Derivation of an $\mathcal{H}_2$ Error Bound for Model Reduction of Second Order Systems*

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Abstract

This note derives an error bound in the $\mathcal{H}_2$ norm for the reduction of a second order dynamical system. The reduction preserves second order form and is based upon a dominant eigenspace of a controllability gramian. An equivalent frequency domain definition of this gramian is obtained from the Parseval theorem and this form is key to the derivation of the bound.

1 Introduction

We consider second order systems of the form

$$\Sigma: \begin{cases} M\ddot{x} + G\dot{x} + Kx = Bu \\ y(t) = Cx(t) \end{cases} \quad (1.1)$$

where $M, G, K \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}$.

One approach to model reduction is to convert this system to an equivalent first order linear time invariant (LTI) system and then to apply existing reduction techniques to reduce the first order system. A difficulty with this approach is that the second order form is lost in the reduction process and there is a mixing of the state variables and their first derivatives. Several authors have noted undesirable consequences and have endeavored to provide either direct reductions of the second order form or structure preserving reductions of the equivalent first order system. This has required several alternative definitions of a controllability gramian. However, while successful structure preserving reductions have been obtained, none of these possess a priori error bounds [1, 2, 3].

Here, we give a direct reduction of the second order system based upon the dominant eigenspace of a controllability gramian that does lead to an error bound in the $\mathcal{H}_2$ norm.

2 Direct Reduction of a Second Order System

The Controllability Gramian:

It is readily verified that the transfer function for (1.1) in the frequency domain is

$$H(s) = C(Ms^2 + Gs + K)^{-1}B.$$

Moreover, in the frequency domain, the input to state and the input to output maps are

$$x(s) = (Ms^2 + Gs + K)^{-1}Bu(s), \quad y(s) = H(s)u(s).$$

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During the remainder of this discussion, we shall assume a single input single output (SISO) system (m = p = 1) to avoid certain technical issues. Then, for impulse response \( u(t) = \delta(t) \) and \( u(s) = 1 \),

\[
x(s) = (Ms^2 + Gs + K)^{-1}B, \quad \text{and} \quad y(s) = H(s).
\]

In the time domain, the state to output map is \( y(t) \leftarrow Cx(t) \) and we consider

\[
\int_0^\infty y^*y \, dt = tr \left\{ \int_0^\infty yy^* \, dt \right\} = tr \left\{ \int_0^\infty Cxx^*C^* \, dt \right\}.
\]

Define \( F(s) := (Ms^2 + Gs + K)^{-1}B \). From the Parseval theorem,

\[
tr \left\{ \int_0^\infty yy^* \, dt \right\} = tr \left\{ \int_0^\infty Cxx^*C^* \, dt \right\} = tr \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} C F(i\omega) F(i\omega)^* C^* \, d\omega \right\}.
\]

The controllability gramian is defined to be the gramian of the state \( x(t) \) under impulse response:

\[
P := \int_0^\infty x(t)x(t)^* \, dt.
\]

We seek a direct formula for this gramian involving the system parameters \( M, G, K, \) and \( B \). There is no useful formula in the time domain but there is one in the frequency domain (again given by the Parseval theorem)

\[
P = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) F(i\omega)^* \, d\omega.
\]

**Direct Second Order Reduction via the Controllability Gramian:**

To achieve model reduction, we consider the eigen-decomposition of the symmetric positive definite matrix \( P \).

Let

\[
P = VSV^T, \quad \text{with} \quad V = [V_1, V_2] \quad \text{and} \quad S = \text{diag}(S_1, S_2),
\]

where the diagonal elements of \( S \) are in decreasing order, and \( V \) is orthogonal.

Now, define \( A(s) := V^T (Ms^2 + Gs + K) V \) and partition

\[
A(s) = \begin{bmatrix}
A_{11}(s) & A_{12}(s) \\
A_{21}(s) & A_{22}(s)
\end{bmatrix}, \quad [C_1, C_2] = CV, \quad \text{and} \quad V^TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\]

according to \( A_{ij}(s) = V_i^T A(s) V_j \) and \( B_i = V_i^T B \). Let us put \( \hat{A}(s) \equiv A_{11}(s) \). Now, to avoid cumbersome notation, we shall re-define \( B \leftarrow V_i^TB, C \leftarrow CV \) and \( F(s) \leftarrow V_i^TF(s) \). Note the relationship

\[
A(s)F(s) \equiv B \quad \text{and partition} \quad F(s) = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}
\]

to conform with the partition of \( A(s) \) and \( V^TB \). Next, since \( S = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega)F(i\omega)^* \, d\omega \), we observe the following

\[
S_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(i\omega)F_1(i\omega)^* \, d\omega, \quad (2.1)
\]

\[
S_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(i\omega)F_2(i\omega)^* \, d\omega, \quad (2.2)
\]

\[
0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(i\omega)F_1(i\omega)^* \, d\omega. \quad (2.3)
\]
Moreover,

\[
\text{tr}\{ S_1 \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \| F_1(i\omega) \|^2 \, d\omega, \quad \text{and} \quad \text{tr}\{ S_2 \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \| F_2(i\omega) \|^2 \, d\omega. \tag{2.4}
\]

Let us now construct a reduced model according to

\[
\check{M} = V_1^T \check{M} V_1, \quad \check{G} = V_1^T \check{G} V_1, \quad \check{K} = V_1^T \check{K} V_1, \quad \check{B} = V_1^T B = B_1, \quad \check{C} = C V_1 = C_1.
\]

We shall define

\[
\check{A}(s) := \check{M}s^2 + \check{G}s + \check{K} = A_{11}(s) \quad \text{and then define} \quad \check{F} \text{ via} \quad \check{A}(s)\check{F}(s) = B_1.
\]

Observe that the controllability gramian of the reduced system is

\[
\check{F} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{F}(i\omega)\check{F}(i\omega)^* \, d\omega.
\]

Also, observe from the defining equation for \( F(s) \) we have

\[
F_1(s) = A_{11}(s)^{-1}[B_1 - A_{12}F_2(s)] = \check{F}(s) - A_{11}(s)^{-1}A_{12}F_2(s).
\]

We shall put \( W(s) := A_{11}(s)^{-1}A_{12} \) and, for the moment, we assume that \( \sup_\omega \| W(i\omega) \|_2 \leq W_0 \) is finite. Thus,

\[
\check{F}(s) = F_1(s) + W(s)F_2(s).
\]

If, as we expect, the reduced system has all poles strictly in the open left half plane, then this is a valid assumption. For \( M, G, K \) all symmetric and positive definite, this condition is automatically satisfied.

**Bounding the \( H_2 \) Norm of the Error System:**

Let \( \tilde{y} = \check{C}\tilde{x} \) be the output from the reduced system and let \( y = Cx \) the the output from the full system where both outputs are the result of the same input \( u \). In the time domain,

\[
y(t) - \tilde{y}(t) = Cx(t) - \check{C}(t)\check{x}(t)
\]

and in the frequency domain

\[
y(s) - \tilde{y}(s) = C(Ms^2 + Gs + K)^{-1}Bu - \check{C}(Ms^2 + Gs + \check{K})^{-1}Bu.
\]

Thus

\[
H_e(s) := C(Ms^2 + Gs + K)^{-1}B - \check{C}(Ms^2 + Gs + \check{K})^{-1}B
\]

is the transfer function of the error system \( E = \Sigma - \tilde{\Sigma} \).

The \( H_2 \)-norm in the of the error system in the frequency domain is given by

\[
\| E \|_{H_2}^2 = \text{tr}\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} H_e(i\omega)H_e(i\omega)^* \, dt \} = \text{tr}\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}\{ CFI(i\omega)(CF(i\omega))^* \} \, dt \} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}\{ CF(i\omega)(\check{C}\check{F}(i\omega))^* \} \, dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}\{ \check{C}\check{F}(i\omega)(\check{C}\check{F}(i\omega))^* \} \, dt
\]

Each of the three terms in this expression can be simplified as follows:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}\{ CF(i\omega)(CF(i\omega))^* \} \, dt = \text{tr}\{ C_1S_1C_1^* \} + \text{tr}\{ C_2S_2C_2^* \},
\]
and
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{C F(i\omega)(CF(i\omega))^* \} dt = tr\{C_1 S_1 C_1^*\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{C F(i\omega)F_2(i\omega)^* W(i\omega)^* C_1^* \} dt,
\]
and
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{\tilde{C} F(i\omega)(\tilde{C} F(i\omega))^* \} dt = tr\{C_1 S_1 C_1^*\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} 2tr\{C_1 F_1 (i\omega) F_2(i\omega)^* W(i\omega)^* C_1^* \} dt
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{(C_1 W(i\omega) F_2(i\omega))(C_1 W(i\omega) F_2(i\omega))^* \} dt.
\]
From this it is possible to derive
\[
\|\mathcal{E}\|_{\mathcal{H}_2}^2 = tr\{C_2 S_2 C_2^*\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{(C_1 W(i\omega) - 2C_2) F_2(i\omega)(C_1 W(i\omega) F_2(i\omega))^* \} dt.
\]
Finally, we can bound
\[
\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} tr\{(C_1 W(i\omega) - 2C_2) F_2(i\omega)(C_1 W(i\omega) F_2(i\omega))^* \} dt \right|
\leq \sup_{i\omega} \| (C_1 W(i\omega))^* (C_1 W(i\omega) - 2C_2) \|_2 tr\{S_2\},
\]
to obtain our main result
\[
\|\mathcal{E}\|_{\mathcal{H}_2}^2 \leq C_0 tr\{S_2\}, \quad \text{with} \quad C_0 = \sup_{i\omega} \| (C_1 W(i\omega))^* (C_1 W(i\omega) - 2C_2) \|_2 + \| C_2 \|_2^2.
\]

3 Summary

We have presented a direct reduction of a second order system based upon the dominant eigenspace of a controllability gramian that preserves second order form. Moreover, we have given an error bound in the $\mathcal{H}_2$ norm for this reduction. An equivalent definition of the gramian was obtained through a Parseval relationship and this was key to the derivation of our bound. Here, we just sketched the derivations. Full details may be found in [] where we also discuss computational details and ideas.

References

