Least-Squares Solution of Toeplitz Systems and Its Applications

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Let \( T = [a_{j-k}]_{j=0}^{m-1} \) be an \( m \times n \) Toeplitz matrix with complex entries and \( m \geq n \). We consider the following least squares problem: given \( b \in \mathbb{C}^m \), determine \( x \in \mathbb{C}^n \) such that \( \|Tx - b\| \) is minimal. Here \( \| \cdot \| \) denotes the 2-norm. The solution is given by \( x = T^\dagger b \), where \( T^\dagger = (T^HT)^{-1}T^H \) is the Moore-Penrose pseudo inverse of \( T \). Here the superscript \( H \) denotes the conjugate transpose matrix.

1 Formulas for the Moore-Penrose Inverse

In the case \( m = n \) the least squares problem reduces to the problem of solving \( Tx = b \) and Moore-Penrose inversion turns into inversion in the usual sense. There are formulas for the inverse of a nonsingular Toeplitz matrix that turned out to be useful for the solution of Toeplitz systems. The Gohberg-Semencul formula [4] is the "classical" and most famous representative of such inversion formulas. This formula represents the inverse as a two-term product sum of triangular Toeplitz matrices.

There are formulas which are more efficient from a computational viewpoint in which the inverse matrix is represented as

\[
T^{-1} = \sum_{k=1}^{2} C_k^+ C_k^-
\]

where \( C_k^+ \) are circulant and \( C_k^- \) are skew-circulants. Such formulas were presented in [1] and later discussed in [3], [6] and other papers. These formulas can be written in a form that contains only diagonal matrices and discrete Fourier or other trigonometric transformations, which implies that the solution can be found with complexity \( \mathcal{O}(n \log n) \), provided that the \( \mathcal{O}(n) \) parameters in the formula are known [10].

In view of the importance of the Toeplitz inversion formulas it is reasonable to ask for such representations of the Moore-Penrose inverse. It turns out that \( T^\dagger \) can be represented
as
\[ T^\dagger = [I_n \ 0] \sum_{k=1}^{4} C_k^+ C_k^-, \]  \hspace{1cm} (2)
where \( C_k^+ \) are circulant and \( C_k^- \) skew-circulant matrices.

A representation (2) also exists for the matrix \((T^HT + \varepsilon I_n)^{-1} T^H\) which provides the solution of a regularized least squares problem to minimize \( \|Tx - b\|^2 + \varepsilon \|x\|^2 \), where \( \varepsilon \geq 0 \) is a regularization parameter.

The existence of a representation (for \( \varepsilon = 0 \)) was, in principle, already shown in [8] and, in a different way, in [9] (including the rank deficient case). In the full rank case it is convenient to derive the representation by means of the augmented matrix approach. This gives more inside into the structure of \( C_k^\pm \). The augmented matrix approach (see [2]) relates \( T \) with the matrix
\[
\text{aug}_\varepsilon(T) = \begin{bmatrix} I_m & T \\ T^H & -\varepsilon I_n \end{bmatrix}.
\]
The proof of (2) relies on the displacement structure of \( \text{aug}_\varepsilon(T) \).

Via diagonalization of the matrices \( C_k^\pm \) the representation (2) can be brought into a form that contains only diagonal matrices and DFTs.

Note that the representation (2) is not unique. The data in the formula can be characterized by a fundamental system which is a basis of a 4-dimensional space. Different choices of this basis will give different representations in which \( C_k^\pm \) will have different magnitudes. For the sake of numerical stability it is desirable to have these magnitudes as small as possible. In the standard representations the norms of \( C_k^\pm \) will be of order \( 1/\sigma^2 \), where \( \sigma \) is the smallest singular value of \( T \), which could cause problems if \( \sigma \) is tiny. However, it can be achieved to choose the fundamental system in such a way that the norms of \( C_k^\pm \) are only of order \( 1/\sigma \), which is the best possible.

The data in the Moore-Penrose inversion formula (2) can be obtained via solving some tangential interpolation problems using, for example, the method in [13] (see also [12]).

## 2 Recursive Algorithms

There are two types of direct fast algorithms for solving a Toeplitz systems \( T x = b \): Levinson-type and Schur-type. These algorithms also compute the data in the inversion formula (1).

In [7] it is shown that these algorithms can be generalized to the solution of least-squares Toeplitz problems. The algorithms in [7] are based on updating formulas for the fundamental system after a modification of the matrix. These modifications include adding a row, adding a column or both. Note that there are some other modifications which are important for applications (for example the applications in Sections 3 and 4).

The recursions for the fundamental systems are of the form
\[ U_{k+1}(t) = U_k(t) D(t) \Theta_k, \]  \hspace{1cm} (3)
Here \( U_k(t) \) is a \( 2 \times 4 \) matrix polynomial formed by the fundamental system, \( D(t) \) is a \( 4 \times 4 \) diagonal matrix with diagonal elements \( t_i^j \) \((i = 0, 1)\) and \( \Theta_k \) is a constant \( 4 \times 4 \) matrix. The matrix \( \Theta_k \) corresponds to the hyperbolic rotation that appears in the classical Levinson algorithm. The Schur algorithm computes the corresponding systems of residuals.
3 Approximate Polynomial GCD Computation

Computation of the approximate GCD of two polynomials is important both theoretically and in applications like control and computer aided design. For this reason it is the subject of many investigations. For an account of the state of the art we refer to [11] and references therein. There are two main approaches for finding an approximate GCD of two polynomials. The first is singular value decomposition, which is reliable but expensive, and the second is Padé approximation which is cheaper but less reliable.

As a third way we propose a least squares approach, which could be a reasonable compromise. Let us briefly explain it. The GCD of the complex polynomial $a(t)$ and $b(t)$ can be found if one knows the solution $(u(t), v(t))$ of the homogeneous Bezout equation

$$a(t)u(t) - b(t)v(t) = 0$$

with smallest possible degree. We replace this by the problem to minimize

$$\|a(t)u_k(t) - b(t)v_k(t)\|$$

for polynomials $u_k(t), v_k(t)$ of fixed degree. If the residual is smaller than a threshold, then with these polynomials an approximate GCD is computed.

We consider 4 normalizations of the minimization problem (4). The corresponding 4 solutions will be collected to a fundamental system. The fundamental system satisfies a recursion of the form (3). This will provide a recursive procedure to find an approximate GCD.

4 Systems Identification

By systems identification we mean to recover the transfer function of a linear time-invariant system from its Markov parameters. This problem is closely related to that of model reduction. Like for GCD computation, there are two main approaches: singular value decomposition and Padé approximation. The first one is expensive and the second has the disadvantage that the obtained approximation might not be stable. It was recently shown in [5] that a least squares approach will provide stable solutions.

The least squares approach reduces the identification problem to a least squares Toeplitz (actually Hankel) problem that can be solved using some modifications of the algorithm derived in [7].

References


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