Existence and Computation of Second Order Reduced Systems using Krylov Subspace Methods

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The semi-discretization of partial differential equations in electrical networks, mechanical systems, aeronautics, civil engineering and micro-electro-mechanical systems leads to a large number of second (or higher) order differential equations. A smaller dimensional model which approximates the behavior of the original system should then preserve its second-order structure, to allow a meaningful physical interpretation of the coefficient matrices and to be compatible with standard software tools which process second order systems.

In [2], the method of balancing and truncation is modified for second order systems. The relatively high computational complexity of balanced truncation method does not in general allow its use for very large systems.

In [5], the original system is reduced by applying a projection to the original second order system, based on Krylov subspace methods. The method in [5] approximates the behavior of the original system by matching at most \( q \) moments of its transfer function, where \( q \) is the dimension of the reduced state space model.

In [3] second order Krylov subspaces are introduced and it is shown that the number of moments that can be matched increases to \( q \).

Often the second order system given as

\[
M \ddot{z}(t) + D \dot{z}(t) + Kz(t) = \bar{b} u(t),
\]

\[
y(t) = \bar{c}^T z(t),
\]

(1)

where \( M \in \mathbb{R}^{r \times r} \) is here assumed to be invertible, is transformed into the equivalent first order system

\[
\begin{bmatrix}
I & 0 & \hat{z}(t) \\
0 & M & \hat{z}(t) \\
E & \bar{x} & z(t) \\
A & \dot{x} & \dot{z}(t) \\
\end{bmatrix} + 
\begin{bmatrix}
0 & 1 & \hat{z}(t) \\
-K & -D & \dot{z}(t) \\
0 & 0 & b \\
0 & 0 & b \\
\end{bmatrix} u(t),
\]

(2)

A Krylov subspace method is then applied to system (2), resulting in a first order system of reduced dimension \( q \). Up to \( 2q \) moments can in principal be matched with such a method. The special form (2), however, which would allow a back transformation of the reduced first order system to a second order structure, is not preserved.

For second order models and their first order counterparts (2) the first Markov parameter is always zero. This property can be preserved by using a Krylov subspace method, which matches in addition to the desired number of moments also the first Markov parameter.

In [4, 1] such a Krylov subspace method, preserving the first Markov parameter zero is used, and the reduced system is subsequently transformed into a second order system of half the dimension. This procedure matches a maximum number of moments preserving the second order structure. It is, however, not proved that such a transformation always exists.

Here we will show how Krylov subspace methods can be modified to match the first Markov parameter zero in addition to a number of moments. Arnoldi- and Lanczos-type methods may be used in such a process. The system (2) can be reduced by applying a projection as follows,

\[
\begin{bmatrix}
W^T E \bar{z}(t) \\
W^T A \bar{x}(t) \\
W^T b \bar{u}(t)
\end{bmatrix} = 
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(3)
where \( V, W \in \mathbb{R}^{n \times q} \). If the columns of \( V \) form a basis of the Krylov subspace \( K_q(A^{-1}E, E^{-1}b) \) then \( q - 1 \) moments and the first Markov parameter are matched \(^2\). The Arnoldi algorithm can be used to compute \( V \). If the columns of \( V \) form a basis of the Krylov subspace \( K_q(A^{-1}E, E^{-1}b) \) and the columns of \( W \) form a basis of the Krylov subspace \( K_q(A^{-T}E^T, A^{-T}c) \) then the first \( 2q - 1 \) moments and the first Markov parameter are matched. The Lanczos algorithm can be used in this case to find matrices \( V \) and \( W \).

Such a Krylov subspace method is applied to the equivalent first order state space system (2) to compute a reduced model (3) of even dimension such that the first Markov parameter remains zero. We assume that the matrix \( E_r \) is invertible and multiply the state equation by \( E_r^{-1} \).

Systems of this kind can now be transformed to second order systems of half the dimension. By a simple construction we can prove the existence of a back transformation to second order structure under very weak conditions in the following theorem:

**Theorem 1** For every controllable single input single output (SISO) system,

\[
\dot{x}(t) = Ax(t) + bu(t), \\
y(t) = c^T x(t),
\]

of even dimension with \( c^T b = 0 \), there exists a nonsingular matrix,

\[
T = \begin{bmatrix}
c^T \\
R \\
c^T A \\
RA
\end{bmatrix},
\]

such that \( Rb = 0 \).

The transformed system

\[
\dot{\tilde{x}}(t) = T^{-1}AT\tilde{x}(t) + T^{-1}bu(t), \\
y(t) = c^T T\tilde{x}(t),
\]

is of form (2) and can thus be transformed to a second order system of half the dimension.

We will sketch the proof of the theorem. Theorem 1 not only proves the existence of the transformation matrix but also suggests a numerically reliable procedure to compute the transformation.

For multiple input multiple output systems the proof is by far more complicated and additional assumptions are needed.

**References**


\(^2\)It is common to choose \( W = V \).