Subtotally positive and Monge matrices

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We start with a class of matrices which is, in a sense, intermediate between the classes of positive and totally positive matrices.

We say that a real matrix is \( k \)-subtotally positive if the determinants of all its submatrices of order at most \( k \) are positive. Clearly, if the matrix is \( m \times n \), then for \( k = 1 \), such matrix is positive, if \( k = \min(m, n) \), the matrix is totally positive.

Analogously to the theory of totally positive matrices, the following result shows that for an \( m \times n \) matrix, only \( mn \) inequalities suffice for distinguishing whether a real matrix is a \( k \)-subtotally positive matrix, if \( k \) is fixed, and, what may be interesting, independently of \( k, 1 \leq k \leq \min(m, n) \).

**Theorem 1.** Let \( A \) be a real \( m \times n \) matrix, let \( k \) be an integer, \( 1 \leq k \leq \min(m, n) \). Then, \( A \) is \( k \)-subtotally positive if and only if for all \( j, 1 \leq j < k \), all \( j \times j \) submatrices whose rows as well as columns are consecutive and either the first row, or the first column (or, both) are in the first row or in the first column of \( A \), have positive determinant, and, in addition, all \( k \times k \) submatrices of \( A \) with consecutive rows and consecutive columns have positive determinant.

**Remark.** Observe that to every entry of \( A \) one can assign exactly one of the mentioned submatrices.

**Theorem 2.** The product of \( k \)-subtotally positive matrices (which can be multiplied) is also a \( k \)-subtotally positive matrix.

**Theorem 3.** Every square \( k \)-subtotally positive matrix has the property that its \( k \) eigenvalues of maximum moduli are simple and positive.

We proceed now to the class of Monge matrices (cf. references in [1]).

An \( m \times n \) matrix \( C = (c_{ik}) \) is called a Monge matrix if it satisfies

\[
c_{ik} + c_{ji} \leq c_{il} + c_{jk} \quad \text{for all} \quad i, j, k, l, \quad i < j, \quad k < l.
\]

In our considerations, it will be simpler to consider the class of matrices called anti-Monge. This class is defined by inequalities analogous to (1) but with opposite signs of inequalities.

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Let us note that there is a simple connection between strict Monge or anti-Monge matrices and the 2-subtotally positive matrices, which in fact led to some of the results of the author in [1]. By strict we mean matrices for which there are always strict inequalities in (1) or the corresponding opposite inequalities.

**Theorem 4.** Let $A$ be a real $m \times n$ matrix, let $J$ be the $m \times n$ matrix of all ones. Then the following are equivalent:

1. $J + \varepsilon A$ is a 2-subtotally positive matrix for all sufficiently small positive numbers $\varepsilon$.
2. $A$ is a strict anti-Monge matrix.

We also called a matrix equilibrated if all its row-sums and all its column sums are equal to zero.

This approach mentioned in Theorem 4 allowed us to prove some spectral properties of equilibrated anti-Monge matrices and the following

**Theorem 5.** If $C_1$ and $C_2$ are equilibrated anti-Monge matrices which can be multiplied, then the product $C_1C_2$ is again an equilibrated anti-Monge matrix.

Let us turn now to further problems on Monge or anti-Monge matrices. We call a matrix incomplete if some of the entries of the matrix are not specified. The set of specified entries forms then a specified pattern, the set of unspecified entries the complementary unspecified pattern. These patterns can easily be described by a bipartite undirected graph which, if the matrix is $m \times n$, has distinguished sets of vertices with cardinality $m$ and $n$. (As usual, edges join vertices from different distinguished sets.)

One can now easily see that the bipartite graph corresponding to an anti-Monge matrix has the property that on every circuit in the corresponding graph (which has, of course, an even number of edges) the sum of the entries corresponding to odd edges is greater than or equal than the sum of entries corresponding to even edges; the first edge has to be taken as that on top of the graph (or, as entry with the smallest sum of indices). We call such circuit positive. Thus: every circuit in an anti-Monge matrix is positive.

**Theorem 6.** Let $A$ be a real incomplete matrix. If every circuit in the pattern of specified entries is positive, then $A$ can be completed to an anti-Monge matrix.

**References**