Nyquist Stability Criterion of Sampled-Data Systems
with the 2-Regularized Determinant and
Its Applications to Robust Stability Analysis

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Abstract: Using the 2-regularized determinant associated with Hilbert-Schmidt operators, this paper derives
a Nyquist stability criterion of sampled-data systems that covers the general setting of sampled-data systems,
which gives a necessary and sufficient condition for internal stability of closed-loop sampled-data systems.
This criterion is applied to the derivation of robust stability theorems such as the small-gain and passivity
theorems with respect to the internal stability of sampled-data systems, by which the usefulness of the derived
Nyquist stability criterion is demonstrated.

1 Introduction

The robustness study of control systems with digital
controllers has attracted great attention since the late
1980’s, and it is widely recognized these days that con-
trol systems with such controllers should be treated as
sampled-data systems [19],[4],[6]. Such treatment is cru-
cial since the uncertainties in the continuous-time plant
act on continuous-time signals and thus it is imperative
that the intersample behavior of continuous-time signals
be taken into account completely. Among the approaches
that the intersample behavior of continuous-time signals
can be applied to the general setting of sampled-data sys-
tems.

Regarding the internal stability problem of sampled-
data systems, robustness against additive/multiplicative
perturbations was first treated in [14] with the FR-
operator approach to sampled-data systems [1],[13]. Also,
the relationship between robust $L_2$-stability and robust
internal stability was studied in [12] based on the spectral
analysis of the transfer operator $\hat{G}(z)$ and the associated
state matrices. Even though a sort of arguments closely
related to a Nyquist stability criterion of sampled-data
systems were developed in the former study [14], no ex-
licit arguments were given as to such a criterion. Fur-
thermore, it was enough in that study to assume that the
subsystem $P_{11}$ from the disturbance input $w$ to the con-
trolled output $z$ of the generalized plant $P$ is zero. This
corresponds to assuming that the transfer operator $\hat{G}(z)$
is of finite rank, or equivalently, the associated so-called
compression operator $\mathcal{D}$ is zero. A Nyquist (internal) sta-
bility criterion of sampled-data systems was first given in
[11] with the lifting technique, but it was derived only
under the assumption that the transfer operator $\hat{G}(z)$ (or
equivalently, the compression operator $\mathcal{D}$) is a trace class
operator [3],[10]. This assumption, however, is still re-
strictive since $\mathcal{D}$ becomes a trace class operator if and
only if the relative degree of the above-mentioned sub-
system $P_{11}$ is at least 2; if the relative degree is 1, then
$\mathcal{D}$ becomes a Hilbert-Schmidt operator and if the relative
degree is 0, then $\mathcal{D}$ even becomes a noncompact bounded
operator. Because of this fact, the usual definition of the
operator determinant [8],[10] could not be applied to the
cases with the relative degree less than 2, which had been
the obstacle to deriving a Nyquist stability criterion of
sampled-data systems with full generality. This paper in-
troduces the notion of the 2-regularized determinant [3],
[10] to get around the difficulty, and derives a new Nyquist
criterion for internal stability of ‘closed-loop’ sampled-
data systems.

The use of the 2-regularized determinant in a Nyquist
stability criterion was first accomplished in the context of
finite-dimensional linear continuous-time periodic-
time-varying systems in [28], but the arguments here are
quite different from the arguments there; the arguments in
[28] are based on the harmonic analysis approach [27] sim-
lar to the FR-operator approach to sampled-data systems
in [1],[13], while the arguments of this paper are based on
the lifting technique, so that the way we apply the spec-
tral properties of operators involved and the way we take
the Nyquist contour are quite different. Also, the results
in [28] correspond to the case of $D_{11} = 0$ in our context,
while our results can be applied to the case of $D_{11} \neq 0$
as well and thus consider the well-posedness issues of the
closed-loop systems based on some spectral properties of
the transfer operators. Furthermore, we show that this
criterion can be used to derive robust stability theorems
such as the small-gain theorem and the passivity theorem
of sampled-data systems with respect to internal stability.

The contents of this paper are as follows. In Sec-

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data systems together with the transfer operator \( \hat{G}(z) \), as well as trace class and Hilbert-Schmidt operators and the 2-regularized determinant associated with Hilbert-Schmidt operators. Using this determinant, we derive a new Nyquist (internal) stability criterion of ‘closed-loop’ sampled-data systems with full generality in Section 3. This new criterion is then applied in Section 4 to the derivation of robust stability theorems for sampled-data systems with respect to robust internal stability.

The notation used in this paper is standard. \( \sigma(\cdot) \) denotes the spectrum of an operator while \( \lambda(\cdot) \) is used to denote the set of the eigenvalues of a finite-dimensional matrix to avoid possible confusion in some context. Also, \( \sigma_e(\cdot) \) is used to denote the essential spectrum of an operator [5]. The adjoint of an operator on a Hilbert space is denoted by \( (\cdot)^* \).

2 Preliminaries

2.1 Lifted Description of Sampled-Data Systems

This subsection reviews the lifted description of the sampled-data system \( \Sigma_0 \) shown in Fig. 1 [25],[2],[22],[23],[26], where \( P \) denotes the continuous-time generalized plant, \( \Psi \) the discrete-time controller, \( \mathcal{H} \) the zero-order hold, and \( S \) the ideal sampler. The underlying sampling period will be denoted by \( h \). We assume that \( P \) and \( \Psi \) are described respectively by

\[
\begin{align*}
\frac{dx}{dt} &= Ax + B_1 w + B_2 u, \quad z = C_1 x + D_{11} w + D_{12} u \\
y &= C_{2} x
\end{align*}
\]

(1)

\[
\xi_{k+1} = A\phi \xi_k + B\phi y_k, \quad u_k = C\phi \xi_k + D\phi y_k
\]

(2)

where \( y_k = y(kh) \) and \( u(t) = u_k (kh \leq t < (k+1)h) \). The Hilbert space of square integrable vector functions over the time interval \([0,h)\) with the standard inner product will be denoted by \( \mathcal{K} \), whatever the dimension of the vector may be. The Euclidean space with dimension \( \dim(x) \) will be denoted by \( \mathcal{F}_x \). We define \( \mathcal{F}_u \) and \( \mathcal{F}_\xi \) in a similar way, and we further define \( \mathcal{F} := \mathcal{F}_u \oplus \mathcal{F}_\xi \).

Now, introduce the matrices \( A_d, B_{d2} \) and \( C_{d2} \), and the operators \( B_1, C_1, D_{11} \) and \( D_{12} \) as follows:

\[
A_d := \exp(Ah), \quad B_{d2} := \int_0^h \exp(A\sigma)B_2 d\sigma, \quad C_{d2} := C_2(3)
\]

Fig. 1: Open-loop sampled-data system \( \Sigma_0 \).

Fig. 2: Closed-loop sampled-data system \( \Sigma_c \).

Then, the ‘lifting-based transfer operator’ \( \hat{G}(z) \) of the sampled-data system \( \Sigma_0 \) is defined by

\[
\hat{G}(z) := \mathcal{C}(zI - A)^{-1} B + D : \mathcal{K} \to \mathcal{K}
\]

(8)

with

\[
A := \begin{bmatrix} A_d + B_{d2} D\phi C_{d2} & B_{d2} C_{d2} \\ B\phi C_{d2} & A\phi \end{bmatrix} : \mathcal{F} \to \mathcal{F}
\]

\[
B := \begin{bmatrix} B_1 \\ 0 \end{bmatrix} : \mathcal{K} \to \mathcal{F}
\]

\[
C := \begin{bmatrix} C_1 & D_{12} \\ D\phi C_{d2} & 0 \end{bmatrix} : \mathcal{F} \to \mathcal{K}
\]

\[
D := D_{11} : \mathcal{K} \to \mathcal{K}
\]

(9)

The importance of \( \hat{G}(z) \) lies in that it captures all the intersample behavior (i.e., the aliasing phenomena) in the sampled-data system \( \Sigma_0 \). With a slight abuse of notation\(^1\), the operator of multiplication by the matrix \( D_{11} \) which maps \( f(\cdot) \in \mathcal{K} \) to \( D_{11} f(\cdot) \in \mathcal{K} \), is also denoted by \( D_{11} \). Then, the operator \( D_{11} \) given in (6) can be rewritten as \( D_{11} = D_{110} + D_{11} \) with \( D_{110} \) defined in an obvious fashion, and thus \( D \) can also be rewritten as \( D = D_0 + D_{11} \). Here, it is well known that \( D_0 = D_{110} \) is compact. Hence, \( \hat{G}(z) \) is compact if and only if \( D_{11} = 0 \) (see, e.g., [26]).

Unless otherwise stated, we assume in the following that \( \dim(w) = \dim(z) \) and thus the matrix \( D_{11} \) is square. This assumption is natural in our context since Fig. 1 is usually considered to follow by taking out a certain signal path

\(^1\)It will be clear from the context whether \( D_{11} \) refers to the operator by multiplication or the underlying matrix.
from inside the continuous-time plant as in $\Sigma_c$ shown in Fig. 2 and then cutting that signal path; by such treatment, we can study robust stability of sampled-data systems under the setting of Fig. 3 as we shall see in Section 4. For lack of better terminologies, $\Sigma_c$ is called the closed-loop sampled-data system, while $\Sigma_0$ is called the open-loop sampled-data system in this paper.

2.2 Trace Class and Hilbert-Schmidt Operators and the 2-Regularized Determinant

In this section, we review some basic properties about trace class and Hilbert-Schmidt operators, as well as some fundamental results about the 2-regularized determinant associated with Hilbert-Schmidt operators [3],[10]. In this subsection, the underlying space that operators act on is suppressed for simplicity, but is assumed to be a separable Hilbert space; in the context of our discussions, the underlying space will usually be $\mathcal{K}$, but it could sometimes be the finite-dimensional space $\mathcal{F}$.

Let $\lambda_i(T)$ denote the $i$th eigenvalue of a linear compact operator $T$, counted up to algebraic multiplicities, and let $s_i(T) := \lambda_i^{1/2}(T^*T)$ be the $i$th singular value of $T$. For $1 \leq p \leq \infty$, the set of compact operators $T$ such that

$$\|T\|_p := \left( \sum_i s_i(T)^p \right)^{1/p} < \infty$$

(10)

is denoted by $c_p$ and referred to as a Schatten-von-Neumann class, which is a Banach space under the norm $\| \cdot \|_p$. In particular, $T$ is said to be a trace class operator if $T \in c_1$ while it is said to be a Hilbert-Schmidt operator if $T \in c_2$. Here, it is a fact that $c_1 \subset c_2$. It is also a fact that $ST \in c_1$ whenever $S,T \in c_2$. Furthermore, if $S \in c_1$ and $T \in c_2$, then $U_1SU_2 \in c_1$ and $U_1TU_2 \in c_2$ for any linear bounded operators $U_1$ and $U_2$. In particular, $\|U_1SU_2\|_1 \leq \|U_1\| \cdot \|S\| \cdot \|U_2\|$ and $\|U_1TU_2\|_2 \leq \|U_1\| \cdot \|T\|_2 \cdot \|U_2\|$

(11)

where $\| \cdot \|$ denotes the norm induced from the norm on the underlying Hilbert space of $U_1$ or $U_2$.

For every trace class operator $T \in c_1$, the operator trace and determinant

$$\text{tr}(T) := \sum_i \lambda_i(T), \quad \text{det}(I+T) := \prod_i (1+\lambda_i(T))$$

(12)

are well-defined in the sense the associated infinite series and infinite product converge absolutely. Note that the above definitions apply even when the underlying space of the operators is finite-dimensional, in which case these definitions coincide with the standard definitions of the trace and determinant for finite-dimensional matrices. If $S$ is a linear bounded operator and $T \in c_1$, then it is true that $\text{tr}(ST) = \text{tr}(TS)$ and $\text{det}(I+ST) = \text{det}(I+TS)$.

As for Hilbert-Schmidt operators $T \in c_2$, the above definitions for trace class operators do not make sense, in general, since the associated infinite series and infinite product may not converge. To get around this difficulty, we use the $p$-regularized determinant instead, which is defined as

$$\text{det}_p(I+T) := \text{det}(I + R_p(T))$$

(13)

where $R_p(T) := (I+T) \exp\left(\sum_{k=1}^{p-1} \frac{(-T)^k}{k}\right) - I$. Since $R_p(T) \in c_p$ whenever $T \in c_p$, $\text{det}_p(I+T)$ is well-defined for $T \in c_p$, and is referred to as the $p$-regularized determinant of $I+T$, where $p > 1$ is an integer. In this paper, we only need to consider the case of $p = 2$. Hence in the sequel, we review some useful properties about the 2-regularized determinant. First, it readily follows from the definition that for $T \in c_2$,

$$\text{det}_2(I+T) = \prod_i (1+\lambda_i(T)) \exp(-\lambda_i(T))$$

(14)

Also, if $ST \in c_2$ (and thus $TS \in c_2$), then

$$\text{det}_2(I+ST) = \text{det}_2(I+TS)$$

(15)

It is also a fact that for $S,T \in c_2$,

$$\text{det}_2(I+S) \text{det}_2(I+T) = \text{det}_2((I+S)(I+T)) \cdot \exp(\text{tr}(ST))$$

(16)

Note that $\text{tr}(ST)$ is well-defined and equals $\text{tr}(TS)$ in the above.

3 Nyquist Stability Criterion of Sampled-Data Systems Using the 2-Regularized Determinant

In this section, we derive a Nyquist stability criterion of sampled-data systems, which provides a condition for stability of the closed-loop sampled-data system $\Sigma_c$ in terms of the transfer operator $G(z)$ of the open-loop sampled-data system $\Sigma_0$. Before proceeding, however, we give some remarks about our previous study [11] on a Nyquist stability criterion for sampled-data systems. In that study, the matrix $D_{11}$ was assumed to be 0 unlike in this paper. It was further assumed that the continuous-time generalized plant $P$ satisfies $C_1B_1 = 0$ so that the operator $\mathcal{D} = \mathcal{D}_0$ is not only compact but is a trace class operator. Under these assumptions, the standard trace and determinant of operators given in (12) were applicable to derive a Nyquist stability criterion. In this section, we remove these two assumptions and derive a Nyquist stability criterion in full generality in the context of sampled-data systems. The key that enables this is the use of the 2-regularized determinant introduced in Subsection 2.2.

Now, we introduce the following assumption:

**A0** The matrix $D_{11}$ does not have an eigenvalue at $-1$.

**Lemma 1** Under the assumption **A0**, the operator $I + \mathcal{D}$ is invertible.

**Proof.** We first note that the essential spectrum $\sigma_e(I + \mathcal{D})$ is given by $\lambda(I + D_{11})$, which can be seen in a similar way to the arguments in Subsection 3.1 of [12]. Now, suppose that $\gamma \notin \lambda(I + D_{11})$. Then,
\[ \gamma I - (I + \mathcal{D}) = \{ \gamma I - (I + D_1) \}(I - \mathcal{D}_\gamma) \]  
(17)

where \( \mathcal{D}_\gamma = \{ \gamma I - (I + D_1) \}^{-1} D_0 \). Since \( \mathcal{D}_\gamma \) is a compact operator, it follows from (17) that \( \gamma \notin \lambda(I + D_1) \) belongs to \( \sigma(I + \mathcal{D}) \) if and only if \( \mathcal{D}_\gamma \) has an eigenvalue at 1. However, since \( \mathcal{D}_\gamma \) is a Volterra operator \( [5] \) (because so is \( D_0 \) and the operator \( (\gamma I - (I + D_1))^{-1} \) is simply an operator of multiplication, whose matrix counterpart simply acts on the associated kernel of the Volterra operator) so that it does not have a nonzero eigenvalue, the latter cannot occur. This implies that \( \sigma(I + \mathcal{D}) \setminus \lambda(I + D_1) \) is an empty set. Since \( \lambda(I + D_1) = \sigma_\nu(I + \mathcal{D}) \subset \sigma(I + \mathcal{D}) \) as mentioned above, it follows that \( \sigma(I + \mathcal{D}) = \lambda(I + D_1) \), where the right hand side does not contain the origin by the assumption \( \textbf{A0} \). This implies that \( I + \mathcal{D} \) is invertible (with the inverse being bounded). Q.E.D.

Hence, under the assumption \( \textbf{A0} \), it is justified to talk about \( (I + \mathcal{D})^{-1} \). Also, it is not hard to see that the closed-loop sampled-data system \( \Sigma_c \) is internally stable if and only if

\[ A_c := A - B(I + \mathcal{D})^{-1}C \]
(18)

is a stable matrix in the discrete-time sense. Now, to study the stability condition for \( A_c \), we introduce, again under the assumption \( \textbf{A0} \), the modified generalized plant \( P' \) described by

\[
\begin{aligned}
\frac{dx}{dt} &= Ax + B_1 w + B_2 u \\
z &= (I + D_1)^{-1} C_1 x + (I + D_1)^{-1} D_{12} u \\
y &= C_2 x
\end{aligned}
\]
(19)

Note that the “\( D_{11} \) matrix” has disappeared in the modified generalized plant \( P' \). Also, let us consider the modified open-loop sampled-data system \( \Sigma'_c \), which is defined as \( \Sigma_c \) with the generalized plant \( P \) replaced by \( P' \), and consider the matrix \( \mathcal{A} \) corresponding to \( \Sigma'_c \), which we denote by \( \mathcal{A}' \). We also define the operators \( B', C' \) and \( \mathcal{D}' \) in a similar fashion. Then, it is easy to see that

\[ \mathcal{A}' = \mathcal{A}, \quad B' = B, \quad C' = (I + D_1)^{-1} C, \quad \mathcal{D}' = (I + D_1)^{-1} D_0 \]
(20)

On the other hand, the matrix \( A_c \) corresponding to the modified generalized plant \( P' \), which we denote by \( A'_c \), can be rearranged as

\[ A'_c = A' - B'(I + \mathcal{D}')^{-1} C' = A - B(I + \mathcal{D})^{-1} C = A_c \]
(21)

This implies that the closed-loop sampled-data system \( \Sigma_c \) is internally stable if and only if the modified closed-loop sampled-data system \( \Sigma'_c \) is internally stable, where \( \Sigma'_c \) is defined as \( \Sigma_c \) with the generalized plant \( P \) replaced by the modified generalized plant \( P' \). Hence, in what follows, we provide arguments that relate internal stability of \( \Sigma'_c \) (and thus \( \Sigma_c \)) with the modified transfer operator \( \tilde{G}'(z) \) given by

\[ \tilde{G}'(z) = C'(zI - A')^{-1} B' + \mathcal{D}' = C'(zI - A)^{-1} B + \mathcal{D}' \]
(22)

associated with the modified open-loop sampled-data system \( \Sigma'_c \). An advantage of the discussions in this direction is that the operator \( \mathcal{D}' \) given in (20) belongs to the class \( \mathcal{C}_2 \) of Hilbert-Schmidt operators since \( \mathcal{D}_\gamma \) does [4], and thus so does the modified transfer operator \( \tilde{G}'(z) \) for any \( z \) that is not an eigenvalue of \( \mathcal{A}' = \mathcal{A} \), and hence we can apply the 2-regularized determinant even when the original \( \mathcal{D} \) does not belong to \( \mathcal{C}_1 \) nor is it a compact operator. Indeed, let us assume that \( z \notin \lambda(\mathcal{A}) \) and compute \( \det_2(I + \tilde{G}'(z)) \) using the properties reviewed in Subsection 2.2. Then, we can show that

\[ \det_2(I + \tilde{G}'(z)) \cdot \exp(\eta(z)) = \frac{\det(zI - A_c)}{\det(zI - A)} \]
(23)

where

\[ \eta(z) = \text{tr}(B C'(zI - A)^{-1}) \]
(24)

The details of this deduction are given in the Appendix.

In the following, we denote by \( \partial \mathcal{D} \) the unit circle, (i.e., the boundary of the annular region \( \mathcal{D} := \{ z : |z| > 1 \} \) including \( z = \infty \), and introduce the following assumptions.

**A1** The matrix \( \mathcal{A} \) has no eigenvalue on \( \partial \mathcal{D} \). That is, \( \tilde{G}'(z) \) has no pole on \( \partial \mathcal{D} \).

**A2** \( \det_2(I + \tilde{G}'(z)) \neq 0 \) \( \forall z \in \partial \mathcal{D} \)

Under these assumptions, the unit circle \( \partial \mathcal{D} \) is called the Nyquist contour, and the locus drawn by \( \det_2(I + \tilde{G}'(z)) \cdot \exp(\eta(z)) \) as \( z \) goes anticlockwise along the Nyquist contour is called the Nyquist locus of the open-loop sampled-data system \( \Sigma_0 \). We are in a position to state the following Nyquist stability criterion of sampled-data systems with full generality (as opposed to our previous result in [11] stated only under the restrictive assumption that \( D_{11} = 0 \) and \( C_1 B_1 = 0 \)).

**Theorem 1** Under the assumptions \( \textbf{A0}, \textbf{A1} \) and \( \textbf{A2} \), let \( \mu_0^* \) denote the number (counted according to the algebraic multiplicities) of the eigenvalues of the matrix \( \mathcal{A} \) in \( \mathcal{D} \), and let \( \nu \) denote the number of the anticlockwise encirclements of the Nyquist locus of \( \det_2(I + \tilde{G}'(z)) \cdot \exp(\eta(z)) \) around the origin. Then, the closed-loop sampled-data system \( \Sigma_c \) is well-posed and internally stable if and only if \( \nu = \mu_0^* \).

**Proof.** \( \Sigma_c \) is well-posed if and only if the system \( P \) with \( w \) set to \( -z \) is well-posed, if and only if the assumption \( \textbf{A0} \) is satisfied. Hence, it is enough to prove that under the assumptions \( \textbf{A0} - \textbf{A2} \), \( \Sigma_c \) is internally stable if and only if only \( \nu = \mu_0^* \). However, under \( \textbf{A0} \), it follows readily from (23) that \( \det_2(I + \tilde{G}'(z)) \cdot \exp(\eta(z)) \) is meromorphic in \( z \), and that the matrix \( A_c \) has \( \mu_0^* - \nu \) eigenvalues on \( \mathcal{D} \) by the argument principle, provided that \( \textbf{A1} \) and \( \textbf{A2} \) are satisfied. Since \( \Sigma_c \) is internally stable if and only if \( A_c \) is a stable matrix (i.e., it has no eigenvalues on \( \mathcal{D} \)), we are led to the stability condition \( \nu = \mu_0^* \). This completes the
Similarly, \( \| \cdot \| \) denotes the induced norm on \( \mathcal{K} \). We can also consider the lifted description of the continuous-time system \( \Sigma \) in completely the same manner as in \( \Sigma_0 \); the corresponding transfer operator is denoted by \( \tilde{\Sigma}(z) \). If \( \tilde{\Sigma} \) is internally stable, then the \( H_{\infty} \) norm \( \| \tilde{\Sigma}(z) \|_{\infty} \) can also be introduced in exactly the same way as in \( \| \tilde{\Sigma}(z) \|_{\infty} \); it is well known that \( \| \tilde{\Sigma}(z) \|_{\infty} \) coincides with the \( H_{\infty} \) norm of the usual (continuous-time) transfer matrix \( \Sigma(s) \) associated with \( \tilde{\Sigma} \), as well as the \( L_2 \)-induced norm of \( \tilde{\Sigma} \).

Similarly, \( \| \tilde{\Sigma}(z) \|_{\infty} \) coincides with the \( L_2 \)-induced norm of \( \tilde{\Sigma} \).

Remark 1 When the assumption \( \text{A1} \) is not satisfied (i.e., \( \bar{A} \) has an eigenvalue at some \( z \in \partial D \)), then we can simply detour around such \( z \) with an infinitesimally small semicircle to form a modified Nyquist contour. On the other hand, \( \text{A2} \) is nothing but the assumption that the Nyquist locus does not go through the origin, and is in fact a necessary condition for stability of \( \Sigma_c \).

Remark 2 It would be worth mentioning that \( \text{A0} \) and \( \text{A2} \), which are necessary conditions for well-posedness and stability of \( \Sigma_c \), can be combined into the condition that \( \sigma(\tilde{\Sigma}(z)) \not\subset -1 \) (\( \forall z \in \partial D \)).

4 Applications to Robust Stability Analysis of Sampled-Data Systems

In this section, we demonstrate that the Nyquist stability criterion derived in full generality in the preceding section is quite useful when it is applied to the study of robust stability of sampled-data systems with uncertainty \( \Delta \) shown in Fig. 3, which we denote by \( \Sigma_\Delta \). In this paper, the uncertainty \( \Delta \), which is a continuous-time system, is assumed to be finite-dimensional and linear time-invariant for simplicity (even though it is not hard to generalize the class to cover finite-dimensional linear \( h \)-periodically time-varying systems without essential difficulties). In this case, by replacing \( P \) with

\[
P_\Delta := P \text{ diag}[\Delta, I]
\]

in Fig. 2, the sampled-data system \( \Sigma_\Delta \) can be identified with \( \Sigma_c \), so that the Nyquist stability criterion applies to \( \Sigma_\Delta \) mutatis mutandis, as long as Fig. 3 makes sense; in general, \( \text{dim}(w) \neq \text{dim}(z) \) is not assumed in this section.

To facilitate the descriptions to follow, we introduce the \( H_{\infty} \) norm of \( \Sigma_\Delta \) (or, to be more precise, \( \tilde{\Sigma}(z) \)), denoted by \( \| \tilde{\Sigma}(z) \|_{\infty} \), which is given by

\[
\| \tilde{\Sigma}(z) \|_{\infty} = \max_{z \in \partial D} \| \tilde{\Sigma}(z) \|
\]

(26)

where \( \| \cdot \| \) denotes the induced norm on \( \mathcal{K} \). We can also consider the lifted description of the continuous-time system \( \Delta \) in completely the same manner as in \( \Sigma_0 \); the corresponding transfer operator is denoted by \( \tilde{\Delta}(z) \). If \( \tilde{\Delta} \) is internally stable, then the \( H_{\infty} \) norm \( \| \tilde{\Delta}(z) \|_{\infty} \) can also be introduced in exactly the same way as in \( \| \tilde{\Sigma}(z) \|_{\infty} \); it is well known that \( \| \tilde{\Delta}(z) \|_{\infty} \) coincides with the \( H_{\infty} \) norm of the usual (continuous-time) transfer matrix \( \Delta(s) \) associated with \( \tilde{\Delta} \), as well as the \( L_2 \)-induced norm of \( \tilde{\Delta} \). Similarly, \( \| \tilde{\Sigma}(z) \|_{\infty} \) coincides with the \( L_2 \)-induced norm of \( \tilde{\Sigma} \).

Next, we give the following result, which corresponds to the passivity theorem but again in the context of internal stability rather than \( L_2 \)-stability.

Theorem 2 Suppose that the open-loop sampled-data system \( \Sigma_0 \) is internally stable, and that \( \Delta \) is finite-dimensional linear time-invariant and internally stable. If

\[
\| \tilde{\Sigma}(z) \|_{\infty} < \gamma, \quad \| \tilde{\Delta}(z) \|_{\infty} \leq 1/\gamma
\]

(27)

for some \( \gamma > 0 \), then \( \Sigma_\Delta \) is well-posed and internally stable.

Remark 3 The transfer operator \( \tilde{\Sigma}(z) \) is said to be strongly positive-real if it is stable and (28) holds, while \( \tilde{\Delta}(z) \) is said to be positive-real if it is stable and (29) holds [21],[15]. We can also show that \( \Sigma_\Delta \) is well-posed and internally stable if \( \tilde{\Sigma}(z) \) is positive-real and \( \tilde{\Delta}(z) \) is strongly positive-real, or more explicitly, if \( \Sigma_0 \) and \( \Delta \) are both internally stable, if

\[
\tilde{\Sigma}(z) + \tilde{\Sigma}(z)^\ast \geq \varepsilon I \quad (\forall z \in \partial D)
\]

(28)

and if

\[
\tilde{\Delta}(z) + \tilde{\Delta}(z)^\ast \geq 0 \quad (\forall z \in \partial D)
\]

(29)

then \( \Sigma_\Delta \) is well-posed and internally stable.

Proposition 1 Suppose that \( \Sigma_0 \) is internally stable. Then, \( \Sigma_0 \) is passive if and only if (30) holds. Similarly, \( \Sigma_0 \) is strictly passive if and only if there exists \( \varepsilon > 0 \) such that (28) holds.

![Fig. 3: Uncertain closed-loop sampled-data system \( \Sigma_\Delta \).](image-url)
We can derive a more general result such that the above two theorems can be regarded as its special cases. To state such a result, we first introduce some notations.

**Definition 1** The set of measurable, essentially bounded, symmetric matrix functions \( \Phi : [0, h] \to \mathbb{R}^{m \times m} \) is denoted by \( \Phi^{m \times m} \). The set of operators of multiplication by \( \Phi \in \Phi^{m \times m} \), i.e., \( \Lambda : f(\cdot) \in \mathcal{K} \to \Phi(\cdot)f(\cdot) \in \mathcal{K} \), is denoted by \( A^{m \times m} \). The set of (linear bounded) operators given by the sum of \( \Lambda \in A^{m \times m} \) and a linear self-adjoint compact operator is denoted by \( \Theta^{m \times m} \). When \( m \) is understood from the context, we simply denote these sets by \( \Phi, \Lambda \) and \( \Theta \), respectively.

Now we are in a position to state the following result.

**Theorem 4** Suppose that the open-loop sampled-data system \( \Sigma_0 \) is internally stable, and that \( \Delta \) is a set such that (i) every \( \Delta \in \Delta \) is finite-dimensional linear time-invariant and internally stable, and (ii) \( \delta \Delta \in \Delta \) whenever \( \Delta \in \Delta \) and \( 0 < \delta \leq 1 \). Then, \( \Sigma_\Delta \) is well-posed and internally stable for every \( \Delta \in \Delta \) if and only if there exists \( \Theta \in \Theta \) possibly dependent on \( z \in \partial \mathcal{D} \) and \( \varepsilon > 0 \) possibly dependent on \( \Delta \) such that

\[
\begin{bmatrix}
    I & \hat{G}(z)^* \\
    -\hat{\Delta}(z)^* & I
\end{bmatrix}
\begin{bmatrix}
    \hat{G}(z) \\
    \hat{\Delta}(z)
\end{bmatrix} \geq \varepsilon I, \forall \Delta \in \Delta, \forall z \in \partial \mathcal{D}
\]

**Proof.** Note that \( \tilde{G}_\Delta(z) := \hat{G}(z)\hat{\Delta}(z) \) is nothing but the transfer operator of the open-loop sampled-data stem \( \Sigma_{0\Delta} \), which is given by replacing \( P \) in \( \Sigma_0 \) by \( P_\Delta \) given by (25), and that \( \Sigma_{0\Delta} \) is nothing but the closed-loop sampled-data system \( \Sigma_0 \) corresponding to the open-loop sampled-data system \( \Sigma_0 \) mentioned just above. Thus, we can apply Theorem 1 by considering \( \tilde{G}_\Delta(z) \).

Sufficiency: Since a linear bounded self-adjoint operator is nonnegative definite if and only if its spectrum lies on the nonnegative real axis (Theorem X.4.2 of [7]), and since the essential spectrum is a subset of the spectrum, it follows from (33) that the essential spectrum of

\[
T_X := \left[ -D_\Delta \right] A \left[ -D_\Delta \right] - \varepsilon I
\]

(34)

consists of only nonnegative numbers for any \( \Delta \in \Delta \), where \( D_\Delta \) denotes the direct feedthrough matrix of \( \Delta \), and \( A \) is the (uniquely determined) portion of the operator of multiplication in \( \Theta \) (recall Definition 1). However, since \( T_X \) is an operator of multiplication by the matrix

\[
X := \left[ -D_\Delta \right] A \left[ -D_\Delta \right] - \varepsilon I
\]

and thus is isometrically isomorphic to the Laurent operator defined by the matrix (function) \( X \), it follows from Corollary XXIII.2.5 of [8] that the spectrum of \( T_X \) coincides with its essential spectrum, which consists of only nonnegative numbers, as mentioned above. Hence, again from Theorem X.4.2 of [7], it follows that \( T_X \) is nonnegative definite. This in particular implies that \( X \geq 0 \), and hence

\[
[ -D_\Delta \quad I ] A \left[ -D_\Delta \quad I \right] > 0, \forall \Delta \in \Delta
\]

(36)

Similarly, it follows from (32) that for each direct feedthrough matrix \( D_\Delta \) of \( \Delta \in \Delta \),

\[
[ I \quad D_{1 \Delta}^T ] A \left[ I \quad D_{1 \Delta} \right] \leq 0
\]

(37)

Now, let us show that (36) and (37) lead to the well-posedness of \( \Sigma_\Delta \) for all \( \Delta \in \Delta \). To see this, suppose the contrary and take \( x \neq 0 \) such that \( (I + D_{1 \Delta}D_\Delta)x = 0 \) for some \( \Delta \in \Delta \). Taking the quadratic forms with (36) and \( x \), as well as (37) and \( D_\Delta x \), and then comparing these results, we are led immediately to contradiction because we have two values that coincide with each other but are positive and nonnegative at the same time. Hence, \( \Sigma_\Delta \) is well-posed for all \( \Delta \in \Delta \).

Applying similar arguments to (32) and (33) leads immediately to that \( I + \tilde{G}_\Delta(z) = I + \bar{G}(z)\bar{\Delta}(z) \) does not have an eigenvalue at \( 0 \) whenever \( \Delta \in \Delta \) and \( z \in \partial \mathcal{D} \). By using the relation \( I + \tilde{G}_\Delta(z) = (I + D_{1 \Delta}D_\Delta)(I + \bar{G}_\Delta(z)) \) (together with the above-established well-posedness), it follows readily that \( \tilde{G}_\Delta(z) \) does not have an eigenvalue at \(-1\) whenever \( \Delta \in \Delta \) and \( z \in \partial \mathcal{D} \), where \( \tilde{G}_\Delta(z) \) is defined in the same manner as the way \( G^*(z) \) is constructed from \( \tilde{G}(z) \). By the definition of the 2-regularized determinant, this implies that \( \det_2(I + \tilde{G}_\Delta(z)) \neq 0 \) and thus the Nyquist locus of \( \det_2(I + \tilde{G}_\Delta(z)) \exp(\eta_\Delta(z)) \), with \( \eta_\Delta(z) \) defined appropriately in an obvious fashion, never goes through the origin whenever \( \Delta \in \Delta \).

On the other hand, for \( \Delta = 0 \), it follows that \( \tilde{G}_\Delta(z) = 0 \) and \( \eta_\Delta(z) = 0 \) so that \( \det_2(I + \tilde{G}_\Delta(z)) \exp(\eta_\Delta(z)) = 1 \) regardless of \( z \in \partial \mathcal{D} \) (and thus the Nyquist locus does not encircle the origin when \( \Delta = 0 \)). The assumption (ii) of the theorem on \( \Delta \) allows us to employ continuity arguments and thus this consequence together with the conclusion in the above paragraph leads to the claim that the Nyquist locus of \( \Sigma_{0\Delta} \) never encircles the origin whenever \( \Delta \in \Delta \). Since \( \Sigma_{0\Delta} \) is internally stable by the internal stability assumptions of \( \Sigma_0 \) as well as \( \Delta \), the assertion of this theorem follows readily from Theorem 1.

Necessity: Suppose that \( \Sigma_\Delta \) is well-posed and internally stable for all \( \Delta \in \Delta \). Then, \( \sigma(I + \tilde{G}_\Delta(z)) \neq 0 \), and thus \( \sigma(I + \tilde{G}_\Delta(z)) \neq 0 \), whenever \( \Delta \in \Delta \), \( z \in \partial \mathcal{D} \) (recall Remark 2). Hence, \( \sigma(\bar{G}_\Delta(z)) \bar{\Delta}(z) \) \( \neq 0 \), \( \forall \Delta \in \Delta, \forall z \in \partial \mathcal{D} \). Noting that (i) a linear bounded self-adjoint operator is nonnegative definite if and only if its spectrum lies on the nonnegative real axis [7], (ii) the spectrum of a linear bounded operator is compact, (iii) the spectrum of a linear self-adjoint operator is continuous under self-adjoint perturbations [16, p. 243], and (iv) \( \partial \mathcal{D} \) is a compact set, it follows that there exists \( \varepsilon > 0 \) dependent only on \( \Delta \) such that \( (I + \tilde{G}_\Delta(z)^*)(I + \tilde{G}_\Delta(z)) \geq \varepsilon I \), \( \forall z \in \partial \mathcal{D} \). Hence, taking

\[
\Theta(z) = \left[ \begin{bmatrix} -\tilde{G}(z)^* & I \end{bmatrix} \right] \left[ \begin{bmatrix} -\tilde{G}(z) & I \end{bmatrix} \right]
\]

(38)

leads to (32) and (33).

Q.E.D.
Remark 4 The above theorem corresponds, to some extent, to the result in the continuous-time case in [18]. Our arguments are particularly significant in having given a class of $\Theta$ that can be used in the sampled-data setting. That is, the class $\Theta$, or more essentially, the class $\Lambda$ (which corresponds to the class of Laurent operators) plays a key role especially in establishing the wellposedness of the closed-loop systems, as seen from the above proof. Also, it is easy to see that the sufficiency assertion of Theorem 4 remains true if $\varepsilon \leq 0$ in (32) and $\varepsilon \geq \varepsilon_0$ in (33) are replaced simultaneously by $\varepsilon \leq -\varepsilon_1$ and $\varepsilon \geq 0$, respectively. In that case, we can see as in [18] that taking $\Theta = \Theta_{SG}$ leads to Theorem 2, while taking $\Theta = \Theta_{PR}$ leads to Theorem 3, where

$$\Theta_{SG} := \left[ \begin{array}{cc} -\gamma^2 I & 0 \\ 0 & I \end{array} \right], \quad \Theta_{PR} := \left[ \begin{array}{cc} 0 & -I \\ -I & 0 \end{array} \right]$$ (39)

On the other hand, taking $\Theta = \Theta_{PR}$ in Theorem 4 leads to the result mentioned in Remark 3; the statement of Theorem 4 that allows $\varepsilon$ to be dependent on $\Delta \in \Delta$ is important in this situation so that the class of all strictly passive systems can be dealt with. Taking $\Theta = \Theta_{SG}$ in Theorem 4, on the other hand, leads to a slight variant of Theorem 2. As for the necessity assertion of Theorem 4, however, the inequality cannot be modified in the way suggested above for the sufficiency assertion, in general, unless $\Theta$ is allowed to be dependent on $\Delta$ as well as $\varepsilon$. This is because $\epsilon$ such that $(I + \hat{G}_\Delta(z))((I + \hat{G}_\Delta(z)) \geq \varepsilon I \ (\forall z \in \partial D)$ is generally dependent on $\Delta$, and hence taking

$$\Theta = \left[ \begin{array}{cc} -\hat{G}(z)^* \\ I \end{array} \right] \left[ \begin{array}{cc} -\hat{G}(z) & I \\ \varepsilon' I & 0 \end{array} \right]$$ (40)

with such sufficiently small $\varepsilon' > 0$ that satisfies (33) makes $\varepsilon$ and thus $\Theta$ dependent on $\Delta$.

Remark 5 In Theorem 4, $\varepsilon$ satisfying (33) can be taken independent of $\Delta$ if $\Delta$ is a compact set (for example, if $\Delta$ is a bounded and closed set of constant matrices). This can be seen easily by slightly modifying the necessity proof.

Remark 6 One might expect that the assumption (ii) on $\Delta$ could be relaxed to the assumption: $\Delta$ is such a set that for every $\Delta \in \Delta$, there exists a continuous path in the $L_2$-induced norm (or, when viewed in the domain of $\hat{G}_\Delta(z)$, in the norm $\| \cdot \|$ on the space of linear bounded operators on $\mathcal{K}$) contained in $\Delta \cup \{0\}$ that connects $\Delta$ and 0. To complete the proof for such a case in entire rigor, however, seems to be unexpectedly involved because one must convert the continuity with respect to $\Delta$ in terms of the above-mentioned norm to that about $\hat{G}_\Delta(z)$ in terms of the norm $\| \cdot \|_2$ on the space of Hilbert-Schmidt operators on $\mathcal{K}$ as well as that about the function $\eta(z)$, which is related to the norm $\| \cdot \|$ on the space of trace class operators on $\mathcal{K}$. However, it is not hard to show that the condition can be indeed relaxed to the condition that $\Delta \cup \{0\}$ is a connected set if $\Delta$ is a set of constant matrices.

5 Conclusion

In this paper, we used the 2-regularized determinant associated with Hilbert-Schmidt operators and derived a new Nyquist stability criterion of sampled-data systems with full generality, which gives a necessary and sufficient condition for internal stability of closed-loop sampled-data systems. Next, the usefulness of this criterion was demonstrated by applying it to the derivation of robust internal stability theorems of sampled-data systems. Even though the continuous-time generalized plant was assumed to be finite-dimensional linear time-invariant in this paper, it is straightforward to generalize the class to accommodate finite-dimensional linear $h$-periodic generalized plants with a measurable essentially bounded direct feedthrough matrix $D_{11}(t)$ (and the internally stable uncertainties $\Delta$ in the same class) as in [20], [14], where $h$ is the underlying sampling period. This is because the lifted description of the generalized plant can readily be extended to the $h$-periodic case mutatis mutandis [2], while the essential spectrum of the transfer operator $\hat{G}(z)$ in such a case only slightly changes from $\lambda(D_{11})$ into

$$\lambda_{[0,h]}(D_{11}) = \{ \lambda : \text{the set of } t \in [0, h] \text{ such that } |\det(\lambda I - D_{11}(t))| < \gamma \text{ has nonzero measure, } \forall \gamma > 0 \}$$ (41)

which follows from [9, Section XXIII.2] (see also [12]); it is not hard to see that with these slight modifications, the arguments of this paper apply even to the $h$-periodic case in a straightforward manner.

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References

This appendix is devoted to deriving (23). Taking account of (20) and (21), we have
\[
\det_2 (I + \hat{G}'(z)) = \det_2 \left( (I + D')(I+(I+D')^{-1}C(zI-A)^{-1}B) \right) \\
= \det_2 \left( (I + (I+D')^{-1}C(zI-A)^{-1}B) \exp(-\eta_1(z)) \right) \\
= \det_2 \left( (I + B(I+D')^{-1}C(zI-A)^{-1}) \exp(-\eta_1(z)) \right) \\
= \det_2 \left( (zI - \Lambda_c)(zI-A)^{-1} \exp(-\eta_1(z)) \right) \\
= \det_2 \left( (I - \Lambda_c/z)(I-A/z)^{-1} \right) \exp(-\eta_1(z)) \\
\] (A1)

where
\[
\eta_1(z) := \text{tr} \left( (D'(I+D')^{-1}C'(zI-A)^{-1}B) \right) \\
= \text{tr} \left( (BD'(I+D')^{-1}C'/z)(I-A/z)^{-1} \right) \\
\] (A2)

In particular, we used the fact that \( \det_2 (I+D') = 1 \), which follows since \( D' \) is a Volterra operator so that it has no (nonzero) eigenvalues. Here, since
\[
(I - A/z)^{-1} = I + (A/z)(I-A/z)^{-1} \\
\] (A3)

(48) and thus \( \det_2 ((I-A/z)^{-1}) \) is well-defined, we have
\[
\det_2 \left( (I-A/z)(I-A/z)^{-1} \right) = \det_2 (I-A/z) \exp(-\eta_2(z)) \\
\] (A4)

where
\[
\eta_2(z) := -\text{tr} \left( (A/z)^2(I-A/z)^{-1} \right) \\
\] (A5)

Obviously, the left hand side of (A4) is 1, so that we have
\[
\det_2 \left( (I-A/z)^{-1} \right) = \exp(\eta_2(z)) / \det_2 (I-A/z) \\
\] (A6)

Also, it follows that
\[
\det_2 \left( (I - \Lambda_c/z)(I-A/z)^{-1} \right) = \det_2 (I - \Lambda_c/z) \det_2 \left( (I-A/z)^{-1} \right) \exp(-\eta_3(z)) \\
\] (A7)

where
\[
\eta_3(z) := -\text{tr} \left( (\Lambda_c/z)(A/z)(I-A/z)^{-1} \right) \\
\] (A8)

Combining (A1), (A6) and (A7) and recalling (14), we obtain
\[
\det_2 (I + \hat{G}'(z)) = \frac{\det_2 (I - \Lambda_c/z)}{\det_2 (I - A/z)} \exp(-\eta'(z)) \\
= \frac{\det_2 (A_c - \Lambda_c/z)}{\det_2 (I - A/z)} \exp(-\eta'(z)) \exp(\eta_4(z)) \\
\] (A9)

where
\[
\eta'(z) = \eta_1(z) - \eta_2(z) + \eta_3(z), \quad \eta_4(z) = \text{tr} ((A_c - A)/A)(A10)
\]

Note in (A9) that \( \det(\cdot) \) is just the standard determinant for finite-dimensional matrices. Since
\[
\eta(z) := \eta'(z) - \eta_4(z) \\
= \eta_1(z) - ((\eta_2(z) - \eta_3(z)) + \eta_4(z)) \\
= \text{tr} \left( (BD'(I+D')^{-1}C' - (A_c - A)(I-A/z)^{-1}/z \right) \\
= \text{tr} \left( (BD'(I+D')^{-1}C' + B(I+D')^{-1}C')(I-A/z)^{-1}/z \right) \\
\] (A11)

by (20) and (21), we are readily led to (23).