Analysis of persistent bounded disturbance rejection for neutral delay systems

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Abstract—This paper deals with the problem of persistent bounded disturbance rejection for uncertain neutral delay systems. Using Lyapunov-Krasovskii functional method, we establish sufficient conditions in terms of linear matrix inequality (LMI) that simultaneously ensure ρ-performance and stability (delay-dependent and delay-independent). Particulary, for the delay-dependent case, an estimation of admissible maximum delay bound is established in terms of a generalized eigenvalue problem which can be solved with the efficient LMI toolbox. Similarly, we study the corresponding problem for the neutral delay systems with norm uncertainty. Finally, a numerical example is worked out to illustrate the efficiency of the proposed approach.

Keywords: LMI, neutral delay systems, persistent bounded disturbance rejection.

1 Introduction

In the past years, much interest has been focused on the problem of disturbance rejection induced by signal input (energy-bounded or peak-bounded) [2], [3]. Since many objectives in control engineering practice involve signal peak and the disturbance signals of the plants are persistent bounded in most cases, much attention has been paid to the problem of persistent bounded disturbance rejection for systems without time delay (see, e.g. [4], [8], [11], and the references therein). Ref. [6], [7] have studied the optimal L_1 (l_1) control problems for continuous- and discrete-time linear systems. Moreover [11] has discussed this problem for nonlinear systems. A recent paper [8] has studied disturbance rejection problem for Lurie system, but very little attention has been drawn to the problem of persistent bounded disturbance rejection for delay systems.

On the other hand, time delay ubiquitously occurs in many dynamical systems such as biological systems, chemical systems and electrical networks. Since small delay may lead to destabilization and poor performance, the stability and performance analysis of delay systems have received significant attention in recent years, see, for example [5], [10], [17]. Two types of stability conditions have been reported in the literature:
the so called delay-dependent conditions (the condition containing delay information) and delay-independent conditions (the condition without containing delay information). Generally, delay-independent conditions are simple and easy to be applied but conservative while those delay-dependent conditions are more complex but less conservative. One type of time-delay systems-neutral delay systems have been a well studied area in recent years [14], [12] because of its theoretical and practical importance. The systems can be described by neutral type systems include steam or water pipes, lumped parameters networks interconnected by transmission lines, systems of turbojet engine, etc. The effect of small delays on the stability properties of some closed-loop neutral systems has been considered in [13] and the references therein. Ref. [9] has developed sufficient conditions on the robust stability of uncertain neutral delay systems in terms of linear matrix inequality (LMI). However, there are few papers concerned with the problem of persistent bounded disturbance rejection of neutral delay systems.

Because of the importance of the neutral delay systems in the real application, we are going to study the problem of persistent bounded disturbance rejection of neutral delay systems in this paper. The organization of the paper is as follows. The preliminary results are given in Section 2. The main work is in Section 3: for both of delay-dependent and delay-independent cases, we give sufficient conditions on guaranteeing stability and achieving persistent bounded disturbance rejection performance of the neutral delay systems under consideration. An example is given in Section 4 to illustrate the efficiency and feasibility of our proposed approach. The last section gives conclusion of this paper.

Notation: In this paper, \( \mathbb{R} \) is the set of all real numbers, \( \mathbb{R}^n \) is the set of all \( n \)-tuples of real numbers, and \( \mathbb{R}^{m \times n} \) is the set of all real matrices with \( m \) rows and \( n \) columns. Denote by \( A^T \) the transpose of a matrix \( A \), by \( \sigma(A) \) the maximum singular value of a matrix \( A \). \( I \) denotes the unit matrix of appropriate dimension. \( \tilde{R}_{n \times \pi} = \tilde{R}([-\pi, 0], \mathbb{R}^p) \) denotes Banach space composed of continuous vector-valued functions from \([-\pi, 0] \) to \( \mathbb{R}^p \). Given a linear operator \( H : L_{\infty} \rightarrow L_{\infty} \), we define the induced \( L_{\infty} \) norm of \( H \) to be

\[
\|H\|_{\infty} := \sup_{\|w\|_{\infty} \leq 1} \|Hw\|_{\infty}
\]

see [1] for more details.

2 Preliminaries

Consider the following neutral delay system:

\[
\begin{cases}
\dot{x} - E\dot{x}(t - \tau) = Ax(t) + A_d x(t - \tau) + B\omega, \\
x(t_0 + s) = \psi(s), & s \in [-\pi, 0], \\
z = Cx + D\omega,
\end{cases}
\]

(1)

where \( x \in \mathbb{R}^n \), \( \omega \in \mathbb{R}^p \) are the state and disturbance input vectors, respectively. \( z \in \mathbb{R}^m \) is controlled output of the system (1). \( \tau > 0 \) is a given constant scalar. \( (t_0, \psi) \in \mathbb{R}^+ \times \tilde{R}_{n \times \pi} \), and \( \psi(\cdot) \) is the smoothly initial condition. \( A, A_d, E, B, C, D \) are all constant matrices with compatible dimensions. The \( L_{\infty} \) norm is defined by
The origin-reachable set \( R_w(0) \) of the system (1) is the set that the states of the system can reach from the origin for all admissible disturbances. A set is said to be a positive invariant set for a dynamical system, if \( x(t_0) \in \Omega \) implies that the trajectory \( x(t) \) of the system (1) remains in \( \Omega \) for all \( t > t_0 \). A set \( \Omega \) is said to be a robust attractor \( \Omega \) of the system (1) with respect to \( w \), if all the state trajectory of the system (1) initiating from any point outside of \( \Omega \) eventually enters and remains in it. Obviously, a robust attractor is positive invariant.

For the system (1), define \( \rho \)-performance set by:

\[
\Omega(\rho) := \{ x : \| z \|_\infty = \| Cx + D\omega \|_\infty \leq \rho, \forall \omega \in W \}.
\]

The system (1) is said to have \( \rho \)-performance, if \( \| z \|_\infty \leq \rho \) for all \( w \in W \). By the performance set defined above, in order to show that system (1) has \( \rho \)-performance, we only need to prove \( \Omega(\rho) \) contains \( R_w(0) \).

**Lemma 1 [15, 16].** For any positive scalar \( \alpha \) and any positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), the following inequalities hold.

\[
\begin{align*}
2x^T y & \leq \frac{1}{\alpha} x^T x + \alpha y^T y, \\
2x^T y & \leq x^T Q^{-1} x + y^T Q y,
\end{align*}
\]

where \( x \in \mathbb{R}^n, y \in \mathbb{R}^n \).

To guarantee that the difference operator \( D : [-\Gamma, 0] \to \mathbb{R}^n \) given by \( D(x_t) = x(t) - EX(t - \tau) + \int_{t-\tau}^t A_d x(t) dt \) is stable, we assume

\[
|\tau| |A_d| + |E| < 1,
\]

where \( |\cdot| \) is any matrix norm.

## 3 Main Results

The object of this section is to develop sufficient conditions on the stability and persistent bounded disturbance rejection performance of the neutral delay systems for both delay-dependent and delay-independent cases.

### 3.1 Analysis of persistent bounded disturbance rejection

Now for the neutral delay systems without uncertainty, we give our conclusion.

First, we consider the delay-independent case.

**Theorem 1.** If there exist a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) and a positive scalar \( \alpha \) satisfying the following matrix inequalities:

\[
\begin{bmatrix}
A^T P + PA + \alpha P & PA_d - A^T PE & PB \\
A_d^T P - E^T PA & -A_d^T PE - E^T PA_d & -E^T PB \\
B^T P & -B^T PE & -\alpha I
\end{bmatrix} < 0,
\]

(2)
then for any $\tau > 0$ the ellipsoid $\Omega_P = \{x : x^T Px \leq 1\}$ is an attractor of the system (1) for any $\tau > 0$ and the system (1) has $\rho$-performance. Furthermore, inequality (2) guarantees that the system (1) is delay-independently stable.

Proof: Let us consider the following Lyapunov-Krasovskii functional:

$$
V(x_t) = (x(t) - Ex(t - \tau))^T P (x(t) - Ex(t - \tau)).
$$

In the following of this paper, we denote $x(t)$ as $x$ and $x(t - \tau)$ as $x_\tau$ if there is no confusion. The time derivative of $V(x_t)$ along the trajectory of the system (1) is given by

$$
\dot{V}(x_t) = 2(Ax + A_d x_\tau + B\omega)^T P (x - Ex_t)
= x^T (A^T P + PA) x + 2x^T PB\omega + 2x^T PA_d x_\tau - 2x^T A^T P E x_t
- 2x^T A_d^T P E x_\tau - 2\omega^T B^T P E x_t
= \begin{bmatrix}
    x \\
    x_\tau \\
    \omega
\end{bmatrix}^T \begin{bmatrix}
    A^T P + PA + \alpha P & PA_d - A_d^T P E & PB \\
    A_d^T P - E^T PA & -A_d^T P E - E^T PA_d & -E^T PB \\
    B^T P & -B^T PE & -\alpha I
\end{bmatrix} \begin{bmatrix}
    x \\
    x_\tau \\
    \omega
\end{bmatrix}
- \alpha x^T Px + \alpha \omega^T \omega.
$$

Because $x^T Px > 1$ for $x \notin \Omega_P$, one obtains $\dot{V}(x) < 0$, if (2) holds. It’s obvious that the ellipsoid $\Omega_P = \{x : x^T Px \leq 1\}$ is an attractor of the system (1).

Further more, the negativity of the Lyapunov functional derivation does not use any information about the delay size and in conclusion, the stability property holds for any positive delay.

On the other hand, by Schur complement, condition (3) is equivalent to the existence of a positive scalar $\alpha$ such that

$$
\begin{bmatrix}
    \alpha P - C^T C & -C^T D \\
    -D^T C & (\rho^2 - \alpha) I - D^T D
\end{bmatrix} > 0.
$$

Thus we obtain

$$
0 < \alpha < \rho^2,
$$

and

$$
\alpha x^T Px + (\rho^2 - \alpha) \omega^T \omega - \|Cx + D\omega\|^2 > 0.
$$

It’s obvious that if $x^T Px \leq 1$ and $\omega^T \omega \leq 1$, we have $\|Cx + D\omega\| \leq \rho$ and thus $\Omega_P \subset \Omega(\rho)$.

Because $\Omega_P$ is a closed attractor which contains origin, it is a positive invariant set. While origin reachable set is the smallest positive invariant set that contains origin, we have $R_{\infty}(0) \subset \Omega_P \subset \Omega(\rho)$. Hence the system (1) has $\rho$-performance.

Remark 1: For any fixed $\alpha > 0$, (2) and (3) in Theorem 1 are LMIs.
In many control systems, stability conditions depend on delay time, it is necessary to study the delay-dependent stability. For the delay-dependent case, we have the following conclusion.

**Theorem 2.** For the neutral delay system (1) and a given positive scalar $\Gamma > 0$, if there exist positive definite matrices $P, Q_1, Q_2 \in \mathbb{R}^{n \times n}$ and a positive scalar $\alpha$ such that (3) and the following matrix inequality hold:

\[
\begin{bmatrix}
(A + A_d)^T P + P (A + A_d) & \Gamma (A + A_d)^T P & - (A + A_d)^T P E & PB \\
2 \Gamma A_d^T Q_1 A_d + Q_2 + \alpha P & \Gamma (A + A_d)^T P & - (A + A_d)^T P E & PB \\
\Gamma P (A + A_d) & - \Gamma Q_1 & 0 & 0 \\
-E^T P (A + A_d) & 0 & - \Gamma Q_1 & 0 \\
B^T P & 0 & - B^T E & - \alpha I \\
0 & 0 & 0 & \Gamma B^T P \\
0 & 0 & 0 & \Gamma PB & - \Gamma Q_1
\end{bmatrix} < 0,
\]

then for any $\tau : \Gamma \geq \tau > 0$ the ellipsoid $\Omega_\rho = \{x : x^T P x \leq 1\}$ is an attractor of the system (1) and the system (1) has $\rho$-performance. Furthermore, inequality (4) guarantees that the system (1) is delay-dependently stable.

**Proof:** Let

\[
\begin{align*}
\dot{z}(t) &= x(t) - E x(t - \tau) + \int_{t-\tau}^t A_d x(v) dv, \\
\dot{z}(t) &= (A + A_d) x(t) + B \omega.
\end{align*}
\]

We consider the following Lyapunov-Krasovskii functional:

\[
V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t),
\]

\[
V_1(x_t) = z^T(t) P z(t),
\]

\[
V_2(x_t) = 2 \int_{t-\tau}^t z^T(v) A_d^T Q_1 A_d x(v) dv ds,
\]

\[
V_3(x_t) = f_{t-\tau}^t x^T(v) Q_2 x(v) dv.
\]

The time derivative of $V(x_t)$ along the trajectory of the system (1) is given by

\[
\dot{V}(x_t) = \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t)
\]

\[
= 2 ((A + A_d) x + B \omega)^T P (x - E x(t - \tau) + \int_{t-\tau}^t A_d x(v) dv) + 2 \tau x^T A_d^T Q_1 A_d x(t) + 2 \tau x^T Q_1 A_d x(t - \tau) + x^T Q_2 x - x^T Q_2 x(t)
\]

\[
= x^T ((A + A_d)^T P + P (A + A_d) + 2 \tau A_d^T Q_1 A_d + Q_2) x + 2 \tau T \rho - x^T (A + A_d)^T P E x(t - \tau) - 2 \omega^T B^T E P x(t) + 2 \tau x^T (A + A_d)^T P \int_{t-\tau}^t A_d x(v) dv + 2 \omega^T B^T P \int_{t-\tau}^t A_d x(v) dv - 2 \tau x^T (A + A_d)^T P \int_{t-\tau}^t A_d x(v) dv - x^T Q_2 x.
\]

By Lemma 1, the following inequalities hold.

\[
2 x^T (A + A_d)^T P \int_{t-\tau}^t A_d x(v) dv \leq \tau x^T (A + A_d)^T P Q_1^{-1} P (A + A_d) x + \int_{t-\tau}^t x^T(v) A_d^T Q_1 A_d x(v) dv,
\]

\[
2 \omega^T B^T P \int_{t-\tau}^t A_d x(u) dv \leq \tau \omega^T B^T P Q_1^{-1} P B \omega + \int_{t-\tau}^t x^T(v) A_d^T Q_1 A_d x(v) dv.
\]

It follows that

\[
\dot{V}(x_t) \leq \begin{bmatrix} x \\ x_t \\ \omega \end{bmatrix}^T \begin{bmatrix} \Pi & -(A + A_d)^T P E & PB \\
-E^T P (A + A_d) & - Q_2 & -E^T P B \\
B^T P & - B^T E & \Sigma \end{bmatrix} \begin{bmatrix} x \\ x_t \\ \omega \end{bmatrix} - \alpha x^T P x + \alpha \omega^T \omega,
\]
where
\[
\Pi = \begin{pmatrix} (A + A_d)^T P + P(A + A_d) + 2\tau A_d^T Q_1 A_d + Q_2 + \alpha P \\
+ \tau(A + A_d)^T P Q_1^{-1} P(A + A_d) \end{pmatrix}, \\
\Xi = \begin{pmatrix} \tau B^T P Q_1^{-1} P B - \alpha I \end{pmatrix}.
\]

Since \(x^T P x > 1\) for \(x \notin \Omega_P\), one obtains \(\dot{V}(x) < 0\), if
\[
\begin{bmatrix}
\Pi & -(A + A_d)^T P E & P B \\
-E^T P (A + A_d) & -Q_2 & -E^T P B \\
B^T P & -B^T P E & \Xi
\end{bmatrix} < 0.
\]

By Schur complement, it is equivalent to (4) where \(\tau\) is replaced by \(\Gamma\).

Hence, if (3) – (4) hold, the ellipsoid \(\Omega_P\) is an attractor of the system (1) for \(\Gamma \geq \tau > 0\), and systems (1) is delay-dependently stable.

As the proof of Theorem 1, for any \(\tau\) satisfying \(\Gamma \geq \tau > 0\), the system (1) has \(\rho\)-performance.

**Remark 2:** Conservative overbounds have been introduced by tacking the same variable \(Q_1\) in the inequalities in this proof, and only sufficient conditions for the solvability of the control problem are given.

### 3.2 Analysis of persistent bounded disturbance rejection with norm-bounded uncertainty

Consider the following system:
\[
\begin{aligned}
\dot{x} - E\dot{x}(t - \tau) &= Ax(t) + A_d x(t - \tau) + B_i p + B\omega, \\
x(t_0 + s) &= \psi(s), \quad s \in [-\tau, 0], \\
z &= Cx + D\omega, \\
q &= C_1 x + D_{11} p, \\
p &= \Delta q, \quad ||\Delta|| \leq 1, \quad \bar{\sigma}(D_{11}) < 1,
\end{aligned}
\]

where \(C_1\) and \(D_{11}\) are matrices with compatible dimensions, \(p\) and \(q\) are uncertain input and output of the system. Such form of description for uncertainties can be found in [4].

Now let us present the sufficient conditions on guaranteeing delay-independent stability and achieving \(\rho\)-performance.

**Theorem 3.** If there exist positive definite matrices \(P, Q \in \mathbb{R}^{n \times n}\), a positive scalar \(\alpha\) such that (3) and the following matrix inequality hold:
\[
\begin{bmatrix}
A^T P + PA + \alpha P + Q & PA_d - A^T P E & PB & PB_1 & C_1^T \\
A_d^T P - E^T PA & -A_d^T P E - E^T PA_d - Q & -E^T P B & -E^T P B_1 & 0 \\
B^T P & -B^T P E & -\alpha I & 0 & 0 \\
B_{11}^T P & -B_{11}^T P E & 0 & -I & D_{11}^T \\
C_1 & 0 & 0 & D_{11} & -I
\end{bmatrix} < 0,
\]

**Remark 4:**

1. The results in Theorems 1–3 can be extended to the case where \(\omega\) is not persistent bounded, but \(\psi(s)\) is persistent bounded disturbance rejection with norm-bounded uncertainty.
2. The approach is based on the Lyapunov-Krasovskii functional method and the use of auxiliary functions to derive the sufficient conditions for stability and performance.
then for any \( \tau > 0 \) the ellipsoid \( \Omega_p = \{ x : x^T P x \leq 1 \} \) is a robust attractor of the system (5) and the system (5) has robust \( \rho \)-performance. Furthermore inequality (6) guarantees that the system (5) is robust delay-independently stable.

**Proof:** Let us consider the following Lyapunov-Krasovskii functional:

\[
\dot{V}(x_t) = V(x_t) + \int_0^t (q^T(s)q(s) - p^T(s)p(s))ds,
\]

where

\[
V(x_t) = (x(t) - Ex(t - \tau))^T P (x(t) - Ex(t - \tau)) + \int_{t-\tau}^t x^T(v)Qx(v)dv.
\]

Because \( \dot{V}(x_t) = V(x_t) + q^T(s)q(s) - p^T(s)p(s) \), \( p = \Delta q \) and \( \| \Delta \| \leq 1 \), one has \( \dot{V}(x_t) < 0 \) if \( \dot{V}(x_t) < 0 \). The time derivative of \( \dot{V}(x) \) along the trajectory of the system (5) is given by

\[
\dot{V}(x_t) = 2(Ax + A_d x_\tau + B\omega + B_1 p)^T P (x - Ex_t) + q^T q - p^T p + x^T Q x - x_\tau^T Q x_\tau
\]

\[
= x^T (A^T P + P A + Q) x + 2x^T P B \omega + 2x^T P A_d x_\tau + 2x^T P B_1 p - 2x^T A^T P E x_t - x_\tau^T Q x_\tau
\]

\[
- 2x^T A_d^T P E x_t - 2x^T B^T P E x_t - 2p^T B_1^T P E x_t + (C_1 x + D_{11} p)^T (C_1 x + D_{11} p) - p^T p
\]

\[
= \begin{bmatrix} x \end{bmatrix}^T \begin{bmatrix} \Lambda & P A_d - A^T P E & PB & PB_1 + C_1^T D_{11} \\ A_d^T P - E^T P A & -A_d^T P E - E^T P A_d^T - Q - E^T P B & -E^T P B - E^T P B_1 \\ B^T P & -B^T P E & -\alpha I & 0 \\ B_1^T P + D_{11}^T C_1 & -B_1^T P E & 0 & D_{11}^T D_{11} - I \end{bmatrix} \begin{bmatrix} x \\ x_\tau \\ \omega \\ p \end{bmatrix}
\]

where \( \Lambda = A^T P + P A + \alpha P + C_1^T C_1 + Q \). Because \( x^T P x > 1 \) for \( x \notin \Omega_p \), one obtains \( \dot{V}(x) < 0 \), if

\[
\begin{bmatrix} \Lambda & P A_d - A^T P E & PB & PB_1 + C_1^T D_{11} \\ A_d^T P - E^T P A & -A_d^T P E - E^T P A_d^T - Q - E^T P B & -E^T P B - E^T P B_1 \\ B^T P & -B^T P E & -\alpha I & 0 \\ B_1^T P + D_{11}^T C_1 & -B_1^T P E & 0 & D_{11}^T D_{11} - I \end{bmatrix} < 0.
\]

By Schur complement, it is equivalent to (6). It’s obvious that the ellipsoid \( \Omega_p = \{ x : x^T P x \leq 1 \} \) is a robust attractor of the system (5).

As the proof of Theorem 1, the system (5) has robust \( \rho \)-performance and is delay-independently stable.

**Remark 3:** For any fixed \( \alpha > 0 \) (3) and (6) in Theorem 3 are LMIs.

For the delay-dependent case, similar analysis is given as follows.

**Theorem 4.** For the neutral delay system (5) and a given positive scalar \( \Gamma > 0 \), if there exist positive definite matrices \( P, Q_1, Q_2 \in \mathbb{R}^{n \times n} \), a positive scalar \( \alpha \) such that (3) and the following matrix inequality hold:

\[
\begin{bmatrix} \Phi & \Gamma (A + A_d)^T P & - (A + A_d) E P E & PB & 0 & PB_1 & 0 & C_1^T \\ \Gamma P (A + A_d) & -\Gamma Q_1 & 0 & 0 & 0 & 0 & 0 \\ -E^T P (A + A_d) & 0 & -E^T Q_2 E & -E^T P B & 0 & -E^T P B_1 & 0 & 0 \\ B^T P & 0 & -B^T P E & -\alpha I & \Gamma B^T P & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma P B & -\Gamma Q_1 & 0 & 0 & 0 \\ B_1^T P & 0 & -B_1^T P E & 0 & 0 & -I & \Gamma B_1^T P & D_{11}^T \\ 0 & 0 & 0 & 0 & \Gamma P B_1 & -\Gamma Q_1 & 0 \\ C_1 & 0 & 0 & 0 & 0 & D_{11} & 0 & -I \end{bmatrix} < 0, \quad (7)
\]
where $\Phi = (A + A_d)^T P + P(A + A_d) + 2\Gamma A_d^T Q_1 A_d + E^T Q_2 E + \alpha P$, then for any $\tau \geq \tau > 0$ the ellipsoid $\Omega_{\tau} = \{x : x^T P x \leq 1\}$ is a robust attractor of the system (5), $\Omega_{\tau} \subseteq \Omega(\rho)$ and the system (5) has robust $\rho$-performance. Furthermore inequality (7) guarantees that the system (5) is robustly delay-dependently stable.

**Proof:** Let
\[
\dot{z}(t) = x(t) - Ex(t - \tau) + \int_{t-\tau}^{t} A_d x(v) dv,
\]
\[
\dot{\zeta}(t) = (A + A_d)x(t) + B\omega + B_1 p.
\]
Take the following Lyapunov-Krasovskii functional:
\[
\dot{V}(x(t)) = V(x(t)) + \int_{t-\tau}^{t} (q^T(s)q(s) - p^T(s)p(s))ds,
\]
where
\[
V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)),
\]
\[
V_1(x(t)) = z^T(t)Pz(t),
\]
\[
V_2(x(t)) = 3\int_{t-\tau}^{t} x^T(t)A_d^T Q_1 A_d x(t)dvds,
\]
\[
V_3(x(t)) = \int_{t-\tau}^{t} x^T(t) E^T Q_2 E x(t)dv.
\]
Because $\dot{V}(x(t)) = \dot{V}(x(t)) + q^T(s)q(s) - p^T(s)p(s)$, $p = \Delta q$ and $\|\Delta\| \leq 1$, one has $\dot{V}(x(t)) < 0$ if $\dot{V}(x(t)) < 0$. The time derivative of $\dot{V}(x(t))$ along the trajectory of the system (5) is given by
\[
\dot{V}(x) = V_1(x) + V_2(x) + V_3(x) + q^T q - p^T p
\]
\[
= 2((A + A_d)x + B\omega + B_1 p)^T P(x - Ex + \int_{t-\tau}^{t} A_d x(v) dv) + x^T E^T Q_2 E x - x^T E^T Q_2 E x
\]
\[
+ 3\tau x^T A_d^T Q_1 A_d x - 3\int_{t-\tau}^{t} x^T(t) A_d^T Q_1 A_d x(t)dv + (C_1 x + D_{11} p)^T (C_1 x + D_{11} p) - p^T p
\]
\[
= x^T (A + A_d)^T P + P(A + A_d) + 3\tau A_d^T Q_1 A_d + E^T Q_2 E x + 2x^T P B\omega - 2x^T (A + A_d)^T P E x
\]
\[
+ 2x^T PB_1 p - 2\omega^T B^T P E x - 2\omega^T B_1^T P E x + 2x^T (A + A_d)^T P \int_{t-\tau}^{t} A_d x(v) dv
\]
\[
+ 2\omega^T B^T P \int_{t-\tau}^{t} A_d x(v) dv + 2p^T B_1^T P \int_{t-\tau}^{t} A_d x(v) dv - 3\int_{t-\tau}^{t} x^T(t) A_d^T Q_1 A_d x(t) dv
\]
\[
- x^T E^T Q_2 E x + (C_1 x + D_{11} p)^T (C_1 x + D_{11} p) - p^T p.
\]

By Lemma 1, the following inequalities hold.

\[
2x^T (A + A_d)^T P \int_{t-\tau}^{t} A_d x(v) dv \leq \tau x^T (A + A_d)^T P Q_1^{-1} P(A + A_d)x + \int_{t-\tau}^{t} x^T(t) A_d^T Q_1 A_d x(t) dv,
\]
\[
2\omega^T B^T P \int_{t-\tau}^{t} A_d x(v) dv \leq \tau \omega^T B^T P Q_1^{-1} P B\omega + \int_{t-\tau}^{t} x^T(t) A_d^T Q_1 A_d x(t) dv,
\]
\[
2p^T B^T P \int_{t-\tau}^{t} A_d x(v) dv \leq \tau p^T B_1^T P Q_1^{-1} P B_1 p + \int_{t-\tau}^{t} x^T(t) A_d^T Q_1 A_d x(t) dv.
\]

It follows that
\[
\dot{V}(x) \leq \begin{bmatrix} x^T & -\gamma^T & -(A + A_d)^T P E & PB & PB_1 + C_1^T D_{11} \\ x \tau & -E^T P (A + A_d) & -E^T Q_2 E & -E^T P B & -E^T P B_1 \\ \omega & B^T P & -B^T P E & -\alpha I + \tau B^T P Q_1^{-1} P B & 0 \\ p & B_1^T P + D_{11}^T C_1 & -B_1^T P E & 0 & \Delta \\ -\alpha x^T P x + \alpha \omega^T \omega & \end{bmatrix} \begin{bmatrix} x \\ x \tau \\ \omega \\ p \\ \rho \end{bmatrix}
\]

where
\[
Y = (A + A_d)^T P + P(A + A_d) + 3\tau A_d^T Q_1 A_d + E^T Q_2 E + \alpha P + C_1^T C_1 + \tau (A + A_d)^T P Q_1^{-1} P(A + A_d),
\]
\[
\Delta = D_{11}^T D_{11} - I + \tau B_1^T P Q_1^{-1} P B_1.
\]
Now let us estimate the admissible maximum delay bound for the system described by:

\[
\begin{bmatrix}
    Y & -(A + A_d)^T P E & PB & PB_1 + C_1^T D_{11} \\
    -E^T P(A + A_d) & -E^T Q_2 E & -E^T P B & -E^T P B_1 \\
    B^T P & -B^T P E & -\alpha I + \tau B^T P Q_1^{-1} P B & 0 \\
    B^T P + D_{11}^T C_1 & -B_1^T P E & 0 & \Delta
\end{bmatrix} < 0.
\]

By Schur complement, it is equivalent to (7) where \( \tau \) is replaced by \( \Gamma \).

Hence, if (3) and (7) hold, the ellipsoid \( \Omega_P \) is a robust attractor of the system (1) for any \( \Gamma \geq \tau > 0 \), and systems (5) is robustly delay-dependently stable.

As the proof of Theorem 1, for any \( \tau \) satisfying \( \Gamma \geq \tau > 0 \), the system (5) has robust \( \rho \)-performance.

**Remark 4:** For any fixed \( \alpha > 0 \), (3) and (7) in Theorem 4 are LMIs.

### 3.3 Estimation for the maximum delay size

Now let us estimate the admissible maximum delay bound \( \Gamma \) such that the system we consider is delay-dependently stable and has \( \rho \)-performance for any \( \tau \) satisfying \( \Gamma \geq \tau > 0 \). We give an estimation of the maximum delay size for the system (5), similar result can be obtained for the system (1). The problem can be described by:

\[
\max \Gamma > 0
\]

s.t. there exist positive definite matrices \( P, Q_1, Q_2 \in R^{n \times n} \), and a positive constant scalar \( \alpha \) satisfying matrix inequalities (3) and (7).

Let \( \delta = \Gamma^{-1} \), this problem can be transformed into a generalized eigenvalue problem:

\[
\min \delta > 0
\]

s.t. there exist positive definite matrices \( P, Q_1, Q_2 \in R^{n \times n} \), and a positive constant scalar \( \alpha \) satisfying matrix inequalities (3) and

\[
\begin{bmatrix}
    \Sigma & A_d^T \delta Q_1 & (A + A_d)^T P & (A + A_d)^T P E & PB & 0 & PB_1 & 0 & C_1^T \\
    Q_1 A_d & \frac{1}{2} \delta Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    P(A + A_d) & 0 & -\delta Q_1 & 0 & 0 & 0 & 0 & 0 \\
    -E^T P(A + A_d) & 0 & 0 & -E^T Q_2 E & -E^T P B & 0 & -E^T P B_1 & 0 & 0 \\
    B^T P & 0 & 0 & -B^T P E & -\alpha I & B^T P & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & PB & 0 & -\delta Q_1 & 0 & 0 \\
    B_1^T P & 0 & 0 & -B_1^T P E & 0 & 0 & -I & B_1^T P & D_{11}^T \\
    0 & 0 & 0 & 0 & 0 & 0 & PB_1 & -\delta Q_1 & 0 \\
    C_1 & 0 & 0 & 0 & 0 & 0 & D_{11} & 0 & -I
\end{bmatrix} < 0,
\]

where \( \Sigma = (A + A_d)^T P + P(A + A_d) + E^T Q_2 E + \alpha P \).
4 Illustrative examples

To illustrate the efficiency of our proposed approach, now we consider the neutral delay system (5) for both delay-independent case and delay-dependent case. We choose the following parameters:

\[
A = \begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 & 0.07 \\ 0.01 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & -0.01 \\ 1 & 0.1 \end{bmatrix},
\]

\[
A_d = \begin{bmatrix} -0.001 & -0.001 \\ -0.001 & -0.01 \end{bmatrix}, \quad C = \begin{bmatrix} 0.01 & -0.1 \\ -0.09 & 0.01 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0.001 & 0.01 \end{bmatrix},
\]

\[
D = \begin{bmatrix} -0.1 & -0.001 \\ -0.1 & 0.1 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.
\]

For the delay-independent case, with \( \rho = 1 \) and \( \alpha = 0.8 \), we obtain \( P, Q \) by solving the LMIs in Theorem 3 with the LMI toolbox [4]:

\[
P = \begin{bmatrix} 0.2168 & 0.0439 \\ 0.0439 & 0.2237 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.9534 & -0.0181 \\ -0.0181 & 0.9860 \end{bmatrix}.
\]

Hence we can obtain a robust attractor \( \Omega_P \) and the delay-independently stable system (5) has robust \( \rho \)-performance with \( \rho = 1 \).

For the delay-dependent case, with the same parameter, solving the generalized eigenvalue problem proposed in section 3.3, we obtain the admissible maximum delay bound \( \Gamma \) and positive definite matrices \( P, Q_1, Q_2 \) that satisfy Theorem 4:

\[
P = \begin{bmatrix} 0.1012 & 0.0150 \\ 0.0150 & 0.0985 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.9913 & 0 \\ 0 & 0.9913 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 11.2696 & -0.5129 \\ -0.5129 & 4.8789 \end{bmatrix}, \quad \Gamma = 1.1474.
\]

So if the delay size is smaller than \( \Gamma \), then under conditions (3) and (7), the system (5) has robust \( \rho \)-performance with \( \rho = 1 \).

5 Conclusion

By Lyapunov-Krosovskii method, we mainly considered the problem of persistent bounded disturbance rejection performance and stability for a class of neutral delay systems. We developed sufficient conditions on guaranteeing stability (delay-dependent and delay-independent) and achieving persistent bounded disturbance rejection performance. Moreover, persistent bounded disturbance rejection for neutral delay systems with norm uncertainty was analyzed using similar technique. For delay-dependent case, an estimation of the maximum admissible delay bound was given in terms of a standard generalized eigenvalue problem which can be solved numerically with the efficient LMI toolbox. Finally, a numerical example was given to illustrate the efficiency of the proposed approach.
References


