A Hamiltonian-based Solution to the Mixed Sensitivity Problem for Stable Pseudorational Plants

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Abstract—In this paper, we consider the mixed sensitivity optimization problem for infinite-dimensional stable plants. This problem is reducible to a one-block $H^\infty$ problem with a structured weighting function, which violates the assumption of the well-known Zhou-Khargonekar formula. We derive a Hamiltonian-based solution to compute the optimal performance level, by extending the formula to such case. A numerical example is given to illustrate the result.

I. INTRODUCTION

Since mid-1980’s several techniques have been developed for the $H^\infty$ control of infinite-dimensional systems. In particular, for the one-block problem to find

$$\rho_{opt} := \inf_{Q \in H^\infty} \| W - m_o Q \|_\infty$$

with general inner function $m_o$ and rational weights $W$, involving time delay systems with rational weights, a beautiful formula, so-called the Zhou-Khargonekar formula, has been established [6], [10], [11]: Let $(A, B, C)$ be a minimal realization of $W$. Define its Hamiltonian $H_\rho$ by

$$H^W_\rho := \begin{bmatrix} A & BB^T/\rho \\ -C^TC/\rho & -A^T \end{bmatrix}.$$  \hspace{1cm} (1)

Suppose that $m_o$ is analytic on the set of eigenvalues of $H_\rho$. Then the optimal sensitivity $\rho_{opt}$ is the maximum $\rho$ such that

$$\det m_o(-H_\rho)_{22} = 0$$

where $M_{22}$ denotes the $(2, 2)$-block of matrix $M$. It also has a natural extension to the two-block case [2]. Furthermore, when the underlying plant structure is specified by the so-called pseudorational transfer functions [8], we do not need explicit numerical computations of roots and zeros of the associated transfer functions, to determine the related inner function $m_o$ [3].

In this paper, we consider the mixed sensitivity optimization problem

$$\gamma_{opt} := \inf_{C \text{ stabilizing} } \rho \left\| \begin{bmatrix} W_1(1 + PC)^{-1} \\ W_2PC(1 + PC)^{-1} \end{bmatrix} \right\|_\infty,$$  \hspace{1cm} (2)

where $W_1$ and $W_2$ are rational weights, and $P$ is a stable pseudorational plant with its inner part $m_o$. This problem is known to be a typical two-block problem. However we cannot apply the Hamiltonian-based formula of [2], since a "generic" assumption of the formula is almost always violated [4]. In view of this, we derive a Hamiltonian-based formula for the mixed sensitivity computation, by reducing the structured two-block problem to a one-block problem. This result can be viewed as an extension of the Zhou-Khargonekar formula to a specific structured one-block problem.

The paper is organized as follows: in the next section we review some preliminary results on pseudorational systems. In Section III, we briefly summarize the observations made in [3] and state drawbacks in more precise term. In Section IV, we derive a Hamiltonian-based solution for the structured one-block problem. A numerical example is given in Section V, and concluding remarks are made in the last section.

NOTATION AND CONVENTION

As usual, $H^p$ and $H^p_\infty$ denote the Hardy spaces on the open right- and left-half complex plane, respectively. Let $q(s) := q(-s)$. For an inner function $m$, $H(m) := H^2 \ominus mH^2$. It is known that

$$H(m) = \{ x \in H^2 : m^*x \in H^2 \}.$$  \hspace{1cm} (3)

For a given distribution $\alpha$, $\text{supp} \alpha$ denotes its support, and

$$\ell(\alpha) := \inf \{ t : t \in \text{supp} \alpha \},$$

$$r(\alpha) := \sup \{ t : t \in \text{supp} \alpha \}.$$  

Let $E^t(\mathbb{R}_{-})$ denote the space of distributions having compact support in $(-\infty, 0]$. $D^t_+(\mathbb{R})$ is the space of distributions having support bounded on the left. Clearly $E^t(\mathbb{R}_{-}) \subset D^t_+(\mathbb{R})$. Provided that a distribution $\alpha$ is Laplace transformable, its Laplace transform is denoted by $\hat{\alpha}(s)$.
II. PRELIMINARIES ON PSEUDORATIONAL SYSTEMS

We first give the definition of pseudorational impulse responses. This class plays a crucial role in realization, modeling, and control of infinite-dimensional systems, particularly delay-differential systems [8], [9].

Definition 2.1: Let \( f \) be a distribution having support in \([0, \infty)\). It is said to be pseudorational if there exist \( q, p \in \mathcal{E}'(\mathbb{R}_-) \) such that

1. \( q^{-1} \) exists over \( \mathcal{D}'_+(\mathbb{R}) \),
2. \( \text{ord} q^{-1} = -\text{ord} q \),
3. \( f \) can be written as

\[
f = q^{-1} \ast p,
\]

where \( \text{ord} q \) denotes the order of a distribution \( q \) [5].

If \( f \) is pseudorational, its associated transfer function \( \hat{f} \) is also said to be pseudorational. From the Paley-Wiener-Schwartz theorem [5], in the Laplace domain, pseudorational transfer functions are also the ratio of entire functions of exponential type—the simplest extension of polynomials. For stable pseudorational plant \( P \),

\[
\rho_{\text{opt}} := \inf_{Q \in H^\infty} \| W - PQ \|_{\infty}
\]

(4)
can be computed by the following:

Lemma 2.2: ([3]) Let stable pseudorational plant \( P = \hat{p}_1 \hat{p}_2 / \hat{q} \) with \( 1/\hat{q} \), \( 1/\hat{p}_1 \) and \( e^{(\hat{p}_2)^*} / \hat{p}_2 \) \( \in H^\infty \) and \( (A, B, C) \) be a minimal realization of \( \hat{W} \) Define the Hamiltonian \( H_p \) of \( W \) by (1). Suppose that \( 1/\hat{p}_2 \) is analytic on the set of eigenvalues of \( H_p \). Then \( \rho_{\text{opt}} \) in (4) is the maximum \( \rho \) that satisfies

\[
\det (e^{LH_p} \hat{p}_2^{-1} (H_p) \hat{p}_2 (H_p)^{-1} |_{22}) = 0
\]

where \( L := -\ell(q) + \ell(p_1) - r(p_2) \).

Notice that the given plant \( P \) is not necessarily inner.

III. MIXED SENSITIVITY PROBLEM

A. Two-block problem

In this section, we show that weighting functions have some specific structure when we reduce the mixed sensitivity problem to standard two-block problem. By the Youla parameterization, all stabilizing controllers are given by \( C = Q (1 - PQ)^{-1}, Q \in H^\infty \). Hence we obtain

\[
\gamma_{\text{opt}} = \inf_{Q \in H^\infty} \left\| \begin{bmatrix} W_1 (1 - PQ) & W_2 PQ \end{bmatrix} \right\|_{\infty}.
\]

(5)

First, introduce the following spectral factorization \( G^*(W_1W_1 + W_2W_2)G = 1 \),

\[
G^*(W_1 W_1 + W_2 W_2)G = 1,
\]

(6)

where both \( G \) and \( G^{-1} \) have no unstable poles. Then it follows that

\[
L_1 := \begin{bmatrix} (W_1 G)^* & (-W_2 G)^* \\ W_2 G & W_1 G \end{bmatrix}, \quad L_2 := \begin{bmatrix} m_d & 0 \\ 0 & 1 \end{bmatrix}
\]

are unitary, where \( m_d \) is a finite Blaschke product consisting of the unstable poles of \( (W_1 G)^* \) [1]. Hence, by multiplying (5) from the left by \( L_2 L_1 \), we obtain

\[
\gamma_{\text{opt}} = \inf_{Q(s) \in H^\infty} \left\| \begin{bmatrix} W - P_0 Q \end{bmatrix} \right\|_{\infty},
\]

(7)

where

\[
\begin{align*}
P_0 & := m_d P_1 \\
V & := W_1 W_2 G \\
W & := W_1 m_d (W_1 G)^*.
\end{align*}
\]

(8), (9), (10)

Note that both \( W \) and \( V \) are finite dimensional and stable. This type of problem is considered in [2], and a Hamiltonian-based solution is derived. However, in [2], it is assumed that \( V \) and \( W \) have no common poles. In the mixed sensitivity problem, the functions \( W \) and \( V \) need to be in the form (9) and (10). That means unless \( W_1 \) and \( W_2 \) are chosen in some specific manner, \( W \) and \( V \) will have common poles, i.e., the assumption in [2] is almost always violated.

B. One-block problem

Again, applying the standard techniques, see e.g. [1], we now reduce the two-block \( H^\infty \) problem (7) to a one-block problem. First, suppose that \( \gamma > \| V \|_{\infty} \) satisfies \( \gamma = \gamma_{\text{opt}} \).

Then there exists \( Q \in H^\infty \) such that

\[
\| W - P_0 Q \|^2 + \| V \|^2 = \gamma^2.
\]

Here, since \( \gamma > \| V \|_{\infty} \), there exists the spectral factorization \( F_\gamma \)

\[
F_\gamma (\gamma^2 - V^* V) F_\gamma = 1, \quad \gamma > \| V \|_{\infty}
\]

(11)

where both \( F_\gamma \) and \( F_\gamma^{-1} \) belong to \( H^\infty \). Therefore, by defining \( W_\gamma := F_\gamma W \), we see that \( \gamma = \gamma_{\text{opt}} \) if and only if

\[
\mathcal{J}(\gamma) := \inf_{Q \in H^\infty} \| W_\gamma - m_d Q \|_{\infty} = 1.
\]

(12)

It looks like that we can handle such structural two-block \( H^\infty \) problem (7) according to Zhou-Khargonekar formula. But, in fact, \( \gamma \)-dependent rational weights \( W_\gamma \) and a finite Blaschke product \( m_d \) are determined by \( W_1 \) and \( W_2 \). Let us consider the Hamiltonian matrix

\[
H_W^{W_\gamma} := \begin{bmatrix} A_\gamma & B_\gamma B_\gamma^T \\ -C_\gamma C_\gamma & -A_\gamma^T \end{bmatrix},
\]

(13)

where \( (A_\gamma, B_\gamma, C_\gamma) \) is a minimal realization of \( W_\gamma \). Notice that from equations (6), (9), (10) and (11)

\[
1 - W_\gamma^{-*} W_\gamma = (\gamma^2 - W_\gamma^{-*} W_1) F_\gamma^{-*} F_\gamma.
\]

Here eigenvalues of \( H_W^{W_\gamma} \) are the zeros of the right hand side, see Appendix and [4] for a detailed discussion. Therefore there always exits nonsingular matrix \( T \) such that

\[
H_W^{W_\gamma} = T^{-1} \begin{bmatrix} H_W^1 & A_d \\ -A_d & -A_d \end{bmatrix} T,
\]

(14)

where
• $H_{W_1}^\gamma$ is the $\gamma$-dependent Hamiltonian matrix of $W_1$,
• $A_d$ is a $\gamma$-independent square matrix, whose eigenvalues are poles of $m_d$, i.e., $(sI - A_d)^{-1} \in H(m_d)$.

This means that there is a coincidence between poles of $m_d$ and eigenvalues of $H_{W_1}^\gamma$, i.e., the assumption of the formula is also almost always violated. In practice, we can circumvent this problem by slightly altering the modified weight $V$ and obtain upper and lower bounds for the optimal value [4]. In what follows, we derive a Hamiltonian-based formula for the optimal mixed sensitivity computation, i.e., the structured one-block problem to find $\gamma$ satisfying (12).

### IV. MAIN RESULT

We first define $m := m_d m_v$ and the compression of $W_\gamma$ to $H(m)$

$$W_c : H(m) \to H(m) : x \mapsto \pi^m W_\gamma x.$$ 

where $\pi^m$ denotes the orthogonal projection from $H^2$ onto $H(m)$. It is known that $\mathcal{F}(\gamma) = ||W_c||_{H(m)}$. Let us consider singular value equation

$$y = W_c x, \quad x = W_c^* y.$$ 

We can show that $x, y$ belong to $H(m)$ and that these are characterized by finite dimensional vector $\xi$ and $\zeta \in \mathbb{R}^n$, where $n$ is the order of $W_c$, as follows:

$$y = W_c x - m(s) C_{\gamma}(sI - A_{\gamma})^{-1} \xi$$

$$x = W_c^* y + B_{\gamma}(sI + A_{\gamma}^T)^{-1} \zeta.$$ 

Then we can obtain the following Hamiltonian-based condition [10]:

**Lemma 4.1:** Under the definition above, 1 is a singular value of $W_c$ if and only if there exists a nonzero vector $[\xi^T \zeta^T]^T \in \mathbb{R}^{2n}$ such that

$$\Phi := (sI - H_{W_1}^\gamma)^{-1} \begin{bmatrix} m(s) \xi \\ \zeta \end{bmatrix},$$

$$\Phi \in H(m).$$

(15)

From this lemma, we can obtain the Zhou-Khargonekar formula by invoking Dunford integral. Assume that $m$ is analytic at eigenvalues of $H_{W_1}^\gamma$. Let $\Delta_1$ be a closed rectifiable contour that encircles all eigenvalues of $H_{W_1}^\gamma$, but none of the singularities of $m^\gamma$. Since $m^\gamma$ is analytic at eigenvalues of $H_{W_1}^\gamma$, this is possible. Consider the integral

$$-\frac{1}{2\pi i} \int_{\Delta_1} (sI - H_{W_1}^\gamma)^{-1} \begin{bmatrix} \xi \\ m^\gamma(s) \zeta \end{bmatrix} ds$$

Notice that, by spectral integral theory, the integral above is

$$\begin{bmatrix} \xi \\ 0 \end{bmatrix} + m^\gamma(H_{W_1}^\gamma) \begin{bmatrix} 0 \\ \zeta \end{bmatrix}.$$ 

On the other hand, for (15) to hold, this integral must be equal to 0. Hence there exists a nonzero vector $[\xi^T \zeta^T]^T$ such that

$$\begin{bmatrix} \xi \\ 0 \end{bmatrix} = -m^\gamma(H_{W_1}^\gamma) \begin{bmatrix} 0 \\ \zeta \end{bmatrix}$$

or equivalently $\det m^\gamma(H_{W_1}^\gamma)_{22} = 0$. However, when eigenvalues of $H_{W_1}^\gamma$ includes some poles of $m(s)$ as in the case of the mixed sensitivity computation, we cannot take such closed contour $\Delta_1$.

In what follows, we assume that $m$ is analytic at eigenvalues of $H_{W_1}^\gamma$ and define

$$\begin{bmatrix} T_1^T & T_2^T & T_3^T \end{bmatrix}^T := T,$$

(16)

partitioned accordingly as (14). Then the main result of this work is the following:

**Theorem 4.2:** Define $H_{W_1}^\gamma$ and $T_i$ ($i = 1,2,3$) by (1), (14) and (16). Suppose that $m$ is analytic at eigenvalues of $H_{W_1}^\gamma$. Then 1 is a singular value of $W_c$ if and only if there exists a nonzero vector $[\xi^T \zeta^T]^T \in \mathbb{R}^{2n}$ such that

$$T_1 \begin{bmatrix} \xi \\ 0 \end{bmatrix} = -m^\gamma(H_{W_1}^\gamma) T_1 \begin{bmatrix} 0 \\ \zeta \end{bmatrix}$$

(17)

$$T_2 \begin{bmatrix} \xi \\ 0 \end{bmatrix} = 0, \quad T_3 \begin{bmatrix} 0 \\ \zeta \end{bmatrix} = 0.$$ 

(18)

**Proof:** First, (15) holds if and only if $T \Phi \in H(m)$, since $T$ is nonsingular. Therefore the condition (15) in Lemma 4.1 can be written as

$$(sI - H_{W_1}^\gamma)^{-1} T_1 \begin{bmatrix} m(s) \xi \\ \zeta \end{bmatrix} \in H(m),$$

(19)

$$(sI - A_d)^{-1} T_2 \begin{bmatrix} m(s) \xi \\ \zeta \end{bmatrix} \in H(m),$$

(20)

$$(sI + A_d)^{-1} T_3 \begin{bmatrix} m(s) \xi \\ \zeta \end{bmatrix} \in H(m).$$

(21)

Similarly to the discussion above, the condition (19) can be characterized as a rank condition of a matrix. Let $\Delta$ be a closed rectifiable contour that encircles all eigenvalues of $H_{W_1}^\gamma$, but none of the singularities of $m^\gamma$. Then, for (19) to hold, we must have

$$-\frac{1}{2\pi i} \int_{\Delta} (sI - H_{W_1}^\gamma)^{-1} T_1 \begin{bmatrix} \xi \\ m^\gamma(s) \zeta \end{bmatrix} ds = 0$$

This leads to (17).

We now consider the condition (20). Recall that we have $(sI - A_d)^{-1} \in H(m_d) \subset H(m)$. Hence (20) is equivalent to saying that

$$m(s)(sI - A_d)^{-1} T_2 \begin{bmatrix} \xi \\ 0 \end{bmatrix} \in H(m).$$

Then, by (3), we have

$$(sI - A_d)^{-1} T_2 \begin{bmatrix} \xi \\ 0 \end{bmatrix} \in H^2.$$ 

Because $A_d$ has no unstable pole, this implies

$$T_2 \begin{bmatrix} \xi \\ 0 \end{bmatrix} = 0.$$
We now move to (21). We have $m(s)(sI + A_d)^{-1} \in H(m)$ and eigenvalues of the matrix $-A_d$ are all unstable. Therefore we must have

$$T_3 \left[ \begin{array}{c} 0 \\ \zeta \end{array} \right] = 0.$$ 

This completes the proof.

**Remark 4.3:** This result includes the Zhou-Khargonkar formula. In other words, when the assumption of the formula is violated, there are additional conditions (18).

V. EXAMPLE

Suppose that the weighting functions are given by

$$W_1(s) = \frac{1}{s + 1}, \quad W_2(s) = \frac{s + 0.5}{s + 1}$$

and a stable pseudorational plant

$$P(s) = \frac{e^s - 2}{2e^{2s} - 1} \in H^{\infty}.$$ 

Then the inner part of the plant $m_v$ is given by

$$m_v := e^{-s} - \frac{2e^{-s} - 1}{2 - e^{-s}}.$$ 

The inner function $m_d$ is given by

$$m_d(s) = \frac{s - \sqrt{5}/2}{s + \sqrt{5}/2},$$

and $V$ and $W$ are

$$V(s) = \frac{1}{(s + 1)(s + \sqrt{5}/2)}, \quad W(s) = \frac{s + 0.5}{(s + 1)(s + \sqrt{5}/2)}.$$ 

We can see that $V$ and $W$ has common poles. Next, $W_\gamma$ are given by

$$W_\gamma = \frac{1}{\gamma(s^2 + bs + a)},$$

where $a = \sqrt{\frac{5 - \gamma^2}{2}}$ and $b = \sqrt{\frac{9}{2} + 2a - \gamma^{-2}}$. Here eigenvalues of $H_1^{W_\gamma}$ are $s = \pm \sqrt{5}/2, \pm \sqrt{1 - \gamma^{-2}}$, including the pole of $m_d$.

In [4], by changing the weighting function $W_\gamma$ slightly, it has been shown that $0.852 < \gamma_{\text{opt}} < 0.857$. Figure 1 shows the smallest singular values of the matrix corresponding to Theorem 4.2 versus $\gamma$. We can see $\gamma_{\text{opt}} \approx 0.8567$ and this satisfies the estimation above.

VI. CONCLUSIONS

We have derived a Hamiltonian-based solution to the optimal mixed sensitivity problem for stable pseudorational plants. This result can be viewed as an extension of the Zhou-Khargonkar formula to a specific structured one-block problem.

REFERENCES


APPENDIX

Here we see the structure of weighting functions. Consider rational weighting functions

$$W_1 = \frac{n_1}{d_1}, \quad W_2 = \frac{n_2}{d_2},$$

where pairs of polynomials $(d_i,n_i)$ ($i = 1,2$) are coprime. For simplicity, we assume that $d_1$ and $d_2$ has no common zeros. Let us take a stable polynomial $d_G$ such that

$$d_G d_G^- = n_1 n_1^- d_2 d_2^- + n_2 n_2^- d_1 d_1^-.$$ 

Then we have $G = d_G d_G^-$ and $m_d = d_G^-$. Hence weighting functions in the two-block problem (7) are given by $W = \frac{n_1 n_1^- d_2 d_2^-}{d_G d_G^-}$ and $V = \frac{n_2 n_2^-}{d_G}$, and have common poles. Now let us define a stable polynomial $d_F$ such that

$$d_F d_F^- = \gamma^2 d_G d_G^- - n_1 n_1^- n_2 n_2^-.$$ 

The function $F_\gamma$ satisfying (11) is given by $F_\gamma = \frac{d_F}{d_F^\gamma}$, and its zeros are poles of $m_d$. 