On the Robustness of Limit Cycles

1 Introduction

Stability and robustness of limit cycles oscillations are properties of fundamental importance in many applications in electronics, mechanics, biology, and physics. In current control applications, limit cycle oscillation is crucial in biological locomotion [7, 4], rhythmic mechanical motion [8] and auto-tuning [2]. Tools for rigorous analysis of stability and robustness of limit cycle oscillations is important in the design and verification of such systems. The classical literature provides several useful results but few, if any of them, extends directly to system descriptions that are subject to various forms of unmodelled dynamics. One problem is that the classic results were derived in a state space formalism which does not extend easily to systems with unknown possible infinite dimension. Another problem is that the introduction of uncertainty in the system dynamics perturb both the period time and the orbit of the limit cycle which is in stark contrast to the traditional problems in robust control where the equilibrium solution remains fixed when the system is perturbed. This makes robust stability analysis of limit cycles a challenging problem.

In this extended abstract we review one of the results in [5]. There we consider systems consisting of a feedback interconnection of an exponentially stable linear time-invariant (LTI) plant with a memoryless nonlinearity. We show that a limit cycle in such a system persists and remains stable after bounded perturbations of the LTI plant if certain invertibility conditions hold for the variational system which is obtained after a linearization along the nominal limit cycle. Neither the nominal system nor the perturbation need to be finite dimensional but they are required to be bounded in an appropriate induced norm. The situation simplifies considerably when the nominal system is finite dimensional. Then the conditions reduce to a well-known stability and robustness condition for finite dimensional systems [3, 6, 1]. The distinction is that we allow the perturbation of the system to be dynamic of any order. Our discussion will focus around this result. Its interest lies in applications where system design and system modeling is based on finite dimensional approximations. The robustness result allows the systems analyst to rigorously verify that a modeled or designed limit cycle will appear also in the true infinite dimensional system.

The result presented here shows that the well-known stability and robustness condition on the characteristic multipliers also is a stability and robustness condition for the case when the system is perturbed by a dynamic uncertainty of arbitrary dimension. We
illustrate how our modeling applies to the Van der Pol oscillator. In [5] we also show how bounds on the robustness margin can be estimated using small gain type results. For proofs and more details we refer to [5].

2 Robustness of Limit Cycles

Consider the system

\[ y(t) = \int_{-\infty}^{t} h(t - \tau, \theta) \varphi(y(\tau)) d\tau, \quad \forall t \]

where the nonlinearity \( \varphi(\cdot) \) is assumed to be \( C^1 \) and the impulse response function \( h(t, \theta) \) is assumed to be exponentially stable. This system equation is suitable for representing stationary solutions such as equilibrium solutions or stationary periodic solutions. The parameter \( \theta \) is a scaling of the size of the uncertainty in the system and we assume it belongs to an open interval \( I_\theta \), which contains 0. The system is called nominal when \( \theta = 0 \) and our assumption is that the nominal system has a \( T_0 \)-periodic solution. We will in this section discuss a condition under which there remains a stable periodic solution when \( \theta \) is perturbed from zero.

We next summarize the assumptions on (1). The assumptions on the impulse response function will be formulated in terms of its Laplace transform \( H(s, \theta) \).

**Assumption 1.** For the system in (1) we assume

(i) The nonlinearity \( \varphi(\cdot) \) is \( C^1 \).

(ii) We assume that for some exponential decay rate \( \alpha > 0 \) and all \( \theta \in I_\theta \) (an interval containing \( \theta = 0 \)) the Laplace transform \( H(s, \theta) \) is (i) strictly proper (ii) analytic in \( \text{Re} \, s > -\alpha \), (iii) continuous on \( \text{Re} \, s \geq -\alpha + i\mathbb{R} \), and (iv) bounded such that for \( \text{Re} \, s \geq -\alpha \) we have \( \max(|sH(s, \theta)|, |H(s, \theta)|) \leq b \) for some number \( b \). The parameter \( \theta \) scales the size of the uncertainty such that the following norm bound holds \( \|H(s, \theta) - H(s, 0)\| \leq \gamma |\theta| \).

(iii) When \( \theta = 0 \) there exists a \( T_0 \)-periodic solution \( y_0 \).

**Example 1.** Consider Van der Pol’s equation with a dynamic uncertainty

\[ \ddot{u}(t) + m(u(t)^2 - 1)\dot{u}(t) + u(t) = \theta(\Delta u)(t). \]

Here \( \Delta(s) \) is an exponentially stable transfer function. To represent this system on the form (1) we introduce the new coordinates

\[ x_1 = -\dot{u} - m(u^3/3 - u) \]
\[ x_2 = u \]

Differentiation gives

\[ \dot{x}(t) = Ax(t) + B_1\varphi(y(t)) + \theta B_2(\Delta y)(t) \]
\[ y(t) = Cx(t) \]
Figure 1: Linear fractional transformation.

where

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \\
B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
B_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
C = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

and \(\varphi(y) = -my^3/3 + (2 + m)y\). This system can be represented as in the block diagram in Figure 1, where

\[
H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C & 0 & 0 \\ C & 0 & 0 \end{bmatrix}
\]

This provides the following more compact notation for (1)

\[
y = H(s, \theta)\varphi(y)
\]

where

\[
H(s, \theta) = H_{11}(s) + \theta H_{12}(s)(I - \theta H_{22}(s)\Delta(s))^{-1}H_{22}(s)
\]

The system representation in (2)-(3) is the linear fractional transformation (LFT) formalism from robust control [9]. It includes many system models of practical interest. To satisfy Assumption 1, the nominal dynamics and \(\Delta(s)\) must be such that \(H(s, \theta)\) is strictly proper and exponentially stable for all \(\theta \in I_\theta = (-\bar{\theta}, \bar{\theta})\). If \(\alpha\) is the chosen exponential decay rate then all the required properties on \(H(s, \theta)\) hold if, for example, \(H_{k,l}(s), \Delta(s)\) \(k, l = 1, 2\) are analytic in \(\text{Re} s > -\alpha\), continuous on \(-\alpha + i\mathbb{R}, H_{11}\) and \(H_{12}\) are strictly proper, and the small gain condition \(\bar{\theta}\|\Delta(s-\alpha)\|\|H_{22}(s-\alpha)\| < 1\) is satisfied.

The system in Figure 1 is called nominal when \(\theta = 0\) and our assumption is that the nominal system has a periodic solution. Our main result, which is stated in the next section, establishes that a unique exponentially stable periodic solution remains for all sufficiently small \(\theta\) if an easy-to-check condition holds for the linearized dynamics. A lower bound on how much perturbation the limit cycle can sustain can be obtained from a small gain condition given in [5].
Main Result

We will next present a local robustness result in the case when the nominal dynamics of (1) is finite dimensional, i.e. \( H(s,0) = C(sI - A)^{-1}B \), where \( A \) is Hurwitz. The result shows that the limit cycle is exponentially stable and persist under small enough perturbations given that all but one characteristic multiplier corresponding to the linearized dynamics is exponentially stable. Here the characteristic multipliers of a periodic matrix \( A(t) = A(t + T_0) \) are the eigenvalues of the monodromy matrix \( \Phi(T_0,0) \), where

\[
\frac{d}{dt}\Phi(t,0) = A(t)\Phi(t,0), \quad \Phi(0,0) = I
\]

In order to define exponential stability we introduce the non-steady-state version of (1), defined as (here we suppress the dependence of the parameter \( \theta \))

\[
y(t) = f(t) + \int_0^t h(t - \tau)\varphi(y(\tau))d\tau, \quad t \geq 0.
\]

In (4), \( f(\cdot) \) represents initial conditions and external disturbances. The choice

\[
f_0(t) = \int_{-\infty}^0 h(t - \tau)\varphi(y_0(\tau))d\tau
\]

(5) gives the \( T \)-periodic solution \( y_0(t) \), since (4) has a unique solution for any locally integrable function \( f(\cdot) \).

By exponential stability of the solution \( y_0 \) we will mean that for all \( f \) close to \( f_0 \) the solution \( y \) of (4) will converge exponentially to \( y_0 \).

**Definition 1.** The \( T \)-periodic solution \( y_0 \) is said to be *locally exponentially stable* if there exists \( \alpha > 0, \delta > 0, \) and \( c > 0 \) such that for any \( f \) satisfying the condition

\[
|f(t) - f_0(t)| \leq \delta, \quad \forall t \geq 0
\]

the corresponding solution \( y \) of (4) satisfies the inequality

\[
\int_0^\infty e^{2\alpha t}|y(t) - y_0(t + d)|^2 dt + |d|^2 \leq c \int_0^\infty e^{2\alpha t}|f(t) - f_0(t)|^2 dt
\]

for some \( d \in \mathbb{R} \).

The presence of the phase shift parameter \( d \) in (7) is necessary. It can be shown that with \( d \) fixed at \( d = 0 \), no non-equilibrium solution \( y_0 \) of (1) satisfies (7).

**Theorem 1.** Suppose Assumption 1 holds and consider the case when in addition \( h(t,0) = Ce^{At}B\nu(t) \), where \( \nu(\cdot) \) is the unit step function and \( \text{Re}(A) < -\alpha \). If the characteristic multipliers corresponding to \( A_{cl}(t) = A + B\varphi'(y_0(t),0)C \) can be sorted as

\[
1 = \rho_1 > |\rho_2| \geq |\rho_3| \geq \cdots \geq |\rho_n| \quad \text{and} \quad \alpha < -\frac{\log|\rho_2|}{T_0}
\]

then there exists a unique (modulo time translation) exponentially stable limit cycle solution to (1) for all sufficiently small \( \theta \). Moreover, the exponential decay rate in (7) can be chosen to be \( \alpha \).
Example 2. For the Van der Pol system Theorem 1 shows that the characteristic multipliers corresponding to

\[ A_{cl}(t) = A + B\varphi'(y_0(t))C = \begin{bmatrix} 0 & 1 \\ -1 & m(1-y_0(t)^2) \end{bmatrix} \]

must be sorted as \( |\rho_2| < \rho_1 = 1 \) in order for the limit cycle of the Van der Pol oscillator to be robustly stable. From Liouville’s formula we have

\[ \rho_2 = \det(\Phi_{cl}(1,0)) = e^{\int_0^1 \text{tr}(A_{cl}(\tau))d\tau} = e^{\int_0^1 m(1-y_0(\tau)^2)d\tau}. \]

If for example \( m = 0.2 \) then a numerical integration shows that \( \rho_2 = 0.34 \) and the Van der Pol system thus has a robustly stable limit cycle for this value of \( m \). This gives a new interpretation to the same condition in [3].

References


