Nyström-Clenshaw-Curtis Quadrature for Integral Equations with Discontinuous Kernels

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Let the integral operator,

\[(Kx)(t) = \int_a^b k(t, s)x(s)ds, \quad a \leq t \leq b,\]

map \(C^q_{[a,b]}\), \(q > 1\), into itself. We consider the numerical solution of the corresponding Fredholm integral equation of the second kind,

\[x(t) + \int_a^b k(t, s)x(s)ds = y(t), \quad y \in C^q, \quad a \leq t \leq b.\]

(1)

When the kernel \(k(t, s)\) has a discontinuity either by itself or in its partial derivatives along the main diagonal \(t = s\), one can not expect a high accuracy Nyström quadrature based on Newton-Cotes or Gaussian integration rules, since, except for the Trapezium rule, the standard error bounds for these rules are not applicable. If the function \(x(t)\) were known then the discretization of the integral operator in (1) would be straightforward: for any fixed \(t\) the interval \([a, b]\) can be partitioned so that in each subinterval the integrand is smooth. When \(x(t)\) is unknown it is generally not possible to get an accurate discretization of (1) with \(x(t)\) and \(x(s)\) sampled at the same support points, without using some sort of interpolation. The main purpose of this paper is to introduce a high accuracy Nyström-Gauss quadrature for a certain class of discontinuous kernels which we call semi-smooth, for which this difficulty can be overcome.

DEFINITION

A kernel \(k(t, s)\) is called \(p - semi - smooth\), if

\[k(t, s) = \begin{cases} k_1(t, s) & \text{if } a \leq s \leq t \\ k_2(t, s) & \text{if } t \leq s \leq b, \end{cases}\]
where \( k_{1,2}(t, s) \in C^p_{[a,b] \times [a,b]} \) for some \( p > 1 \).

The Gauss type integration rule which we use here is the Clenshaw-Curtis rule, [2], and hence the resulting quadrature is called Nyström-Clenshaw-Curtis, or NCC for short.

Note that for our purpose each of the auxiliary kernels \( k_1(t, s) \) and \( k_2(t, s) \) must be defined in the whole square \([a, b] \times [a, b]\). The convergence of our method is of \( O(n^{1-r}) \), where \( r = \min\{p, q\} \). When \( r = \infty \), the convergence is super-algebraic, or spectral.

A well-known example of semi-smooth kernels are the semi-separable kernels, with \( k_1 \) and \( k_2 \) being of a low rank \( \alpha \),

\[
  k_i(t, s) = \sum_{j=1}^{\alpha} f_i^{(j)}(t) g_i^{(j)}(s), \quad i = 1, 2.
\]

In fact, semi-smooth kernels can be approximated with semi-separable kernels using the Singular Value Decomposition of \( k_{1,2}(t, s) \). However, NCC is less costly and more accurate than the discretization based on the semi-separable approximation.

Another useful example of semi-smooth kernels are the displacement kernels, \( k(t, s) = k(|t - s|) \). Such kernels occur in radiative transfer, Wiener filter theory, resonance scattering, etc. Our discretization technique is highly accurate for such kernels and at the same time preserves the displacement structure in the discrete equations.

References


