On Strong Stabilization for Linear Time-Varying Systems

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1 Introduction

The strong stabilization problem for linear systems has been studied in various frameworks (See [5] for a discussion of the origins of the problem.) The common point of reference is the well-known "Youla" characterization of all stabilizing compensators of a given plant ([5],[1]). In this context it was natural to ask whether among these compensators there exist stable systems. The deepest result in this area up to this point is that of S. Treil ([4]). For single input single output LTI distributed systems on the unit circle, every stabilizable linear system (the underlying field is the complex numbers) is strongly stabilizable, that is, stabilizable by a stable compensator. These results were recently extended to the half-plane as well in some other directions by A. Quadrat ([2]). The multi-input multi-output problem is still open.

I will consider here the discrete time-varying analogue of this problem and will begin from the operator theoretic formulation of the problem without getting into the derivation of this formulation from first principles. This can be seen in [1]. Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space and let $\mathcal{N} = \{P_k\}$ be a sequence of increasing projections on $\mathcal{H}$ which includes 0 and I and is closed under the standard lattice operations for orthogonal projections on $\mathcal{H}$. This sequence defines a notion of causality for operators on $\mathcal{H}$. A bounded linear operator $T$ on $\mathcal{H}$ is causal if for each projection $P_k \in \mathcal{N}$, $P_k T = P_k T P_k$. The set of causal operators satisfying this requirement is a weakly closed algebra $\mathcal{C}$ containing the identity.

Suppose $A, B \in \mathcal{C}$ such that $\begin{bmatrix} A & B \end{bmatrix}$ is an operator from $\mathcal{H} \oplus \mathcal{H}$ into $\mathcal{H}$ has a right inverse $\begin{bmatrix} X & Y \end{bmatrix}$ with $X, Y \in \mathcal{C}$. Such an inverse is generally not unique. The strong stabilization problem is: Can $X$ be chosen to be an invertible operator from $\mathcal{C}$? Equivalently: Does there exist $C \in \mathcal{C}$ such that $A + BC$ is invertible in $\mathcal{C}$? For $\mathcal{N} = \{0, I\}$, $\mathcal{C} = \mathcal{L}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$ and in this case the answer is negative. The most complete analysis is given in [3], Theorem 2.

Theorem 1.1 let $A, B \in \mathcal{L}(\mathcal{H})$ such that $\begin{bmatrix} A & B \end{bmatrix}$ is right invertible. (a) If $B$ is compact, there exists $C \in \mathcal{L}(\mathcal{H})$ such that $A + BC$ is invertible if and only if $A$ is Fredholm with $\text{ind} S = 0$. 

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(b) If $B$ is not compact there exists $C \in \mathcal{L}(\mathcal{H})$ such that $A + BC$ is invertible if and only if there exists a non-compact operator $G$ such that $\text{Im}AG \subset \text{Im}B$.

We consider the case where $\mathcal{H}$ is the sequence space $\mathcal{H} = l^2(0, \infty; \mathbb{C}^n)$, and the nest $\mathcal{N}$ is the sequence of projections associated with the standard co-ordinate spaces of $\mathcal{H}$. We present the main result.

**Theorem 1.2** Let $A, B \in \mathcal{C}$ such that $[A, B]$ is right invertible. (a) If $B$ is compact then there exists $C \in \mathcal{C}$ such that $A + BC$ is invertible in $\mathcal{C}$.

(b) If $B$ is not compact there exists a not necessarily causal $C \in \mathcal{L}(\mathcal{H})$ such that $A + BC$ is invertible in $\mathcal{L}(\mathcal{H})$.

This of course leaves the problem open for the case where $B$ is not compact.

**References**


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