} = 2 \sum_{j=1}^{d} H_j(t,x) \frac{\partial}{\partial x_j} + H_0(t,x) \]  

(1b)

\[ \mathcal{A} = \sum_{j,l=1}^{d} A_{jl}(t,x) \frac{\partial^2}{\partial x_j \partial x_l} - \sum_{j=1}^{d} A_j(t,x) \frac{\partial}{\partial x_j} - A_0(t,x) \]  

(1c)

(1d)

where \( \{A_{jl}\}_{j,l=1}^{d} \), \( \{A_j\}_{j=0,1,...,d} \) and \( \{H_j\}_{j=0,1,...,d} \) are matrix functions whose entries define the physical properties of the medium through which waves propagate. The operator \( P \) is of hyperbolic type \([7]\).

Let \( \mathcal{B} \) be a matrix of differential operators defined on a neighbourhood of the boundary and let \( g(t,x) \) be a real valued function in \( \mathcal{I} \times \partial \mathcal{D} \); \( q_0(x) \) and \( q_1(x) \) are real valued functions in \( \mathcal{D} \). The state \( q \) is a solution of the following initial boundary value problem

\[ Pq(t,x) = f(t,x) \quad (t,x) \in \mathcal{I} \times \mathcal{D} \]  

(2a)

\[ \mathcal{B}q(t,x) = g(t,x) \quad (t,x) \in \mathcal{I} \times \partial \mathcal{D} \]  

(2b)

\[ q(0,x) = q_0(x) \quad x \in \mathcal{D} \]  

(2c)

\[ \frac{\partial q}{\partial t}(0,x) = q_1(x) \quad x \in \mathcal{D} \]  

(2d)

where \( f(t,x) \) represents the forcing term imposed from an external experimenter.

In what follows we shall restrict to a prototypical 1-dimensional (\( d = 1 \)) scalar-state case, and to propagation taking place in a time-invariant but possibly non-homogeneous medium. The matrix parameters previously introduced reduce to real valued functions independent of \( t \) but dependent on \( x \in \mathcal{D} \) which we denote by \( a_{11}(x) \), \( a_0(x) \), \( a_1(x) \), \( h_0(x) \) and \( h_1(x) \). Moreover we shall assume \( a_{11}(x) \geq 0 \) for all \( x \) (which is sufficient condition in order for \( P \) to be an hyperbolic operator).

From a physical point of view, we are dealing with plane waves travelling in only one space direction \( x \) along which, however, the properties of the medium are allowed to vary. Much of what we are going to do in this simplified setting extends to two or three spatial dimensions.

The differential operator \( P \) has now the following expression

\[ P = \frac{\partial^2}{\partial t^2} + \left( 2h_1(x) \frac{\partial}{\partial x} + h_0(x) \right) \frac{\partial}{\partial t} - \left( a_{11}(x) \frac{\partial^2}{\partial x^2} - a_1(x) \frac{\partial}{\partial x} - a_0(x) \right). \]  

(3)
We shall assume that the external sources are concentrated in a finite set of points \( S = \{ s_1, s_2, \ldots, s_m \} \subset \mathcal{D} \) so that we can represent \( f \) as

\[
f(t, x) = \sum_{l=1}^{m} u_l(t) \delta(x - s_l)
\]  

and likewise, that a finite number, \( p \), of sensors are located at the positions \( R = \{ r_1, r_2, \ldots, r_p \} \subset \mathcal{D} \). The sensors can measure the state or its time derivatives, e.g.

\[
y_i(t) = q(t, r_i) \quad \text{or} \quad \frac{\partial}{\partial t} q(t, r_i) \quad i = 1, 2, \ldots, p
\]  

The problem we want to solve then is: given the initial boundary value problem (2) and input and output data sampled from a finite interval, say \( \{ u_l(kT_c) \}_{l=1,2,\ldots,m} \) and \( \{ y_i(kT_c) \}_{i=1,2,\ldots,p} \) for \( kT_c \in [0, T] \), determine a (best in some sense) approximation of the function coefficients that define the operator \( \mathcal{P} \) in (3).

In order to solve this problem, as a first step we must deal with the following questions:

(a) if a solution exists, is it unique (identifiability)?
(b) how does the solution change if we are given noisy data (stability)?
(c) how to express the unknown functions using a finite number of variables (parametrization)?
(d) which is the most suitable numerical procedure that can be used in order to transform an infinite dimensional problem into a finite dimensional one (discretization)?

These points are briefly touched below, while in the next section the estimation algorithm is discussed.

### 2.1 Identifiability and well posedness

Assessing under which conditions it is possible to determine the coefficient of a PDE given its solution in part of the domain, is in general a hard task [8]. Here we only sketch some aspects of this problem which are specific of hyperbolic PDEs. In particular, we shall assume that an initial estimate of the function is available and will be concerned about its refinement.

The first aspect to keep in mind is that of finite velocity of propagation. A constant \( v_{max} \) can be defined such that

\[
v_{max} \geq |h_1(x)| + \sqrt{h_2^2(x) + a_{11}(x)} \quad \forall x \in \mathcal{D}
\]  

provided all the functions involved are bounded. Moreover we can define a domain of influence at \( (t_0, x_0) \in \mathcal{T} \times \mathcal{D} \) as the smallest set \( \mathcal{Y}(t_0, x_0) \) such that, for arbitrary \( (t, x) \in \mathcal{Y}(t_0, x_0) \), the value of \( q(t, x) \) changes if we change
suitably the value of data (initial and/or boundary data and input) in an arbitrary neighbourhood of \((t_0, x_0)\). It can be shown that

\[
\mathcal{Y}(t_0, x_0) \subseteq \{(t, x) : |x - x_0| \leq v_{\text{max}}(t - t_0)\}
\]

i.e. in a finite time interval perturbations propagate in space with a finite speed.

The second aspect is that of conservation of energy. Define the total energy of the solution \(q\) as

\[
E(q; t) = \frac{1}{2} \int_{\mathcal{D}} e(q; t, x) \, dx
\]

where

\[
e(q; t, x) = \left( \frac{\partial q}{\partial t}(t, x) \right)^2 + a_{11}(x) \left( \frac{\partial q}{\partial x}(t, x) \right)^2
\]

If \(h_1(x) = h_0(x) = a_1(x) = a_0(x) = 0 \quad \forall x \in \mathcal{D}\) then \(E(q; t)\) is constant. On the contrary if, for example, \(h_0(x) \neq 0\) then \(E(q; t)\) is a decreasing function of \(t\).

In practical applications the recording time is finite and dissipation always occurs; thus (theoretical) identifiability can be guaranteed only in a suitable bounded region of space since the values of the coefficients outside of a bounded region do not affect the solution of the PDE and hence the measured data.

In addition to this, practical data are always sampled both in space and time and recovering uniquely parameters which are arbitrary functions of \(x\) from a finite set of data is of course hopeless. For this reasons, a finite parametrization of the coefficients is mandatory.

A desirable property of inverse problems is that of stability of the solution. Together with existence and uniqueness of the solution, this property defines what is called a well-posed problem. Roughly speaking, stability requires that slightly perturbed data should not change substantially the solution with respect to the unperturbed ones. In other words, we would like the estimates of the coefficients to depend continuously on the data. This is also a difficult question and, to the author’s knowledge, for hyperbolic problems no simple general answer exists.

In summary, the coefficient identification problem for the system (3)-(5) is an intrinsically ill-posed problem. For this reason we will have to use a suitable finite-dimensional parametrization of the unknown coefficients and statistical (regularization) methods to obtain reasonably well-behaved solutions.

\subsection*{2.2 Parametrization and discretization}

Parametrization of the unknown coefficients should be chosen as simple and versatile as possible. It is clear that a trade-off has to be found between
the number of parameters and the ability to represent a large variety of spatial behaviours. The choice is therefore application-dependent and in the following we shall only ask the coefficients to be linearly parametrized, i.e.

$$a_k(x) = \sum_l \theta^k_l \varphi^l_k(x), \quad k = 0, 1, 11, \ldots$$  \hspace{1cm} (10)

and similarly for the coefficients $h_0, h_1$. We shall use the notation $\theta$ to collect in a single vector all the $n_p$ scalar parameters involved.

A linear parametrization can be disadvantageous, in terms of number of parameters required, in some situations such as, for example, in describing two known homogeneous media separated by an interface whose position is unknown. However under this hypothesis it is possible to set up an iterative descent procedure for the identification of the parameters (as explained in the next section) which does not require computation of gradients.

We also note that, being $a_{11}$ a positive quantity, constraints on the parameters $\theta_{11}$ should be taken into account once the functions $\varphi_{11}^l$ have been chosen. The same also holds for the other coefficients whenever physical arguments dictate it. We keep track of this fact by imposing that $\theta \in \Theta$ where $\Theta$ is a suitable subset of $\mathbb{R}^{n_p}$.

We shall quickly review the concept of a weak solution, $q(t, \cdot)$, of (2) in the Sobolev space $H^s(D)$ of the open domain $D$. The latter is a Hilbert space defined, for a positive integer $s$, as

$$H^s(D) = \{ f \in L^2(D) : \partial^\alpha f / \partial x^\alpha \in L^2(D), \ \alpha \leq s \}$$  \hspace{1cm} (11)

It is well-known that problem (2) can be reformulated as a variational problem: find $q(t, \cdot) \in H^s(D)$ such that for all $v \in H^s(D)$

$$\langle \mathcal{P}q - f, v \rangle_{L^2(D)} + \langle \mathcal{B}q - g, v \rangle_{L^2(\partial D)} = 0$$  \hspace{1cm} (12)

where $\langle \cdot, \cdot \rangle$ is the notation for the inner product. Existence and uniqueness of the solution have been investigated using tools from functional analysis [9, 10].

Discretizations of the solution can be obtained by Galerkin methods. Let $q$ be approximated by a linear combination of $N$ independent functions $\psi_k(x) \in H^s(D)$ as

$$q(t, x) \approx \sum_{j=1}^{N} q_j(t) \psi_j(x) = \psi^T(x)q(t)$$  \hspace{1cm} (13)

and let the same hold for $v$. Then insted of (12), one solves

$$\langle \mathcal{P}\psi^Tq - f, \psi_k \rangle_{L^2(D)} + \langle \mathcal{B}\psi^Tq - g, \psi_k \rangle_{L^2(\partial D)} = 0, \ \forall \psi_k$$  \hspace{1cm} (14)

We shall assume that convergence of the approximation to the true solution as $N \to +\infty$ is guaranteed (this actually depends on the choice of the $\{\psi_k\}$...
and should be investigated on a case by case basis). Performing the necessary substitutions and calculation, the above expression can be rewritten in matrix form as

\[ \mathbf{M} \ddot{\mathbf{q}}(t) + \mathbf{C}(\mathbf{\theta}) \dot{\mathbf{q}}(t) + \mathbf{K}(\mathbf{\theta}) \mathbf{q}(t) = \mathbf{G}(\mathbf{S}) \mathbf{u}(t) \]  

where \( \mathbf{M} \) is a known matrix, \( \mathbf{G} \) is a \( N \times m \) matrix function of sources locations \( \mathbf{S} \), \( \mathbf{u} \) is a column vector collecting all the inputs \( u_l \) (see (4)) and \( \mathbf{K} \) and \( \mathbf{C} \) depend linearly on the parameters in the vector \( \mathbf{\theta} \):

\[ \mathbf{K}(\mathbf{\theta}) = \mathbf{K}_0 + \sum_{l=1}^{n_p} \theta_l \mathbf{K}_l \quad \mathbf{C}(\mathbf{\theta}) = \mathbf{C}_0 + \sum_{l=1}^{n_p} \theta_l \mathbf{C}_l \]  

The vector \( \mathbf{q} \) has dimension \( N \). The elements of the \( N \times N \) matrices \( \mathbf{M}, \mathbf{K}_l \) and \( \mathbf{C}_l \) are obtained by computing \( L^2 \) inner products between functions of \( \psi_k \) and their derivatives. Therefore they are fixed quantities once those functions have been chosen.

Since \( \mathbf{\theta} \) is not known, if we replace it in (15) with \( \hat{\mathbf{\theta}} = \mathbf{\theta} + \tilde{\mathbf{\theta}} \) then the following model is obtained

\[ \mathbf{M} \ddot{\mathbf{q}}(t) + \mathbf{C}(\hat{\mathbf{\theta}}) \dot{\mathbf{q}}(t) + \mathbf{K}(\hat{\mathbf{\theta}}) \mathbf{q}(t) = \mathbf{G}(\mathbf{S}) \mathbf{u}(t) + \mathbf{e}_0(t) \]  

where

\[ \mathbf{e}_0(t) = \left( \sum_{l=1}^{n_p} \tilde{\theta}_l \mathbf{C}_l \right) \dot{\mathbf{q}}(t) + \left( \sum_{l=1}^{n_p} \tilde{\theta}_l \mathbf{K}_l \right) \mathbf{q}(t) \]  

Moreover, since our data consist of sampled versions of the input and output signals we must consider discretization of the time variable as well. Choosing the simplest explicit Euler discretization

\[ \dot{\mathbf{q}}(k) = \frac{\mathbf{q}(k+1) - \mathbf{q}(k)}{T_c} + \mathbf{e}_1(k) \quad \ddot{\mathbf{q}}(k) = \frac{\dot{\mathbf{q}}(k+1) - \dot{\mathbf{q}}(k)}{T_c} + \mathbf{e}_2(k) \]  

(here \( \mathbf{q}(k) \) is an abbreviation for \( \mathbf{q}(kT_c) \)), defining \( \mathbf{x}(k) = [\mathbf{q}^T(k) \ \dot{\mathbf{q}}^T(k)]^T \), a vector whose dimension is \( n = 2N \), and assuming that the input is constant between two sampling instants all the equations obtained can be cast in the following state space form

\[ \mathbf{x}(k+1) = \mathbf{A}(\hat{\mathbf{\theta}}) \mathbf{x}(k) + \mathbf{B} \mathbf{u}(k) + \mathbf{v}(k) \]  

with

\[ \mathbf{A}(\hat{\mathbf{\theta}}) = \mathbf{I}_n + T_c \begin{bmatrix} \mathbf{O} & \mathbf{I}_N \\ -\mathbf{M}^{-1} \mathbf{K}(\hat{\mathbf{\theta}}) & -\mathbf{M}^{-1} \mathbf{C}(\hat{\mathbf{\theta}}) \end{bmatrix} \]  

\[ \mathbf{B} = T_c \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1} \mathbf{G} \end{bmatrix} \]
and $v(k)$ is an error term which accounts for the contribution of $e_0(k)$, $e_1(k)$ and $e_2(k)$. For the validity of this model we obviously require that the sampling period $T_c$ should be much smaller than the reciprocal of the Nyquist frequency of the input signal.

Under the hypothesis (13) the counterpart of the measurement equations (5) can be written as

$$y(t) = H_0(\mathcal{R})q(t) + H_1(\mathcal{R})\dot{q}(t) + w(k) = Cx(k) + w(k)$$

(23)

where $H_0$ and $H_1$ are two $p \times N$ matrix functions of sensors locations $\mathcal{R}$ while the term $w(k)$ is another error term.

3 Parameter estimation

3.1 Description of the algorithm

In the previous section the following discrete-time model has been obtained:

$$\Sigma(\hat{\theta}) : \begin{cases} x(k + 1) = A(\hat{\theta})x(k) + Bu(k) + v(k) \\ y(k) = Cx(k) + w(k) \end{cases}$$

(24a)

(24b)

We have added two error sequences $v(k)$ and $w(k)$ to take into account all the effect of all approximations made in the discretization phase. Both of them are unknown and then will be modelled in the following as two processes of uncorrelated random variables with zero mean values and covariance

$$\mathbb{E} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \begin{bmatrix} v^T(j) \\ w^T(j) \end{bmatrix} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta(k - j).$$

(25)

The matrices $Q$, $S$ and $R$ are constant quantities; roughly speaking, they fix upper bounds on the errors introduced by discretizing the full-continuous problem (2).

Assuming for a moment that a measure $x(k)$ of the state $x(k)$ is available for all $k = 0, \ldots, N_c$ and at the same time $y(k)$ and $u(k)$ are the measurements of the output $y(k)$ and input $u(k)$ sequences, an estimate of $\theta$ could be obtained by minimizing the errors computed from (24a), namely

$$\hat{\theta} = \min_{\vartheta \in \Theta} \sum_{k=0}^{N_c-1} \|A(\hat{\theta})x(k) + Bu(k) - x(k + 1)\|_2^2$$

(26)

and the solution of this minimization problem could be obtained by least squares, because the parameters enter linearly the relation $A(\hat{\theta}) = A_0 + \sum_{i=1}^{n_\mu} \vartheta_i A_i$.

Since however the state is not known, we shall instead use the best state estimate which can be obtained from the available measurements of the
input and output signals and the initial knowledge about \( \theta \). Assuming an initial estimate \( \hat{\theta}^{(0)} \) can be guessed, one may calculate recursively by Kalman filtering the estimate \( \hat{x}^{(0)}(k|k - 1) \) of \( x(k) \) given the past history, \( u(j) \) and \( y(j), j = 0, \ldots, k - 1 \), and then set up an innovation model, say \( \Sigma(\hat{\theta}^{(0)}) \), which is formally of the same structure as (24) but has state estimates in place of the true state variables [11, 12, 13]. From this model an improvement of the initial parameter estimate \( \hat{\theta}^{(0)} \) can be computed by least squares as described above. The procedure can then be iterated using the new estimate, say \( \hat{\theta}^{(1)} \), to set up new Kalman filter estimates of \( x(k) \), etc.

The algorithm consists of the following steps:

1. choose an initial estimate \( \hat{\theta}^{(0)} \);
2. given \( \hat{\theta}^{(i)} \) compute the estimates \( \hat{x}^{(i)}(k|k - 1) \) running a Kalman filter for the model \( \Sigma(\hat{\theta}^{(i)}) \) and input/output data \( y(k) \) and \( u(k), k = 0, \ldots, N_c \);
3. update the estimate by solving

\[
\hat{\theta}^{(i+1)} = \min_{\vartheta \in \Theta} \sum_{k=0}^{N_c-1} \left\| \left[ A(\vartheta) - \text{SR}^{-1}C \right] \hat{x}^{(i)}(k|k - 1) + Bu(k) + \text{SR}^{-1}y(k) - \hat{x}^{(i)}(k + 1|k) \right\|^2_2 \quad (27)
\]

using least squares;
4. repeat from step 2 until convergence.

### 3.2 Theoretical motivation

If the complete statistical description of the random variables \( v(k) \) and \( w(k) \) were known, say through their joint probability density \( p_{v,w} \), the log-likelihood function with respect to the parameters vector \( \vartheta \) could be written as (see (24)):

\[
l(\vartheta|X,Y,U) = \sum_{k=0}^{N_c-1} \log p_{v,w}(x(k + 1) - A(\vartheta)x(k) - Bu(k), y(k) - Cx(k)) \quad (28)
\]

where \( X = \{x(k), k = 0, \ldots, N_c\} \), \( Y = \{y(k), k = 0, \ldots, N_c - 1\} \) and \( U = \{u(k), k = 0, \ldots, N_c - 1\} \) are realizations of the random variables \( x(k) \), \( y(k) \) and \( u(k) \) respectively. The maximum likelihood estimate would be computed then as

\[
\hat{\theta} = \max_{\vartheta \in \Theta} l(\vartheta|X,Y,U) \quad (29)
\]
Practically one looks for stationary points solving
\[ \nabla l(\vartheta|X, Y, U) = 0 \]  

being \( \nabla l \) the \( n_p \) component gradient (column) vector of \( l \) with respect to the elements of \( \vartheta \), also called score-function. This is justified by the well-known result that for all \( \theta \in \Theta \)
\[ \mathbb{E}_\theta \nabla l(\theta|X, Y, U) = 0 \]  

where we have introduced the sets of random variables \( X = \{x(k), k = 0, \ldots, N_c\} \), \( Y = \{y(k), k = 0, \ldots, N_c-1\} \) and \( U = \{u(k), k = 0, \ldots, N_c-1\} \). Moreover
\[ \mathbb{E}_\theta [\nabla l(\theta|\cdot) \nabla l^T(\theta|\cdot)] = -\mathbb{E}_\theta \nabla (\nabla l(\theta|\cdot)) = I(\theta) \]  

being \( \nabla (\nabla l) \) the Jacobian matrix of the vector \( \nabla l \) and it is also well known that the variance of any unbiased estimator is bounded from below by \( I(\theta)^{-1} \), i.e. by the inverse of the information matrix.

In the present case \( X \) is not available. A way to deal with such incompleteness in the data is to use the EM algorithm which starting from an initial guess \( \hat{\theta}^{(0)} \) guarantees convergence to a local maximum of \( l \). The original procedure presented in [3] is equivalent to the following one (see [4] for details) which consists in an iteration between two steps:

E-step: given \( \hat{\theta}^{(i)} \) compute
\[ h^{(i)}(\vartheta) = \mathbb{E}_{\hat{\theta}^{(i)}}[\nabla l(\vartheta|X, Y, U) | Y, U] \]  

M-step: define \( \hat{\theta}^{(i+1)} \in \Theta \) as the root of \( h^{(i)}(\vartheta) = 0 \).

Since it is hard to decide in favour of a particular expression for the probability density \( p_{v,w} \) we try to further generalize the E-step of the algorithm in such a way that only hypotheses on the first and second moments of the random variables \( v(k) \) and \( w(k) \) are needed (see (25)). To this end we substitute the exact score-function with a new function \( g_s^*(\vartheta|X, Y, U) \) chosen optimally in a convenient space \( \mathcal{G} \) of random vectors of dimension \( n_p \) with finite second moments. In strict analogy with (31) it is required that \( \mathbb{E}_\theta g(\theta|X, Y, U) = 0 \) for any \( g \in \mathcal{G} \) and \( \theta \in \Theta \). Moreover each \( g \in \mathcal{G} \) can be standardized in order to satisfy also a relation like (32). In fact \( g_s = -\mathbb{E}_\theta \nabla g(\theta|X, Y, U)^T(\mathbb{E}_\theta \nabla g(|X, Y, U))^{-1}g \) satisfies \( \mathbb{E}_\theta [g_s(\theta|\cdot)g_s(\theta|\cdot)^T] = -\mathbb{E}_\theta \nabla g_s(\theta|\cdot) \) for all \( \theta \in \Theta \) so that
\[ \mathcal{E}_{g_s}(\theta) = \mathbb{E}_\theta [g_s(\theta|X, Y, U)g_s(\theta|X, Y, U)^T] \]  

is a natural generalization of the information matrix \( I(\theta) \).

The element \( g_s^* \) is then defined as the one which satisfies \( \mathcal{E}_{g_s}^*(\theta) \geq \mathcal{E}_{g_s}(\theta) \) for
The E-step requires the computation of $h^s$ presented. The algorithm can be considered as a variant of the well-known

A promising approach for solving inverse scattering problems has been pre-

accomplished by solving a linear least squares problem.

$$
\mathbf{g}(\vartheta|\mathbf{X}, \mathbf{Y}, \mathbf{U}) = \sum_{k=0}^{N_c-1} \mathbf{P}(k, \vartheta) \begin{bmatrix} \mathbf{v}(k) \\ \mathbf{w}(k) \end{bmatrix}
$$

(35)

the optimal $n_y \times (n + p)$ matrices $\mathbf{P}^s(k, \vartheta)$ can be determined and the quasi-score function is

$$
\mathbf{g}^s(\vartheta|\mathbf{X}, \mathbf{Y}, \mathbf{U}) = \sum_{k=0}^{N_c-1} \begin{bmatrix} \vdots \\ \mathbb{E}_{\vartheta}[\mathbf{x}(k)]^T \mathbf{A}^T_k \\ \vdots \end{bmatrix} (\mathbf{Q} - \mathbf{SR}^{-1}\mathbf{S}^T)^{-1}.
$$

(36)

The E-step requires the computation of $h^{(i)}(\vartheta) = \mathbb{E}_{\vartheta^{(i)}}[\mathbf{g}^s(\vartheta|\mathbf{X}, \mathbf{Y}, \mathbf{U})|Y, U]$. Since $\mathbf{g}^s$ is a linear combination of random variables, this operation results in the substitution in (36) of $\mathbf{x}(\cdot)$ with $\mathbb{E}_{\vartheta^{(i)}}[\mathbf{x}(\cdot)|Y, U]$, and of $\mathbf{y}(\cdot)$ and $\mathbf{u}(\cdot)$ with $\mathbb{E}_{\vartheta^{(i)}}[\mathbf{y}(\cdot)|Y, U]$ and $\mathbb{E}_{\vartheta^{(i)}}[\mathbf{u}(\cdot)|Y, U]$ respectively. Moreover we shall consider in place of the random variables $\mathbf{x}(\cdot)|Y, U$, the corresponding minimum variance linear estimates given $Y$ and $U$ or, in other words, the projection on the Hilbert space generated by $Y$ and $U$. For these variables we use the notation $\hat{\mathbf{x}}(\cdot)|Y, U$. The advantage is that again only hypotheses on the first and second moments of the random variables $\mathbf{v}(k)$ and $\mathbf{w}(k)$ are needed. The terms $\mathbb{E}_{\vartheta^{(i)}}[\mathbf{x}(k)|Y, U]$ are then the smoothed estimates of the state and can be calculated by standard routines [12]. In the algorithm presented above, however, these are replaced by the Kalman filtered estimates $\mathbb{E}_{\vartheta^{(i)}}[\hat{\mathbf{x}}(k)|Y_{k-1}, U_{k-1}]$ in order to limit the computational burden. Here $Y_{k-1} = \{y(i), i = 0, \ldots, k - 1\}$, $U_{k-1} = \{u(i), i = 0, \ldots, k - 1\}$ and the notation $\hat{\mathbf{x}}^{(i)}(k|k-1)$ is also used for these estimates.

The M-step requires solving $h^{(i)}(\vartheta) = 0$. Since the terms $\mathbb{E}_{\vartheta}[\mathbf{x}(k)]^T$ in (36) cannot be computed, the suboptimal procedure of finding a $\vartheta^{(i+1)}$ such that

$$
\left[ \mathbf{A}(\vartheta^{(i+1)}) - \mathbf{SR}^{-1}\mathbf{C} \right] \hat{\mathbf{x}}^{(i)}(k|k-1) + \mathbf{Bu}(k) + \mathbf{SR}^{-1}\mathbf{y}(k) - \hat{\mathbf{x}}^{(i)}(k+1|k) \approx 0
$$

(37)

for each $k = 0, \ldots, N_c - 1$ has been proposed in the algorithm. This task is accomplished by solving a linear least squares problem.

4 Conclusions

A promising approach for solving inverse scattering problems has been presented. The algorithm can be considered as a variant of the well-known
EM algorithm used in statistics for finding maximum likelihood estimates. The methodology is general enough to be used in a large variety of inverse scattering problems. Convergence of the algorithm has been tested by numerical experiments. Experimental evidence shows also that the quality of the estimates is affected by the parametrization, i.e. by the choice of the functions $\varphi^k_l$ in (10). This aspect will be further investigated in the future along with the generalization of the algorithm for the case of unknown mean and covariance of the random processes $v$ and $w$.

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References


