HOW NONCOMMUTING
ALGEBRA ARISES IN SYSTEMS
THEORY

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\[
\frac{dx(t)}{dt} = Ax(t) + Bv(t) \\
y(t) = Cx(t) + Dv(t)
\]

\(A, B, C, D\) are matrices

\(x, v, y\) are vectors

Asymptotically stable

\[
\text{Re}(\text{eigvals}(A)) < 0 \iff A^T E + EA < 0 \quad E > 0
\]

Energy dissipating

\[
G : L^2 \to L^2
\]

\[
\int_0^T |v|^2 dt \geq \int_0^T |Gv|^2 dt
\]

\(x(0) = 0\)

\[
\exists \quad E = E^T \succeq 0
\]

\[
H := A^T E + EA + EBB^T E + C^TC = 0
\]

\(E\) is called a storage function

Two minimal systems

\([A, B, C, D]\) and \([a, b, c, d]\)

with the same input to output map.

\[
\exists \quad M \text{ invertible, so that}
\]

\[
MAM^{-1} = a \\
MB = b \\
CM^{-1} = c
\]

\(\exists M \text{ invertible, so that}
\]

\[
(B \ AB \ A^2B \cdots) : \ell^2 \to X
\]

is onto
**H∞ Control Problem**

Given
\[ A, B_1, B_2, C_1, C_2, D_{12}, D_{21} \]

Find
\[ K \]

\[ \frac{dx}{dt} = Ax + B_1w + B_2u \]
\[ \text{out} = C_1x + D_{12}u + D_{11}w \]
\[ y = C_2x + D_{21}w \]

\[ D_{21} = I \quad D_{12}D_{12}' = I \quad D_{12}'D_{12} = I \quad D_{11} = 0 \]

**PROBLEM:** Find a control law \( K : y \rightarrow u \) which makes the system dissipative over every finite horizon:

\[ T \int_0^T |\text{out}(t)|^2 dt \leq \int_0^T |w(t)|^2 dt \]

The unknown \( K \) is the system

\[ \frac{d\xi}{dt} = a\xi + b \quad u = c\xi \]

So \( a, b, c \) are the critical unknowns.
CONVERSION TO ALGEBRA

Engineering Problem: Make a given system dissipative by designing a feedback law.

Given

\[
\begin{pmatrix}
A, B_1, C_1, \\
B_2 C_2
\end{pmatrix}
\|
D
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

Find

\[a \ b \ c\]

DYNAMICS of “closed loop” system: BLOCK matrices

\[
\begin{pmatrix}
A & B & C & D
\end{pmatrix}
\]

ENERGY DISSIPATION:

\[
H := \mathcal{A}^T \mathbf{E} + \mathbf{E} \mathcal{A} + \mathbf{E} \mathcal{B} \mathcal{B}^T \mathbf{E} + \mathcal{C}^T \mathcal{C} = 0
\]

\[
\mathbf{E} = \begin{pmatrix}
\mathbf{E}_{11} & \mathbf{E}_{12} \\
\mathbf{E}_{21} & \mathbf{E}_{22}
\end{pmatrix}
\quad \mathbf{E}_{12} = \mathbf{E}_{21}^T
\]

\[
H = \begin{pmatrix}
H_{xx} & H_{xy} \\
H_{yx} & H_{yy}
\end{pmatrix}
\quad H_{xy} = H_{yx}^T
\]
$H^\infty$ Control Problem

ALGEBRA PROBLEM:
Given the polynomials:

\[
H_{xx} = E_{11} A + A^T E_{11} + C_1^T C_1 + E_{12}^T b C_2 + C_2^T b^T E_{12}^T + \nonumber
\]
\[
E_{11} B_1 b^T E_{12} + E_{11} B_1 B_1^T E_{11} + E_{12} b b^T E_{12}^T + E_{12} b B_1^T E_{11} \nonumber
\]
\[
H_{xz} = E_{21} A + \frac{a^T (E_{21} + E_{12}^T)}{2} + c^T C_1 + E_{22} b C_2 + c^T B_2^T E_{11}^T + \nonumber
\]
\[
E_{21} B_1^T (E_{21} + E_{12}^T) + E_{21} B_1 B_1^T E_{11} + \frac{E_{22} b b^T (E_{21} + E_{12}^T)}{2} + E_{22} b B_1^T E_{11} \nonumber
\]
\[
H_{zx} = A^T E_{21}^T + C_1^T c + \nonumber
\]
\[
\frac{(E_{12} + E_{21}^T)}{2} a + E_{11} B_2 c + C_2^T b^T E_{22}^T + \nonumber
\]
\[
E_{11} B_1 b^T E_{22}^T + E_{11} B_1 B_1^T E_{21} + \frac{(E_{12} + E_{21}^T)}{2} b b^T E_{22}^T + \frac{(E_{12} + E_{21}^T)}{2} b B_1^T E_{21} \nonumber
\]
\[
H_{zz} = E_{22} a + a^T E_{22}^T + c^T c + E_{21} B_2 c + c^T B_2^T E_{21}^T + E_{21} B_1 b^T E_{22}^T + \nonumber
\]
\[
E_{21} B_1 B_1^T E_{21}^T + E_{22} b b^T E_{22}^T + E_{22} b B_1^T E_{21} \nonumber
\]

(HGRAIL) $A, B_1, B_2, C_1, C_2$ are knowns.

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \succeq 0$ for unknowns $a, b, c$ and for $E_{11}, E_{12}, E_{21}$ and $E_{22}$

When can they be solved?

If these equations can be solved, find formulas for the solution.
TEXTBOOK SOLUTION TO THE \( H^\infty \) PROB

DGKF = Doyle-Glover Kargonekar - Francis 1989 ish

KEY Riccatis

\[
DGKF_X := (A - B_2 C_1)'X + X(A - B_2 C_1) + X(\gamma^{-2} B_1 B'_1 - B_2^{-1} B'_2)X
\]

\[
DGKF_Y := A^\times Y + Y A^\times' + Y(\gamma^{-2} C'_1 C_1 - C'_2 C_2)Y
\]

here \( A^\times := A - B_1 C_2 \).

THM DGKF There is a system \( K \) solving the control problem if there exist solutions

\[
X \succeq 0 \quad \text{and} \quad Y \succ 0
\]

to inequalities the

\[
DGKF_Y \preceq 0 \quad \text{and} \quad DGKF_X \preceq 0
\]

which satisfy the coupling condition

\[
X - Y^{-1} \prec 0.
\]

This is iff provided \( Y \succeq 0 \) and \( Y^{-1} \) is interpreted correctly.
Riccati Inequalities

\[
A_1'X + XA_1 + XQ_1X + R_1 \preceq 0 \\
A_2'X + XA_2 + XQ_2X + R_2 \preceq 0 \\
X \succeq 0
\]

These are “matrix convex” in the unknown \( X \) provided \( Q_1, Q_2 \) are positive semidefinite matrices. If such an \( X \) exists, then can simultaneously control or stabilize several systems.

**Numerical Solution** Can solve convex (especially linear) matrix inequalities numerically with \( X \) smaller than 150 \( \times \) 150 matrices using interior point optimization methods - called [semidefinite programming](https://en.wikipedia.org/wiki/Semidefinite_programming).

**Main Algebra Problem** “Convert” your engineering problem to a set of equivalent ‘convex matrix inequalities” .