Feedback and positive feedback stabilizability and holdability of linear discrete-time systems

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1. Introduction

Consider the following problem. Given a discrete-time linear system, find, if possible, linear state-feedback control laws such that under these laws all trajectories originating in the non-negative orthant of the state space remain non-negative while asymptotically deteriorating to the origin. This problem is called feedback stabilizability-holdability problem (FSH). If, in addition, the requirement of non-negativity is imposed on controls, the problem is a positive feedback stabilizability-holdability (PFHS).

The feedback stabilizability-holdability problem has been studied for continuous-time linear systems with scalar controls in [1, 2], where conditions requiring that all principal minors of the closed-loop system matrix are positive or, equivalently, the closed-loop system matrix is a non-singular M-matrix have been obtained. Related results are given in [3] for discrete-time periodic linear systems. Stabilization of positive linear systems by state-feedback is considered in [5] but the author does not take into account the restriction on non-negativity of controls in the closed loop. This restriction is essential because the control variables in many real-life systems represent quantities, which do not have meaning unless being non-negative.

In all the aforementioned works the requirement on positivity (non-negativity) of controls in the closed loop is not considered. The geometry of the problems and the related computational aspects are not at all exposed either. At the same time the requirement on positivity of controls is important since it imposes additional restrictions on the closed-loop system dynamics. The computational aspects of the problem are also quite important for the design of stable and holdable systems.

2. Preliminaries and problem formulation

Consider the linear discrete-time system

\[ x(t + 1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots, \] (1)

where \( x(t) \in \mathbb{R}^n \) is the system state, \( u(t) \in \mathbb{R}^m \) is the control, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( \mathbb{R}^{n \times s} \) is the space of all \( n \times s \) real matrices (vectors).

**Definition 1:** The system (1) is said to be **holdable** if there exists a control sequence \( \{u(t)\}_{t=0}^{\infty} \) such that \( x(t) \in \mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \mid x_i \geq 0, i = 1, 2, \ldots, n\}, t = 0, 1, 2, \ldots, \) whenever \( x(0) = x_0 \in \mathbb{R}^n_+ \).
**Definition 2:** The system (1) is said to be *feedback holdable* if there exists a feedback law
\[ u(t) = Kx(t) \]  
(2)
such that the trajectory \( x(t) \) of the corresponding closed-loop system
\[ x(t + 1) = (A + BK)x(t) \]  
(3)
remains in \( \mathbb{R}_+^n \), i.e. \( x(t) \in \mathbb{R}_+^n \) for \( t = 0, 1, 2, \ldots \), whenever \( x(0) = x_0 \in \mathbb{R}_+^n \).

**Definition 3:** The system (1) is said to be *positive feedback holdable* if it is feedback holdable and the control in the closed loop is nonnegative, i.e.
\[ u(t) = Kx(t) \in \mathbb{R}_+^m \text{ for } t = 0, 1, 2, \ldots \]  
(4)

It is shown in [8] that the system (1) is feedback holdable if and only if there exists a controller \( K \in \mathbb{R}^{m \times n} \) such that the closed-loop system matrix \( \tilde{A} \) is a non-negative real matrix (that is, all of its entries are nonnegative),
\[ \tilde{A} = A + BK \geq 0 \quad (\tilde{a}_{ij} \geq 0 \text{ for all } i, j = 1, \ldots, n), \]  
(5)
and positive feedback holdable if and only if (5) holds and the controller is nonnegative too,
\[ K \geq 0. \]  
(6)

Note that under the condition (5) the closed-loop system (3) becomes a positive linear system [4, 6]. Positive linear systems are defined as systems in which the system trajectory \( \{x(t)\}_{t=0}^\infty \) remains nonnegative whenever the initial state \( x_0 \) and the controls are nonnegative.

It is proved in [8] that the control in the closed-loop of a feedback or a positive feedback holdable system is nonnegative at any time if and only if the controller \( K \) is non-negative too, i.e. (6) must hold.

It is well known that the system (3) is asymptotically stable if and only if the dominant eigenvalue of the closed-loop system matrix \( \tilde{A} \) satisfies the inequality
\[ \rho(\tilde{A}) = \rho(A + BK) \leq 1 - \varepsilon. \]  
(7)
where \( \varepsilon \in (0, 1] \) is the stability margin.

A non-negative matrix \( A \geq 0 \) always has a dominant (maximal) eigenvalue \( \rho(A) \) which is real and nonnegative [7], i.e.
\[ \rho(A) \geq 0. \]  
(8)

Let, now, \( r_i \) and \( c_j \) denote the \( i \)-th row sum and the \( j \)-th column sum of \( A \geq 0 \), respectively, that is
\[ r_i = \sum_{j=1}^{n} a_{ij}, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad c_j = \sum_{i=1}^{n} a_{ij}, \quad j = 1, 2, \ldots, n, \]

and let

\[ r = \min r_i, \quad \bar{r} = \max r_i, \quad \underline{c} = \min c_j \text{ and } \bar{c} = \max c_j. \]

Then the following estimates of the dominant eigenvalue \( \rho(A) \) take place [7]

\[ \underline{c} \leq \rho(A) \leq \bar{c}, \tag{9} \]

and

\[ r \leq \rho(A) \leq \bar{r}. \tag{10} \]

**Feedback stabilizability-holdability problem (FSH):** Find, if possible, a linear state-feedback control law \( u(t) = Kx(t) \in \mathbb{R}^m \) such that the discrete-time system (1) is asymptotically stable and feedback holdable.

**Positive feedback stabilizability-holdability problem (PFSH):** Find, if possible, a non-negative linear state-feedback control law \( u(t) = Kx(t) \in \mathbb{R}_+^m \) such that the system (1) is asymptotically stable and positive feedback holdable.

The paper is organized as follows. In the next section 3 some of the properties of polyhedral sets relevant to the problems under study are summarized. Geometry of FSH and PFSH problems for linear discrete-time systems is studied in section 4. A constructive linear programming based approach and related computational procedures for solving the FSH and PFSH problems are also considered in section 4. Finally, section 5 concludes the paper.

### 3. Polyhedral sets

A non-empty set \( C \) of vectors is called a convex cone if \( a_1x + a_2y \in C \) whenever \( x, y \in C \) and \( a_1, a_2 \geq 0 \). The convex cone \( C \) is **polyhedral** if \( C(A) = \{ x \mid Ax \geq 0 \} \) for some real matrix \( A \). The matrix \( A \) is called a representation matrix of the polyhedral cone \( C \). A cone generated by the vectors \( e_1, e_2, \ldots, e_m \) is the set

\[
C(E) = \{ x \mid x = \gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_m e_m, e_j = \text{col}_j E, \gamma_j \geq 0, j = 1, 2, \ldots, m \}
= \{ x \mid x = E\gamma, \gamma = (\gamma_1, \ldots, \gamma_m)' \geq 0 \}.
\]

A cone arising in this way is called **finitely generated**. The matrix \( E \) is a generating matrix for the cone \( C \) and its columns \( e_j \) are called **generators** of the cone. A convex cone is polyhedral if and only if it is finitely generated. There is a unique minimal generating set of vectors \( e_j \) for every polyhedral cone. The members of this
generating set are called minimal generators (edges) of the polyhedral cone. Thus, the polyhedral cone can be viewed as

(i) an intersection of a finite number of closed halfspaces each of them containing the origin on its boundary,
(ii) a finite set of homogeneous linear inequalities, and
(iii) a non-negative linear combination of vectors.

A non-empty set \( P \) is called a convex polyhedron if \( P = \{ x \mid Ax \geq b \} \) for some real matrix \( A \) and vector \( b \). Geometrically the polyhedron can be viewed as the intersection of a finite number of affine halfspaces as long as this intersection is non-empty. Clearly, each polyhedral cone is a particular case of polyhedron if \( b = 0 \). Any polyhedron has a nice dual (internal) representation in terms of its vertices \( v_j \) and edges \( e_i \), namely

\[
P(\mathbf{V}, \mathbf{E}) = \{ x \mid \sum_{j=1}^{k} \eta_j v_j + \sum_{i=1}^{m} \gamma_i e_i, \eta_j \geq 0, \gamma_i \geq 0 \}
= \mathbf{V} \eta + \mathbf{E} \gamma. \quad (11)
\]

A non-empty set is a polytope if and only if it is a bounded polyhedron. The decomposition theorem of polyhedra states that a set of points (vectors) is a polyhedron if and only if \( P = Q + C \) for some polytope \( Q \) and polyhedral cone \( C \) [9].

4. Main results

The necessary and sufficient conditions for feedback stabilizability-holdability and positive feedback stabilizability-holdability follow directly from the necessary and sufficient conditions for feedback holdability, positive feedback holdability and asymptotic stability. The proofs of the following results are quite straightforward.

**Lemma 1.** The system (1) is feedback stabilizable-holdable if and only if there exists a matrix (controller) \( K \in \mathbb{R}^{m \times n} \) such that the inequalities (5) and (7) are satisfied.

**Lemma 2.** The system (1) is positive feedback stabilizable-holdable if and only if there exists a matrix (controller) \( K \in \mathbb{R}^{m \times n} \) such that the inequalities (5), (6) and (7) are satisfied.

The linear matrix inequality (5) can be rewritten in the form

\[
-Bk_{*j} \leq a_{*j}, \quad j = 1, \ldots, n \quad (12)
\]

where \( k_{*j} = (k_{1j}, \ldots, k_{mj})' \) and \( a_{*j} = (a_{1j}, \ldots, a_{nj})' \) are the \( j \)th columns of the controller \( K \) and the system matrix \( A \), respectively, and the symbol “’” denotes the transpose. On the other hand (7) holds true if

\[
c_j = \sum_{i=1}^{n} (a_{ij} + b_i k_{ij}) \leq 1 - \varepsilon, \quad j = 1, \ldots, n, \quad (13)
\]
or, alternatively,
\[ r_i = \sum_{j=1}^{n} (a_{ij} + b_j k_{sj}) \leq 1 - \varepsilon, \quad i = 1, \ldots, n, \tag{14} \]

where \( b_{i*} = (b_1, \ldots, b_m) \) is the \( i \)th row of \( B \) since the dominant eigenvalue that is non-negative satisfies the inequalities (9) and (10), respectively.

The system of linear inequalities (13) can be rewritten as

\[ \alpha_j + \beta k_{sj} \leq 1 - \varepsilon, \quad j = 1, \ldots, n, \tag{15} \]

where

\[ \alpha_j = \sum_{i=1}^{n} a_{ij}, \]

\[ \beta = (\beta_1, \ldots, \beta_m) \text{ with } \beta_s = \sum_{j=1}^{m} b_{js}, \]

\[ \varepsilon \in (0, 1], \]

and \( k_{sj} = (k_{1j}, \ldots, k_{mj})' \) the \( j \)th columns of \( K \).

Thus, the FSH problem has a solution if and only if the system of linear inequalities (12) and (15) is consistent. Similarly, the PFSH problem has a solution if and only if the system of linear inequalities (12) and (15) has nonnegative solutions

\[ k_{sj} \geq 0, \quad j = 1, \ldots, n. \tag{16} \]

Given a pair \((A, B)\) and a stability margin \( \varepsilon \) it is interesting to know whether the open-loop system is not stabilizable and holdable under feedback or, in other words, whether the system of linear inequalities (12) and (15) is inconsistent (the set of solutions is an empty set). Theorem 3 below give such conditions in terms of the pair \((A, B)\) and the stability margin \( \varepsilon \).

**Theorem 3.** The FSH problem has no solution if and only if there exist numbers \( y^{(j)}_i \) such that

1. \( y^{(j)}_i \geq 0 \) for all \( i = 0, 1, \ldots, n \) and \( j = 1, \ldots, n; \)
2. \( y^{(j)}_0 \beta_s - \sum_{j=1}^{n} y^{(j)} b_{is} = 0 \) for all \( j = 1, \ldots, n \) and \( s = 1, \ldots, m; \)
3. \( \sum_{j=1}^{n} y^{(j)}_0 (1 - \varepsilon - \alpha_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} y^{(j)}_i a_{ij} < 0, \)

where \( \varepsilon \in (0, 1] \) is the stability margin, \( \alpha_j = \sum_{i=1}^{n} a_{ij} \), and \( \beta_s = \sum_{j=1}^{m} b_{js} \).

**Proof:**

**Only if part.** Multiply each inequality in (14) and in (12) by \( y^{(j)}_0 \geq 0 \) and by \( y^{(j)}_i \geq 0 \), respectively, to obtain

\[ y^{(j)}_0 \sum_{s=1}^{m} \beta_s k_{sj} \leq y^{(j)}_0 (1 - \varepsilon - \alpha_j) \text{ for all } j = 1, \ldots, n, \tag{17} \]
respectively,

\[ y_i^{(j)} (- \sum_{s=1}^{m} b_{is} k_s) \leq y_i^{(j)} a_j \text{ for all } i, j = 1, \ldots, n, \tag{18} \]

and add all the inequalities in (17)-(18) to form their (nonnegative) linear combination

\[ \sum_{j=1}^{n} (y_0^{(j)} \sum_{s=1}^{m} \beta_s k_s - \sum_{s=1}^{m} y_i^{(j)} b_{is} k_s) \leq \sum_{j=1}^{n} (y_0^{(j)} (1 - \varepsilon - \alpha_j) + \sum_{i=1}^{n} y_i^{(j)} a_j). \tag{19} \]

The inequality (19), clearly, can be rewritten as

\[ \sum_{j=1}^{n} \sum_{s=1}^{m} (y_0^{(j)} \beta_s - \sum_{i=1}^{n} y_i^{(j)} b_{is}) k_s \leq \sum_{j=1}^{n} (y_0^{(j)} (1 - \varepsilon - \alpha_j) + \sum_{i=1}^{n} y_i^{(j)} a_j). \tag{20} \]

Let the system of inequalities (12) and (15) has a solution. Every solution of this system must, obviously, satisfy the nonnegative linear combination (20). Assume now that the conditions of the proposition hold. Then, there exist nonnegative numbers \( y_i^{(j)} \) such that

\[ y_0^{(j)} \beta_i - \sum_{i=1}^{n} y_i^{(j)} b_{is} = 0, \ i = 0, 1, \ldots, n \text{ and } j = 1, \ldots, n, \]

so (20) becomes

\[ 0.k_{11} + \ldots + 0.k_{m1} + \ldots + 0.k_{mn} \leq \sum_{j=1}^{n} (y_0^{(j)} (1 - \varepsilon - \alpha_j) + \sum_{i=1}^{n} y_i^{(j)} a_j). \]

Since

\[ \sum_{j=1}^{n} y_0^{(j)} (1 - \varepsilon - \alpha_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} y_i^{(j)} a_j < 0, \]

by hypothesis (iii), one can obtain

\[ 0.k_{11} + \ldots + 0.k_{m1} + \ldots + 0.k_{mn} < 0, \]

that is a contradiction. The only if part is thus proved.

The if part is rather trivial. The proposition is proved.

Note that because of the strict inequality (iii) the polyhedral set (in terms of the variables \( \{ y_0^{(j)}, y_i^{(j)} \} \)) is not closed. It can be closed by introducing a small parameter \( \delta > 0 \) and considering the inequality

\[ \sum_{j=1}^{n} y_0^{(j)} (1 - \varepsilon - \alpha_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} y_i^{(j)} a_{ij} \leq \delta \tag{iv} \]
instead of (iii). Thus, to establish that a given pair \((A, B)\) is not feedback stabilizable-holdable with a prescribed stability margin \(\varepsilon\) one has to find out that the polyhedral set represented by (i), (ii) and (iv) is not empty. Theorem 4 below gives a direct constructive approach to the solution of FSH problem. By reducing the FSH problem to a linear programming problem this approach provides a simple procedure to find out not only whether the FSH problem has a solution or not but also to determine a linear state feedback controller that endows the closed-loop (positive) system with a maximal stability margin that guarantees the fastest possible convergence to the origin.

**Theorem 4.** The FSH problem has a solution if and only if the problem

\[
\text{(LP 1)} \quad \max_{k, \theta, \varepsilon} \varepsilon \quad \text{subject to} \quad \begin{align*}
\sum_{s=1}^{m} \beta_s \hat{k}_{sj} - \hat{\beta} \theta & + \varepsilon \leq 1 - \alpha_j, \\
- \sum_{s=1}^{m} b_s \hat{k}_{sj} - \hat{\beta}_i \theta & \leq a_{ij} \\
0 & \leq \varepsilon \leq 1 \\
\hat{k}_{sj}, \theta, \varepsilon & \geq 0, \quad s = 1, \ldots, m \text{ and } i, j = 1, \ldots, n,
\end{align*}
\]

has a solution.

Moreover, the optimal linear state-feedback controller \(K^* = [k_{sj}]_{m \times n}\), where \(k_{sj} = \hat{k}_{sj} - \theta^*\), that is the solution to (21) – (25), guarantees a rate of convergence of the closed-loop system (3) to the origin not greater than \(1 - \varepsilon^*\).

**Proof:**
The substitution \(k_{sj} = \hat{k}_{sj} - \theta\) with \(\hat{k}_{sj}, \theta \geq 0\) reduces the inequalities (12) and (15), in which the variables \(k_{sj}\) are unrestricted in sign, to the equivalent inequalities (25) and (26) in nonnegative variables. The linear programming problem (LP1) does not have a solution if only if the system of linear inequalities (12) and (15) is inconsistent, that is the (FSH) problem does not have a solution. Otherwise, (LP1) always has a solution with \(\varepsilon^* = \max \varepsilon \in [0, 1]\). For \(\varepsilon^* = 0\) the solution of the (FSH) follows from the continuity argument.

The rate of convergence of the closed-loop system (3) is determined by the dominant eigenvalue of \(\tilde{A}^* = A + BK^*\), where \(K^*\) is the optimal linear state feedback controller, so \(\rho(\tilde{A}^*) \leq 1 - \varepsilon^*\) if a solution to problem LP1 exists. The proposition is thus proved.

Thus, to solve the FSH problem one has to solve the linear programming problem LP1 that can be done with any of the available LP packages. Notice that the feasible region (polyhedral set) of LP1 represented by the inequalities (22)–(25) is restricted in the direction of the variable \(\varepsilon\), so problem LP1 does not have a solution if and only if the feasible region is empty. Consequently, the FSH problem does not
have a solution either. Otherwise, the solution of LP1 gives the optimal linear state feedback controller that makes the open-loop system holdable and in addition stabilizes it with a maximal stability margin.

The control variables, quite often, do not have meaning unless they are non-negative. For example, concentrations in chemical reactors cannot be negative. In such cases the non-negativity restriction must be taken into account in the design procedure for asymptotically stable and holdable systems. Since the trajectory of a holdable closed-loop system (3) is always non-negative one can stay within the frame of a linear system design scheme.

Notice, that the restriction of non-negativity on control in the closed loop has been reduced to a non-negativity of the linear state feedback controller \[ K \] [8]. Theorem 5 below resolves the PFSH problem

**Theorem 5.** The PFSH problem has a solution if and only if the problem

\[(LP\ 2)\quad \max_{e, \xi} \varepsilon \quad \text{s.t.} \]

\[ \sum_{s=1}^{m} \beta_s k_{ij} + \varepsilon \leq 1 - \alpha_j, \]

\[ - \sum_{s=1}^{m} b_{is} k_{ij} \leq a_{ij} \]

\[ 0 \leq \varepsilon \leq 1 \]

\[ k_{ij}, \varepsilon \geq 0, \quad s = 1, \ldots, m \quad \text{and} \quad i, j = 1, \ldots, n, \]

where

\[ \beta_s = \sum_{s=1}^{m} b_{is}, \quad \beta_s = \sum_{s=1}^{m} b_{is}, \quad \beta_s = \sum_{i=1}^{n} b_{is}, \quad \text{and} \quad \alpha_j = \sum_{i=1}^{n} a_{ij}, \]

has a solution.

Moreover, the optimal linear state-feedback controller \[ K^* = \left[ k_{ij}^* \right]_{m \times n} \], that is the solution to \( LP\ 2 \), guarantees a rate of convergence of the closed loop system (3) to the origin not greater than \( 1 - \varepsilon^* \).

5. Conclusions

The FSH and PSFH problems are formulated and studied in this paper for discrete-time linear systems. Geometry of the problems is examined in details. It is shown that the set of all linear state feedback controllers that make the open-loop system holdable and stabilizable is a polyhedron and the external and its dual (internal) representations of this polyhedron are found. A constructive linear programming based approach to the solution of FSH and PFSH problems is proposed and developed in the paper. This approach provides a simple computational procedure to find out not only whether the FSH, respectively the PFSH, problem has a solution or not but also to determine a linear state feedback controller, respectively a non-negative linear state feedback controller, that endows the closed-loop (positive) system with a maximal stability margin and guarantees the fastest possible convergence to the origin. The sensitivity analysis procedures built-in in any of the commercial LP packages available on the market can be used to perform sensitivity analysis of the FSH and PFSH problems.
Necessary and sufficient conditions for non-existence of the solution to FSH and PFSH problems are obtained in terms of the parameters of the system for any given stability margin.

References


