de Branges Rovnyak spaces and Schur functions: the hyperholomorphic case

D. Alpay, M. Shapiro and D. Volok

1. Introduction. We define de Branges Rovnyak spaces in the quaternionic setting, that is when one replaces the complex numbers by quaternions and analytic functions of one complex variable by hyper-analytic functions.

We first briefly recall the one complex variable setting. Schur functions (that is, functions analytic and contractive in the open unit disk \( \mathbb{D} \)) play an important role in operator theory and related topics; see [1] for a review. For such a function \( s \), the kernel \( k_s(z, w) = \frac{1-s(z)s(w)^*}{1-z^{-1}w} \) is positive in \( \mathbb{D} \). The associated reproducing kernel Hilbert space (denoted by \( \mathcal{H}(s) \)) is contractively included in the Hardy space \( \mathcal{H}_2(\mathbb{D}) \) and is the state space for a coisometric realization of \( s \), that is, \( s(z) = H + zG(I - zT)^{-1}F \) where the operator matrix

\[
\begin{pmatrix}
T & F \\
G & H
\end{pmatrix} : \begin{pmatrix}
\mathcal{H}(s) \\
\mathbb{C}
\end{pmatrix} \rightarrow \begin{pmatrix}
\mathcal{H}(s) \\
\mathbb{C}
\end{pmatrix}
\]

is defined by \( Gf = f(0) \) (the point evaluation at the origin),

\[
 Tf = R_0f \quad Fx = \frac{s(z) - s(0)}{z}x, \quad Hx = s(0)x,
\]

and is coisometric; see [7].

When leaving the setting of one complex variable there are numerous counterparts of de Branges Rovnyak spaces; we mention in particular the cases of compact Riemann surfaces, upper triangular operators and of the unit ball \( \mathbb{B}_N \) of \( \mathbb{C}^N \); see [2] for the latter.
2. The quaternions and hyperholomorphic functions. Let $\mathbb{H}$ denote the skew–field of real quaternions
\[ \mathbb{H} = \{ x = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 ; (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \} \]
where the imaginary units $e_1, e_2, e_3$ satisfy the conditions
\[ e_\ell e_k = \begin{cases} -e_k e_\ell & \text{if } \ell \neq k, \\ -1 & \text{if } \ell = k. \end{cases} \]
and $e_1 e_2 = e_3$.

Writing $x = z_1 + z_2 e_2$ where $z_1 = x_0 + e_1 x_1$ and $z_2 = x_2 + e_1 x_3$ we identify the space of quaternions $\mathbb{H}$ with $\mathbb{C}^2$ endowed with the special structure $ze_2 = e_2 z$.

The map $x \mapsto (z_1, z_2)$ sends the unit ball of the quaternions onto the unit ball of $\mathbb{C}^2$.

The function $f : \Omega \subset \mathbb{R}^4 \to \mathbb{H}$ is called left–hyperholomorphic if
\[ D f := \frac{\partial}{\partial x_0} f + e_1 \frac{\partial}{\partial x_1} f + e_2 \frac{\partial}{\partial x_2} f + e_3 \frac{\partial}{\partial x_3} f = 0. \] (1)

The operator $D$ is called the Cauchy–Riemann or Cauchy–Fueter operator; see [9, p. 37]. Because of the non–commutativity of the multiplication of quaternions there is also the dual notion of right–hyperholomorphic function.

Three important examples of left–hyperholomorphic functions are
\[ \zeta_j(x) = x_j - e_j x_0, \quad j = 1, 2, 3. \]

The product of two left–hyperholomorphic functions will not be hyperholomorphic in general. Define the symmetrized product of $a_1, \ldots, a_n \in \mathbb{H}$ by
\[ a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}, \]
where $S_n$ is the set of all permutations of $\{1, \ldots, n\}$. Furthermore, for $\nu, \mu \in \mathbb{Z}_+^3$ let
\[ |\nu| = \nu_1 + \nu_2 + \nu_3, \quad \nu! = \nu_1! \nu_2! \nu_3!, \]
\[ \begin{align*}
\partial^\nu &= \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \partial x_3^{\nu_3}}, \\
e_1 &= (1 \ 0 \ 0), \quad e_2 = (0 \ 1 \ 0), \quad e_3 = (0 \ 0 \ 1). 
\end{align*} \]

Using the above notation, one has

\[ f(x) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \zeta^\nu(x) f_\nu, \] (2)

where

\[ \zeta^\nu(x) = \zeta_1(x)^{\nu_1} \times \zeta_2(x)^{\nu_2} \times \zeta_3(x)^{\nu_3} \] (3)

\[ f_\nu = \frac{1}{\nu!} (\partial^\nu f)(0). \] (4)

The polynomials \( \zeta^\nu \), defined by (3), are called the Fueter polynomials; see [8], [6].

The Cauchy–Kovalevskaya product of two left–hyperholomorphic functions is defined by

\[ \zeta^\nu c \odot \zeta^\mu d = \zeta^{\nu+\mu} cd, \quad c, d \in \mathbb{H}, \quad \nu, \mu \in \mathbb{N}^3. \]

It is still a left–hyperholomorphic function.

3. de Branges Rovnyak spaces. The counterpart of the Hardy space is now played by the reproducing kernel Hilbert space \( \mathcal{H}(k) \) with reproducing kernel

\[ k_y(x) = (1 - \zeta_1(x)\overline{\zeta_1(y)} - \zeta_2(x)\overline{\zeta_2(y)} - \zeta_3(x)\overline{\zeta_3(y)})^{-\odot} = \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \zeta^\nu(x)\overline{\zeta^\nu(y)} \]

where \( ^{-\odot} \) denotes the inverse with respect to the Cauchy–Kovalevskaya product. The kernel \( k_y(x) \) is positive in the ellipsoid

\[ \Omega = \{ x \in \mathbb{H} \mid 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \}. \]

A Schur function is defined as a Schur multiplier (with respect to the Cauchy–Kovalevskaya product) in the space \( \mathcal{H}(k) \). This is equivalent to the positivity of the kernel

\[ k_s(x, y) = \sum_{\nu \in \mathbb{N}^3} \frac{|\nu|!}{\nu!} \left( \zeta^\nu(x)\overline{\zeta^\nu(y)} - (s \odot \zeta^\nu)(x)(s \odot \zeta^\nu)(y) \right). \]
The de Branges Rovnyak space is the associated reproducing kernel Hilbert space. We will discuss the properties of such spaces in the talk.


References


