Stability and bifurcation analysis of some fluid flow model of TCP behavior

Wim Michiels*  
Department of Computer Science,  
K.U. Leuven,  
Celestijnenlaan 200A,  
B-3001 Heverlee, Belgium.

Silviu-Iulian Niculescu†  
HEUDIASYC (UMR CNRS 6599)  
Université de Technologie de Compiègne,  
Centre de Recherche de Royallieu,  
BP 20529, 60205, Compiègne, France.

Abstract

This note focuses on the stability analysis of some classes of nonlinear time-delay models, encountered as fluid models for TCP/AQM network. By combining analytical and numerical tools, the attractors of these models, as well as the local and global behavior of the solutions are studied. Among others, the presence of a chaotic attractor is shown, which supports the proposition that TCP itself as a deterministic process can cause or contribute to chaotic behavior in a network.

The main goals of the paper are firstly to provide qualitative and quantitative information on the dynamics of the models under consideration, and secondly to illustrate the capabilities of computational tools for stability and bifurcation analysis of delay differential equations to analyze fluid flow models.

1 Introduction

The analysis of fluid flow models for describing high-speed network behavior represents a subject of recurring interest in the last years, mainly for the simplicity of the models, and for the nice and easy interpretation as standard control feedback problems.

Recently, in [10, 4, 5] some models describing accurately the behavior of congested routers in TCP/AQM networks were presented. As it is expected, such kind of models are described by nonlinear differential equations with time-delay where the delay represents the corresponding round-trip time in the network.

*Corresponding author; E-mail: Wim.Michiels@cs.kuleuven.ac.be.
†E-mail: silviu@hds.utc.fr.
The model of [10, 4, 3] consists of the following coupled nonlinear differential equations with time-varying delay:

\[ W(t) = \frac{1}{R(t)} - \frac{1}{2} \frac{W(t) W(t - R(t))}{R(t - R(t))} p(t - R(t)), \]  

\[ Q(t) = \left\{ \begin{array}{ll}
N(t) \frac{W(t)}{R(t)} - C, & q > 0 \\
\max \left( N(t) \frac{W(t)}{R(t)} - C, 0 \right), & q = 0
\end{array} \right. \]  

where \( W(t) \) denotes the average of TCP windows size (packets), \( Q(t) \) is the average of queue length (packets), \( R(t) \) is the round-trip time (secs), \( C \) is the queue capacity (packets/sec), \( N(t) \) is the number of TCP sessions and \( p(\cdot) \) is the probability function of a packet mark. The queue length \( Q(t) \) and window-size \( W(t) \) are positive. The probability function of a packet mark \( p(\cdot) \) takes values only in \([0, 1]\). The round trip-time can be decomposed as

\[ R(t) = \frac{Q(t)}{C} + \tau_p, \]

where \( \tau_p \) is the propagation delay (secs).

The first differential equation describes the TCP window control dynamic. Indeed, the first term \( \frac{W(t)}{R(t)} \) describes the window’s additive increase phase, and the second term \( \frac{W(t) W(t - R(t))}{2R(t - R(t))} \) the multiplicative decreasing phase (including the packet marking probability). Different AIMD continuous-time models can be found in [5, 6]. Note also the excellent overview of existing fluid approximations based approaches proposed by Low et al. [6]. Equation (2) describes the bottleneck queue length as the difference between the packet arrival rate \( \frac{NW(t)}{R} \) and the link capacity \( C \), assuming that there are no internal dynamics in the bottleneck (roughly speaking, a simple integrator).

Using fluid flow models like (1)-(3), Active Queue Management can be interpreted as a feedback control problem, where the control action consists of marking packets (with probability \( p \)) as a function of the measured queue length \( Q \), see [4].

As in [3] we shall assume that the TCP load \( N(t) \) and the round-trip time \( R(t) \) are time-invariant, i.e., \( N(t) \equiv N \) and \( R(t) \equiv R \). The latter may be a good approximation when the round-trip-time is dominated by the propagation delay. This occurs when the capacity \( C \) of the link is large [3]. Furthermore, as it is presented in [3, 9], considering that that the probability marking function \( p(\cdot) \) is proportional to the queue length, i.e., \( p(t) = K Q(t) \), the system under consideration becomes:

\[ W(t) = \frac{1}{R} - \frac{W(t) W(t - R)}{2R} K Q(t - R) \]  

\[ Q(t) = \left\{ \begin{array}{ll}
N(t) \frac{W(t)}{R} - C, & Q > 0 \\
\max \left( N(t) \frac{W(t)}{R} - C, 0 \right), & Q = 0
\end{array} \right. \]
The unique equilibrium point of (4)-(5) is given by:

\[ W^* = \frac{RC}{N}, \quad Q^* = \frac{2N^2}{R^2C^2K}. \]

In [4, Section 5] a linearized stability analysis of the equilibrium point was performed in the frequency domain, where the variation of round trip time was taken into account, yet some of the delay effects were treated as high frequency uncertainty. The references [3, 9] contain a Lyapunov-based (non-)local stability analysis of the equilibrium when making the additional simplification of Equation (4) to

\[ \dot{W}(t) = \frac{1}{R} - \frac{W(t)^2}{2R} K q(t - R). \]  

(6)

In [3] the authors proved that when the the delay is equal to zero the equilibrium point of the system (6) and (5) is asymptotically stable for all \( K > 0 \). When the delay is different from zero a Lyapunov-Razumikhin approach was used to show the asymptotic stability of the equilibrium point of (6) for sufficiently small \( \frac{2N}{R} > 0 \). In [9] this result was refined and sufficient conditions on the parameters for local stability and estimates of the attraction domain were derived using a less conservative Lyapunov-Krasovskii approach.

The structure of the paper is as follows. After commenting on a transformation of state and time, we completely characterize the linear stability region of the steady state solution of (4)-(5) as a function of the model parameters. More explicitly, only one delay interval guarantees the asymptotic stability of the linearized model. Then we take the nonlinearities into account and study the global behavior of the solutions.

2 A transformation

When the round-trip \( R \) time is assumed to be constant one can apply a transformation of state and time to (4)-(5), yielding:

\[ \dot{w}(t) = 1 - \frac{w(t)w(t-1)}{2}kq(t-1) \]  

(7)

\[ \dot{q}(t) = \begin{cases} \frac{w(t) - c}{\max(w(t) - c, 0)}, & q > 0 \\ \max(w(t) - c, 0), & q = 0 \end{cases} \]  

(8)

where

\[ w = W, \quad q = Q/N, \quad \ell^{(\text{new})} = \ell^{(\text{old})}/R \]

and

\[ c = \frac{RC}{N}, \quad k = KN. \]
The importance of this transformation lies in the fact that the four model parameters \((K, N, C, R)\) are reduced to only two parameters \((k, c)\). This facilitates the study of the dependence of the attractors and their stability properties on the system’s parameters. It also allows to display stability regions w.r.t. all parameters in only one figure.

3 Linear Stability Analysis

In the normalized coordinates \((w, q)\) the unique equilibrium point is given by

\[
(w^*, q^*) = (c, \frac{2}{kc^2}) .
\]  

(9)

Linearization around it results in the second order differential equation in \(\ddot{q} = q - q^*\),

\[
\ddot{q}(t) + \frac{1}{c} \dot{q}(t) + \frac{1}{c} \dot{q}(t - 1) + \frac{ke^2}{2} \ddot{q}(t - 1) = 0 ,
\]

whose characteristic equation is given by

\[
H(\lambda) \triangleq \lambda^2 + \frac{1}{c} \lambda + \frac{1}{c} ke^{-\lambda} + \frac{ke^2}{2} e^{-\lambda} = 0 .
\]  

(10)

To characterize the stability region in the \((k, c)\)-plane we first fixed \(c > 0\) and consider the stability region as a function of \(k\). We have the following result:

**Proposition 1** For each value of \(c\), there exists exactly one stability interval as a function of \(k\), i.e. \(k \in (0, \bar{k}(c))\) with \(\bar{k}(c) \in \mathbb{R}^+\).

**Proof.** For \(k = 0\) the characteristic equation reduces to \(H(\lambda) = \lambda(\lambda + \frac{1}{c} + \frac{1}{ke^{-\lambda}})\). Following from [1], the rightmost eigenvalue is equal to zero and isolated. The continuity of this eigenvalue w.r.t. \(k\) implies the existence of a root function \(r(k)\), satisfying \(r(0) = 0\) and

\[
H(r(k)) = 0 .
\]

Differentiating this identity w.r.t. \(k\) at the point \(k = 0\) we arrive at:

\[
r'(0) = -\frac{e^3}{4} < 0 .
\]  

(11)

Therefore, the linearized system has one unstable real eigenvalue for small \(k < 0\). Instability follows for all \(k < 0\), because a zero eigenvalue cannot occur for \(k \neq 0\) and, as a consequence, eigenvalues can only cross the imaginary axis in complex conjugate pairs as \(k\) is varied.
Equation (11) also implies asymptotic stability for small $k > 0$. The stability can only be lost when eigenvalues cross the imaginary axis. When an imaginary eigenvalue $\lambda = j\omega$ would occur for, say $k = k^*$, one can compute:

$$\frac{d\Re(\lambda)}{dk} \bigg|_{\lambda = j\omega, k = k^*} = \frac{\omega^2 c^2}{2} \left( \frac{1}{e} - \frac{k^* c}{2} + \frac{\omega^2}{e^2} + \frac{k^* c^2 \omega^2}{2} \right) > 0. \quad (12)$$

Therefore, eigenvalues can only cross the imaginary axis towards instability as $k$ is increased, and thus only one stability interval is possible. Notice that (12) holds under the implicit assumption that the imaginary eigenvalues are simple. An easy calculation, which is omitted, excludes the non-generic case of having imaginary eigenvalues with a multiplicity larger than one.

The critical value $\tilde{k}(c)$ in Proposition 1 corresponds to a subcritical Hopf bifurcation of the original nonlinear system. By numerical continuation of such Hopf bifurcation in the two-parameter space $(k, c)$, the stability region can be computed and the result is shown in Figure 1. See [13] for theory on continuation and bifurcation analysis and [2] for a numerical tool.

While the technique of numerical continuation of Hopf bifurcations to separate stability/instability regions of a steady state solution in a two-parameter space is applicable in general, the method of D-subdivision [7] only applies to specific problems, as the example above, but allows to obtain analytical expressions for the boundary: when substituting $\lambda = j\omega$ in (10), some simple computations yield an implicit expression of the relation $\tilde{k}(c)$:

$$\begin{align*}
\tilde{k} &= \frac{c}{\omega \sin \omega} \left( \frac{1}{\omega^2 (\sin \omega)^2} \right), \\
\omega &\in (0, \pi).
\end{align*}$$

The above analysis of the linearized fluid model illustrates how analytical and numerical tools can complement each other to obtain a complete solution of an analysis problem. Numerically, we computed a curve separating stable and unstable parameter pairs in the $(k, c)$-plane. Analytically, we obtained qualitative information which proves that this curve bounds the whole stability region.

4 Nonlinear stability analysis

A first observation is that solutions of (7)-(8) cannot grown unbounded, even when the steady state solution is (locally) exponentially unstable:

**Proposition 2** All the solutions of the system (7)-(8) are bounded.

**Proof.** First we show by contradiction that the function $t \rightarrow w(t)$ is bounded along a solution. Therefore, assume that $w$ is unbounded and denote by $t = t_m$ the smallest time such that $w(t) = w_m$, where

$$w_m = |w_0| + c + 3 + 2/\sqrt{\tilde{k}}.$$
Figure 1: Linear stability region of the steady state solution in the \((k, c)\)-plane. The curve is obtained by computing a branch of Hopf-bifurcations, using the software package DDE-BIFTOOL [2].

Since \(\dot{w}(t) < 1\) for \(t \geq 0\), we have \(t_m \geq 2\) and \(w(t) \geq w_m - 2\), \(\forall t \in [t_m - 2, t_m - 1]\). Consequently

\[
q(t_m - 1) = q(t_m - 2) + \int_{t_m-2}^{t_m-1} \dot{q}(t) \, dt \geq \min_{t \in [t_m-2, t_m-1]} \dot{q}(t) \geq |w_0| + 1 + 2/\sqrt{k} \geq 1.
\]

This implies

\[
\dot{w}(t_m) \leq 1 - \frac{w_m(w_m - 2)k}{2} < 0
\]

and we have a contradiction, thus \(w\) is bounded. Similarly, assume that \(q\) is unbounded and denote by \(t = t_m\) the first time such that \(q(t) = q_m\), where \(q_m\) is a sufficiently large number. The boundedness of \(w\) implies the boundedness \(\dot{q}\), thus \(t_m\) must grow unbounded as \(q_m \to \infty\). Furthermore, there exist numbers \(\Delta t, M\) such that in the time-interval \([t_m - \Delta t, t_m]\), the first equation of (7) can be written as

\[
\dot{w}(t) = 1 - w(t)w(t-1)\alpha(t), \quad \alpha(t) > M. \quad (13)
\]

By taking \(q_m\) sufficiently large, also \(\Delta t\) and \(M\) can be chosen arbitrarily large. For sufficiently large values, (13) and the boundedness of \(w\) imply that \(w(t_m) < c\), as can be shown by the method
of steps. Thus \( \dot{q}(t_m) < 0 \) and we have a contradiction. ■

As a consequence there exist other attractors than the equilibrium point. Now we provide some qualitative and quantitative information on these attractors. Since a complete bifurcation analysis is beyond the scope of this paper, we focus on the particularities, due to the delayed damping term in (7) (period doubling route to chaos) and the discontinuity in the right-hand side of (8) (superstable limit cycles).

**Chaotic behavior.** In Figure 2 we show a bifurcation diagram of (7)-(8), when \( k \) is the free parameter and \( c = 1 \) is fixed, computed with the help of DDE-BIFTOOL [2]. Recall that for small \( k > 0 \) the unique steady state solution is locally asymptotically stable and that stability is lost in a subcritical Hopf bifurcation as \( k \) is increased. In the Hopf bifurcation a branch of stable periodic solutions emanates. The latter become unstable after a period doubling bifurcation, where a new branch of stable, period doubled periodic solutions emanates. A sequence of period doubling bifurcations ultimately leads to chaos. In Figure 3 we plot the chaotic attractor for \( k = 5.3 \).

Chaotic behavior is inherent to TCP/IP traffic. The work of Veres and Boda [15], where chaotic behavior was detected and analyzed in simulations with the ns-2 simulator [12], showed that TCP itself can cause or contribute to chaotic behavior as a deterministic system (in previous works a large number of ON-OFF sources with random periods were rather seen as a source of chaos in TCP). The analysis above shows that already the simple second-order deterministic model (4)-(5) exhibits chaos, which thus supports this proposition.

The chaotic behavior in (7)-(8) is clearly caused by the nonlinear delayed damping term in the first equation, which is proportional to \(-w(t)w(t-1)\) (notice that \( q \) is strictly larger than zero along the attractor shown in Figure 3, hence the discontinuity in the right hand size of (8) does not contribute). Therefore, it is expected that also the model proposed in [5],

\[
\dot{x}(t) = k(w - x(t - \tau)p(x(t - \tau))),
\]

which describes the dynamics of a collection of flows all using a single resource and sharing the same gain parameter \( k \), may exhibit chaotic behavior for particular choices of the function \( p() \), which can again be interpreted as the fraction of packets indicating congestion. Finally, an analogous instability mechanism leading to chaos occurs in the delayed logistic equation [14].

**Superstable limit-cycles.** When one lets the size of the attractor grow, by changing the system’s parameters, it may ultimately simplify to a non-smooth limit-cycle, which contains a segment where \( q \equiv 0 \). When \( q = 0 \) for a sufficiently large time, this limit-cycle is superstable, meaning that the effect of perturbations around it disappears in a finite time. In Figure 4 we plot such a superstable limit-cycle. The discontinuity in the right-hand side of (8) creates a mechanism which re-sets the state to the same value, each time before starting a new loop (more precisely to the segment \((w,q) = (c + \theta,0), \theta \in [-1,0]\)). Notice that such non-smooth solutions have a physical
3. Conclusions

A stability and bifurcation analysis of a fluid flow like model was performed. The stability region in the parameter space of the unique equilibrium point was completely characterized by combining analytical and numerical tools. A direct computation of periodic solutions and a continuation procedure revealed a period doubling route to chaos. The presence of a chaotic attractor in the low order, deterministic model supports the assumption that chaotic behavior is inherent to the TCP mechanism.
Figure 3: Chaotic attractor of the system (7)-(8) for $c = 1$ and $k = 5.3$.

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References


Figure 4: Attractor of (7)-(8) for $c = 5$ and $k = 0.06$ (full line), as well as a few trajectories (dashed lines) and the unstable steady state solution (‘+’). Around the attractor perturbations disappear in a finite time.


