Riccati Equations in Delay Systems

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Abstract

We give an overview of how the Riccati equation makes its appearance in the stability analysis of linear systems with delays. We avoid complexities as time-invariance, added perturbations, distributed delays etc, to get to the main issues. Most generalizations can be or have been carried out, but do not add substantially to our understanding. We also show the connection between quadratic stability and an optimization problem. The notion of Riccati stability is introduced as the existence of a positive definite triple of matrices satisfying a certain Riccati equation. While having its origin in the problem of delay systems, the partial characterization of this Riccati stability is carried out as an independent endeavor.

1 Lyapunov Krasovskii Functionals in Delay Systems

The past decade has seen a flurry of activity on systems with delays (See [6,8,9] and references therein). One important direction of this activity involved the stability analysis of delay systems using the Lyapunov-Krasovskii approach. In this connection the Riccati equation made its entrance into the study of delay systems. In this paper we present a stripped down version of the main results on stability and its tangency to the ubiquitous Riccati equation. With stripped down, we mean that we will not be concerned with the time varying, nonlinearly perturbed, or uncertain versions. These inclusions only seem to clutter up the main theory with additional complexities and technicalities, without requiring essentially new techniques or providing greater insight.

We shall consider the linear autonomous continuous time retarded system

\[ \dot{x}(t) = Ax(t) + \sum_i Bx(t - \tau_i), \]

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and the linear continuous time neutral system,

\[ \dot{x}(t) = Ax(t) + \sum_i Bx(t - \tau_i) + \sum_i C\dot{x}(t - \tau_i). \]  

(2)

The autonomous discrete delay system has the form

\[ x_{k+1} = Ax_k + \sum_i B_i x_{k-n_i}. \]  

(3)

We may assume \( \tau_1 < \tau_2 < \cdots < \tau_N \). In continuous time, the dynamical equations are infinite dimensional. Relatively simple sufficient conditions for asymptotic stability have been obtained in [13, 14], using a Lyapunov-Krasovskii functional of the form

\[ V(\phi) = \phi'(0)P\phi(0) + \sum_i \int_{-\tau_i}^{t-\tau_i} \phi'(t)Q_i^\phi(t) \, dt. \]

It is shown that the existence of positive definite matrices \((P, Q_i, R)\) satisfying the matrix Riccati equation

\[ A'P + PA + \sum Q_i + \sum_i PB_iQ_i^{-1}B_i'P + R = 0, \]  

(4)

implies the asymptotic stability for all values of \( \tau \geq 0 \), of the retarded equation.

Consider now the system with a single delay \((N = 1)\). The equation (4) reduces to a standard Riccati equation known in \(H_\infty\) control:

\[ A'P + PA + Q + PBQ^{-1}B'P + R = 0, \]  

(5)

In [3], we defined the notion of Riccati stability of the pair \((A, B)\) by the existence of a positive definite solution \((P, Q)\) to the Riccati equation (5). Note that since (1) has to be stable for \( \tau = 0 \) and \( \tau \to \infty \), the matrices \( A + B \) and \( A \) have to be Hurwitz stable, i.e., have their spectrum in the open left half plane. Recall also that a matrix \( C \) is Schur-Cohn stable, if its spectrum lies in the open unit disk.

If \( B = 0 \), the delay system simplifies to a finite dimensional time-invariant system, for which the Riccati equation reduces to the ubiquitous Lyapunov equation,

\[ A'P + PA + S = 0, \]  

(6)

where we have set \( Q + R = S \). It is well known that a positive definite pair \((P, S)\) exists if and only if \( A \) is Hurwitz. This condition is necessary and sufficient.

Likewise, if \( C \) is Schur-Cohn stable, asymptotic stability of the (single delay) neutral equation (2) is implied by the existence of a triple of positive definite matrices \((P, Q, R)\) satisfying the the Riccati equation

\[ (A' + PN)(I + BQ^{-1}B')(A + NP) + Q + B'B - PNP + R = 0 \]  

(7)
where \( N = \left[ \sum_{k=0}^{\infty} C^k C'^k \right]^{-1} \). See [15, 16] for details.

For the discrete delay equation (3), the quadratic Lyapunov-Krasovskii functional

\[
V_k(\phi) = \phi_0^T P \phi_0 + \sum_{i=-n+1}^{0} \phi_i^T Q \phi_i. \tag{8}
\]

enables to show (see [13]) that the system (3) is robustly asymptotically stable, if either of the following two conditions hold:

i) there exists a triple of positive definite (symmetric) matrices \( P, R \) and \( W \) such that

\[
A'PBW^{-1}B'PA + A'PA + W + B'PB + R = P, \tag{9}
\]

ii) there exists a triple of positive definite (symmetric) matrices \( \Pi, S \) and \( Z \) such that

\[
B'\Pi AZ^{-1}A'\Pi B + B'\Pi B + Z + A'\Pi A + S = \Pi. \tag{10}
\]

Note: These discrete Riccati equations can be transformed into each other.

## 2 Riccati Stability

In the rest of this paper we will just be concerned with the algebraic Riccati equation. Given two \( n \times n \) real matrices, \( A \) and \( B \), consider the matrix Riccati equation

\[
A'P + PA + Q + PBQ^{-1}B'P + R = 0. \tag{11}
\]

Can one characterize the pairs \((A, B)\) for which the above equation has a solution for positive definite symmetric matrices \( P, Q \) and \( R \)? Because of its significance, we provide the following

**Definition 2.1** [12, 3] A pair \((A, B)\) is defined to be Riccati stable if a triple of positive definite matrices \( P, Q, R \) exists such that the equation (11) is satisfied.

Riccati stability may be reformulated as an LMI (this notion was introduced by Jan Willems, see [2]):

Can one characterize all pairs \((A, B)\), without invoking additional matrices, for which there exist positive definite matrices \( P \) and \( Q \) such that

\[
\begin{bmatrix}
    A'P + PA + Q & PB \\
    B'P & -Q
\end{bmatrix} < 0. \tag{12}
\]

By analogy, we call the pair \((A, B)\) discrete Riccati stable if positive definite triples of matrices can be found, such that (9) or (10) hold.
3 Properties of Riccati Stable Pairs

In [12], where Riccati-stability was called "d-stability", in reference to "delay", the following connections with the spectral properties of $A$ and $B$ were obtained.

**Theorem 3.1** If there exists a triple of symmetric positive definite matrices $P, Q$ and $R$, satisfying (4), then $A$ is Hurwitz and $A^{-1}B$ is Schur-Cohn.

There is no complete converse of this theorem, however two partial converses are easily proven:

**Theorem 3.2** If the matrix product $A^{-1}B$ is Schur-Cohn, then there exists an orthogonal matrix $\Theta$ such that $\Theta A$ is Hurwitz, and the pair $(\Theta A, \Theta B)$ is Riccati-stable.

**Theorem 3.3** If the matrix $A$ is Hurwitz, then there exists a matrix $B$ such that $A^{-1}B$ is Schur-Cohn and $(A, B)$ is Riccati-stable.

In addition the following scaling properties are shown in [12]:

**Lemma 3.4** If $(A, B)$ is Riccati-stable, then $(\alpha A, \alpha B)$ is Riccati-stable for all $\alpha > 0$.

**Lemma 3.5** If $(A, B)$ is Riccati-stable, then $(SAS^{-1}, SBS^{-1})$ is Riccati-stable, for all non-singular $S$.

**Lemma 3.6** If $(A, B)$ is Riccati-stable, and $B$ has full rank, then $(A', B')$ is Riccati-stable.

The full rank condition on $B$ can be relaxed. Lemma 3.6 is a duality result.

In [12] a detailed construction was given for a subset of Riccati-stable pairs for the case $n = 2$. It leads to an (over-)parameterization, but the construction readily extends to arbitrary dimensions, by using the following theorem,

**Theorem 3.7** Assume that the pairs $\{(A_i, B_i) | i = 1 \ldots N\}$ are Riccati-stable for the same $P$-matrix. i.e., there exist $Q_i, R_i > 0, i = 1 \ldots N$ such that $A'_iP + PA_i + Q_i + PB_iQ_i^{-1}B'_iP + R_i = 0$. Then all pairs in the positive cone generated by the above pairs are Riccati-stable. i.e., $\forall \alpha_i \geq 0$, but not all zero, the pair $(\sum_i \alpha_i A_i, \sum_i \alpha_i B_i)$ is Riccati-stable.

The invariance of Riccati-stability under similarity, Lemma 3.5, ensures that if $(A, B)$ is Riccati-stable, one can transform the system to one for which the new $P$ matrix, i.e., $S^{-T}PS^{-1}$ is the identity. Thus motivated, we provide a simplified form:

Given $B$, denote by $A_B$ the set of matrices $A$ for which $(A, B)$ is Riccati stable, with $P = I$, i.e.,

\[ A_B = \{ A \mid \exists Q = Q' > 0, \text{ s.t. } A + A' + Q + BQ^{-1}B' < 0 \}. \]
By Theorem 3.7, a necessary condition for $A \in \mathcal{A}_B$ is that its symmetric part $A_s$ satisfies
\[ A_s < -\frac{1}{2}(Q + BQ^{-1}B'), \]
for some $Q > 0$. If for each $B$ the set $\mathcal{A}_B$ can be determined, the characterization of Riccati stable pairs is solved. The following special case is shown in [3]. We reproduce its proof for self containedness:

**Theorem 3.8** If $B$ is in the real-diagonal form,
\[ B = \text{Blockdiag}\{\Lambda_+, 0, -\Lambda_-, B_1, \ldots, B_c\}, \]
where $\Lambda_+ = \text{diag}\{\lambda_1, \ldots, \lambda_p\}$ are the positive real eigenvalues, $-\Lambda_- = \text{diag}\{-\lambda_{p+1}
\ldots, -\lambda_{p+m}\}$ are the negative real ones, and the $B_k$'s are $2 \times 2$ blocks $B_k = \begin{bmatrix} \sigma_k & \omega_k \\ -\omega_k & \sigma_k \end{bmatrix}$, associated with the complex eigenvalues $\sigma_k \pm i\omega_k$, then the set $\mathcal{A}_B$ is characterized by the set of all matrices, $A$, whose symmetric part satisfies $A_s < -2\text{Blockdiag}\{\Lambda_+, 0, -\Lambda_-; |\sigma_1|I_2, \ldots, |\sigma_c|I_2\}$.

**Proof:** In this block diagonal form, it is clear that it suffices to choose the same blockdiagonal structure for $Q$ and the problem decouples. For real eigenvalues in the sets $\Lambda_+$ and $-\Lambda_-$, observe that $q + \frac{\lambda_k^2}{q} \geq 2|\lambda_k|$ and equality is obtained for $q = |\lambda_k|$. Likewise for a zero eigenvalue, the corresponding $q$ may be taken infinitesimally small. For complex conjugate eigenvalue pairs, observe that
\[ \begin{bmatrix} q_1 & q \\ q & q_2 \end{bmatrix} + \begin{bmatrix} \sigma_k & \omega_k \\ -\omega_k & \sigma_k \end{bmatrix} \begin{bmatrix} q_1 & q \\ q & q_2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_k & \omega_k \\ -\omega_k & \sigma_k \end{bmatrix} \geq 2|\sigma_k| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Equality is achieved with $\begin{bmatrix} q_1 & q \\ q & q_2 \end{bmatrix} = \begin{bmatrix} |\sigma| & \omega \\ \omega & |\sigma| \end{bmatrix}$, if $|\sigma| \geq |\omega|$ and, switching to polar form, with $q_1 = \frac{\rho}{|\cos \phi|}(1 + \cos \chi)$, $q_2 = \frac{\rho}{|\cos \phi|}(1 - \cos \chi)$, and $q = \rho\sqrt{\tan^2 \phi - \frac{\cos^2 \chi}{\cos^2 \phi}}$, where $\rho$ and $\phi$ are respectively the modulus and the argument of the complex eigenvalue $\sigma + i\omega$, and $\chi$ arbitrary with $|\cos \chi| < |\sin \phi|$ if $|\sigma| < |\omega|$. In the latter case, the solution was obtained by direct optimization of the minimal eigenvalue of the matrix $Q + B_kQ^{-1}B_k'$ over all positive definite matrices $Q$. Hence $Q + BQ^{-1}B' \geq 0$ if $B$ is singular, and $Q + BQ^{-1}B' \geq 2zI$, where $z = \min\{|\lambda_k|; k = 1 \ldots p + m\} \bigcup\{|\sigma_\ell|, \ell = 1, \ldots, c\}$ if $B$ has full rank, from which the theorem follows. ♦

**Remarks:** Equations related to (4) are also discussed in [1, 7, 11]. Datko presents an explicit solution method of (4) via a line integral over the imaginary axis [5].

### 4 Connection with Optimal Control

The Riccati equation (4) is further realized in the following optimal control problem: Given the linear time invariant finite dimensional system
\[ \dot{x} = Ax + Bu, \]
with initial state $x_0$, determine the control $u(t)$ with given energy $E_U = \frac{1}{2} \int_0^T u'(\tau)Qu(\tau) d\tau$ which maximizes the quadratic form $J = \frac{1}{2} x_T' S_j x_T + \frac{1}{2} \int_0^T x'(t)P x(t) dt = \Phi + E_Y$.

### 4.1 Solution of the control problem

Restate the problem as a fixed terminal time optimal control problem, with performance index
$$J = \frac{1}{2} x_T' S_j x_T + \frac{1}{2} \int_0^T x'(t)P x(t) dt.$$ Imagine a stored amount of energy of size $E_U$ available from which the input is generated. Consider likewise $E_Y$ as an accumulation of potential energy available at the output side. The objective is then to shape the control input $u(\cdot)$ so that the energy is optimally transferred from the input to the output, while the state of the system evolved from $x_0$ to an unspecified final state $x_T$. Note that control ceases if the input energy storage is depleted. Hence at $t = 0$ we have $[\text{state, input energy, output energy}] = [x_0, E_U, 0]$, while at $t = T$ we have $[x_T, 0, E_Y]$.

Treat $U = -\frac{1}{2} \int_0^\tau u'(\tau)Qu(\tau) d\tau$ as a new state, satisfying $\dot{U} = -\frac{1}{2} u'Qu$, with initial condition $U(0) = E_U$, and final constraint $U(T) = 0$. For this problem adjoin the terminal constraint with multiplier $\nu$ giving the augmented terminal cost $\Phi + \nu U$. The Hamiltonian is:
$$H = \frac{1}{2} x'P x - \lambda_U \frac{1}{2} u'Qu + \lambda_x' (Ax + Bu),$$ giving the Euler - Lagrange Equations:
$$\begin{align*}
\dot{\lambda}_U &= 0 \quad (14) \\
\dot{\lambda}_x &= -Qx - A'\lambda_x, \quad (15)
\end{align*}$$
with final conditions
$$\begin{align*}
\lambda_U(T) &= \frac{\partial \Phi}{\partial U} = \nu \quad (16) \\
\lambda_x(T) &= \frac{\partial \Phi}{\partial x} = S_j x(T). \quad (17)
\end{align*}$$

It follow that $\lambda_U$ is constant (= $\nu$). The optimality condition gives
$$\lambda_U Qu - B'\lambda_x = 0 \quad \Longrightarrow \quad u = \frac{Q^{-1}B'\lambda_x}{\lambda_U},$$ and the state equation is thus
$$\dot{x} = Ax + \frac{BB'}{\lambda_U} \lambda_x.$$
We use the sweep method to represent its solution: Postulate a solution in the form 
\[ \lambda_x(t) = S(t)x(t), \]
The resulting Riccati equation is
\[ \dot{S} + A'S + SA + \frac{1}{\lambda_U} SBRQ^{-1}B'S + P = 0, \tag{18} \]
with final condition \( S(T) = S_f \), consistent with \( \lambda_x(T) = S_f x(T) \).

The maximizing input is obtained in feedback form:
\[ u(t) = \frac{B'S(t)x(t)}{\lambda_U}, \tag{19} \]
giving the optimal autonomous system
\[ \dot{x}_{opt} = \left( A + \frac{BQ^{-1}B'}{\lambda_U} \right) x_{opt} \tag{20} \]
Note that \( S(t) \) also depends on \( \lambda_U \). The latter is determined by the constraint
\[ \mathcal{E}_U - \frac{1}{2\lambda_U^2} \int_0^T x_{opt}'SBQ^{-1}B'Sx_{opt} dt = 0. \tag{21} \]
Once \( \lambda_U \) has been determined, the optimal output energy storage is
\[ \mathcal{E}_Y^{opt} = \frac{1}{2} \int_0^T x_{opt}'Px_{opt} dt. \tag{22} \]
Finally, note that the Riccati equation expressed in terms of the closed loop system (20) is
\[ \dot{S} + A_{cl}'S + SA_{cl} - \frac{1}{\lambda_U} SBRQ^{-1}B'S + P = 0, \tag{23} \]
and leads to the identity
\[ \frac{1}{2} x_T' S_f x_T - \frac{1}{2} x_0' S_0 x_0 + \mathcal{E}_Y^{opt} - \lambda_U \mathcal{E}_U = 0. \tag{24} \]
Note that \( \lambda_U \) and \( S \) depend on both \( \mathcal{E}_U \) and \( x_0 \). Moreover, even if \( S_f > 0 \) there is no guarantee that \( S(t) > 0 \) for \( 0 < t < T \). However, if anything but the optimal control is used, but still satisfying the energy constraint, we know that
\[ \mathcal{E}_Y + \frac{1}{2} x_T' S_f x_T \leq \frac{1}{2} x_0' S_0 x_0 + \lambda_U \mathcal{E}_U = 0. \tag{25} \]
4.2 Delay system interpretation

To reconcile this with the delay system,

\[ \dot{x}(t) = Ax(t) + Bx(t - \tau), \]  \hspace{1cm} (26)

interpret \( x(t - \tau) \) as an input \( u(t) \) of dimension \( n \). Let further \( P = Q \), and suppose \( S_f \) is the steady state solution \( \bar{S} \) of the algebraic Riccati equation

\[ A'\bar{S} + \bar{S}A + \frac{\bar{S}BQ^{-1}B'S}{\lambda_U} + Q = 0, \]  \hspace{1cm} (27)

then the above identity (25) gives the relation

\[ x'_\tau \bar{S}x_\tau + \int_0^\tau x'Qx \, dt \leq x'_0 \bar{S}x_0 + \lambda_U \int_{-\tau}^0 x'Qx \, dt. \]  \hspace{1cm} (28)

If a positive definite solution \( \bar{S} \) and a scalar \( \lambda_U \in (0, 1) \) exists, satisfying the above Riccati equation, then the accumulated state energy augmented by a quadratic form drops from interval \((-\tau, 0)\) to \((0, \tau)\), and by extension \( W(x_k) = x'_k \bar{S}x_k + \int_{(k-1)\tau}^{k\tau} x'Qx \, dt \), decreases as \( k \) increases. Hence we conclude asymptotic stability. The same conclusion follows if we take the Lyapunov-Krasovskii functional

\[ V(x_k) = x'(0)\bar{S}x(0) + \lambda_U \int_{-\tau}^0 x'(\sigma)Qx(\sigma) \, d\sigma, \]  \hspace{1cm} (29)

since under the same conditions, \( \bar{S} > 0 \) and \( 0 < \lambda_U < 1 \), we get

\[ \dot{V} \leq x' \left[ A'\bar{S} + \bar{S}A + Q + \frac{\bar{S}BQ^{-1}B'S}{\lambda_U} + (\lambda_U - 1)Q \right] x = (\lambda_U - 1)x'Qx \leq 0. \]  \hspace{1cm} (30)

5 Linearization and Extensions

5.1 A generalized Lyapunov theorem

Lemma 5.1 The matrix equation

\[ A'P + PA + \sum B'_iPB_i = -Q \]

has a positive definite solution, \( P > 0 \), if and only if the matrix

\[ A' \otimes I + I \otimes A' + \sum B'_i \otimes B'_i \]

is Hurwitz.
Proof: See Zelentsovskii [19].

The continuous and discrete Lyapunov equations are special cases of the above equation. For

\[ A'P + PA = -Q \]

the criterion matrix, \( A' \otimes I + I \otimes A' \), has spectrum consisting of all pairwise sums of eigenvalues of \( A \). This reduces to \( A \) being Hurwitz.

In the discrete case,

\[ A'PA + Q = P, \]

the matrix \( P \) is positive definite if and only if \( -I + A' \otimes A' \) is Hurwitz. Since the spectrum of \( A' \otimes A' \) consists of the pairwise products of eigenvalues of \( A \), the criterion reduces to the Schur-Cohn stability of \( A \).

5.2 Sufficient Conditions for Riccati Stability

Consider in this subsection the special case of (4) for nonsingular \( B \). This case leads to an interesting linear equation by letting for some positive scalar \( \beta \)

\[ Q = \frac{1}{\beta} B'PB. \]  (31)

This reducing the Riccati equation (4) to the Sylvester equation

\[ A'P + PA + \frac{1}{\beta} B'PB + \beta P + R = 0. \]  (32)

For this equation, the method of Zelentsovskii [19] gives directly the following corollary

Corollary 5.2 If for some \( \beta > 0 \), the matrix \( M = (A + \frac{\beta}{2} I) \otimes I + I \otimes (A + \frac{\beta}{2} I) + \frac{1}{\beta} B \otimes B \) is Hurwitz, then the pair \( (A, B) \) is Riccati stable.

Proof: The solution \( P \) to the Sylvester equation is given by \( M'vec(P) = vec(R) \), and is positive definite if \( M' \) is Hurwitz by Zelentsovskii’s result and the property \( (U \otimes V)' = U' \otimes V' \).

5.3 Extensions: Time Varying Functionals

Taking a Lyapuniv-Krasovskii functional with time varying weight matrix \( P \) yields

\[ \dot{V} = [x', x_r'] \begin{bmatrix} A'P + PA + Q + \dot{P} B & PB \\ B'P & -Q \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} \]  (33)

which leads to the LMI and its associated Riccati equation

\[ -\dot{P} = A'P + PA + Q + PBQ^{-1}B'P + R = 0. \]  (34)
For this Riccati differential equation, the following asymptotic conditions, similar to the LQ Riccati equation surveyed in [4] can be derived.

Its connection with delay systems stems from the fact that if we choose $T = \tau$ for the system (1), then the delayed state, $x(t - \tau)$ may be considered as the input $u(t)$. Clearly, if then $E_Y = \int_0^T x' P x \, dt < \int_{-T}^0 x' P x \, dt$, for all nonzero initial conditions, the delay system state should converge to the nullsolution, i.e., the system is asymptotically stable. Hence if we can establish that the max $\int_0^T x' P x \, dt < 1$ for all inputs with energy $E_U = 1$, the pair $(A, B)$ is Riccati stable.

5.4 Scale Delay Equations

These are equations of the form

$$\dot{x}(t) = Ax(t) + Bx(\alpha t)$$

with $0 < \alpha \leq 1$. It was shown in [17] that the above system is asymptotically stable if the pair $(A, \frac{1}{\sqrt{\alpha}} B)$ is Riccati stable.

5.5 Stochastic Delay Equations

Here we are only interested in perturbations with multiplicative noise, as essentially new terms appear in the characterization of their stochastic stability via Lyapunov-Krasovskii functionals. See [18] and references therein. The prototype stochastic delay system in Itô-form is

$$dx(t) = [Ax(t) + Bx(t - \tau)] \, dt + [C(x(t) + D x(t - \tau))] \, dw(t)$$

One form of the corresponding Riccati equation is

$$A'P + PA + Q + C'PC + D'PD + (PB + C'PD)Q^{-1}(B'P + D'PC) + R = 0.$$  

If $D = 0$, it results in the addition of one term to the standard Riccati equation we have considered in our analysis.

$$A'P + PA + Q + C'PC + PBQ^{-1}B'P + R = 0.$$  

If a triple $(P,Q,R)$ of positive definite matrices satisfying the above exists, then we call the triple $(A, B, C)$ s-(for stochastically) Riccati-stable. The linearized version results in

**Corollary 5.3** If for some $\beta > 0$, the matrix $M = (A + \frac{\beta}{2} I) \otimes I + I \otimes (A + \frac{\beta}{2} I) + \frac{1}{\beta} B \otimes B + C \otimes C$ is Hurwitz, then the triple $(A, B, C)$ is s-Riccati stable.

**Proof:** Similar to corollary (5.2).
6 Conclusions

We have shown how some forms of the Riccati equation make their entrance in the study of stability for linear time-delay systems. We have also shown a connection between the quadratic stability and an optimization problem. Most of the insight is gained by limiting the study to its standard form: i.e., one single delay. However, extensions are readily made (partially explaining the current proliferation of delay systems papers in the literature). For instance, one could define analogously the Riccati stability for the \((N + 1)\)-tuple \((A, B_1, \ldots, B_N)\) by the existence of a positive definite solution to

\[ A'P + PA + \sum_i Q_i + \sum_i PB_i Q_i^{-1} B'_i P + R = 0 \]

for some positive definite \(Q_i; i = 1 \ldots N\), and \(R\). Choosing \(Q_i = \frac{1}{\beta_i} B'_i PB_i\) leads to the linearized form

\[ A'P + PA + \sum_i \frac{1}{\beta_i} B'_i PB_i + \sum_i \beta_i P + R = 0, \]

thus establishing Riccati stability if the matrix

\[ M = \left( A + \sum_i \frac{\beta_i}{2} \right) \otimes I + I \otimes \left( A + \sum_i \frac{\beta_i}{2} I \right) + \sum_i \frac{1}{\beta_i} B'_i PB_i \]

is Hurwitz.

Similarly, the time varying systems, and stochastic delay systems may be treated in the same way, although some additional technicalities will appear. Further extensions can be made for distributed delay systems. The stability of scale delay systems, which are of a completely different type, can also be treated by the same type of Riccati equation.

Rather than focusing on the stability of time delay systems, we have opted to restrict attention to the notion of Riccati stability as it pertains directly to the algebraic theory of the Riccati equation. Although a complete characterization of Riccati stability is still elusive, we have shown some progress towards the solution of an open problem [3], by identifying a subclass of Riccati stable systems. Some parallels with neutral and discrete delay systems were presented as well.

References


