Differential Forms and 0-dimensional Super Symmetric Field Theories

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1 Introduction

Two of us [ST] spent the last years to find a precise notion of super symmetric Euclidean field theories of (super) dimension $d|\delta$ and relate it to certain multiplicative cohomology theories. We showed that in dimension $1|1$ the relevant cohomology theory is K-theory and conjectured that for dimension $2|1$ one gets elliptic cohomology, or more precisely, the cohomology theory TMF of topological modular forms. In this paper we fill the gap in dimension $0|1$ by showing that de Rham cohomology arises in this easiest case.

Moreover, the geometric cocycles we actually get from EFTs (which is short for *Euclidean field theories*) of dimension $0|1$ are closed differential forms, just like vector bundles with connection can be used to get Euclidian field theories of dimension $1|1$, see [?]. Our goal remains to show that EFTs of dimension $2|1$ are cocycles for TMF.

Our results are consistent with the formal group point of view towards (complex oriented) cohomology theories, where the additive formal group gives ordinary rational cohomology, the multiplicative group gives K-theory and the formal groups associated to elliptic curves lead to elliptic cohomology.

The precise definition of EFTs is unfortunately pretty involved, so we won't repeat it here but refer instead to our survey [ST2]. We will summarize in Section ?? the necessary information for dimension $0|1$. In our definition, an EFT has a *degree* $n \in \mathbb{Z}$ which is related to the central charge as well as to the degree of a cohomology class. If X is a smooth manifold, we also define EFTs *over* X, which can be thought of as *families* of EFTs parametrized by

X. In this case, the degree n can be generalized to a *twist* over X which relates very well to twisted cohomology but will not be discussed in this paper. An EFT over X should be thought of as a *geometric object* over X. This is best explained by our main result below, Theorem thm:0EFT 1, which says that a closed differential form over X can be interpreted as a $0|1$-dimensional EFT over X and vice versa.

Like differential forms or vector bundles with connection, EFTs over X of the same dimension $d|\delta$ can be added and multiplied. Addition preserves the degree n, whereas multiplication adds degrees as expected. Moreover, $d|\delta$-dimensional EFTs over a manifold X of degree n form a category $d|\delta\text{-EFT}^n(X)$ (in fact, a d-category, an issue we'll ignore in this paper) and can be *pulled back* via smooth maps: a smooth map $f\colon Y \to X$ determines a functor

$$f^*\colon d|\delta\text{-EFT}^n(X) \longrightarrow d|\delta\text{-EFT}^n(Y)$$

and these functors compose strictly, unlike the case of vector bundles where $(fg)^*E$ is isomorphic, but not equal to g^*f^*E.

We call two EFTs $E_0, E_1 \in d|\delta\text{-EFT}^n(X)$ *concordant* if there exists a field theory $E' \in d|\delta\text{-EFT}^n(X \times \mathbb{R})$ such that $\iota_t^* E' \cong E_t$ for $t = 0, 1$, where $\iota_t\colon X \to X \times \mathbb{R}$ is the inclusion map $x \mapsto (x, t)$. We observe that concordance give anequivalence relation which can be defined for geometric objects over manifolds for which *pull-backs* and *isomorphisms* make sense. We note that by Stokes' Theorem two closed n-forms on X are concordant if and only if they represent the same de Rham cohomology class; two vector bundles with connections are concordant if and only if they are isomorphic as vector bundles (i.e., disregarding the connections). Passing from an EFT over X to its concordance class forgets the geometric information while retaining the topological information. We will write $d|\delta\text{-EFT}^n[X]$ for the set of *concordance classes* of $d|\delta$-dimensional supersymmetric EFTs of degree n over X.

thm:0EFT **Theorem 1.** *For smooth manifolds X, there are natural ring isomorphisms*

$$0|1\text{-EFT}^n(X) \cong \begin{cases} \Omega^{ev}_{cl}(X) & n \text{ even} \\ \Omega^{odd}(X) & n \text{ odd} \end{cases}$$

where $\Omega^{ev}_{cl}(X)$, respectively $\Omega^{odd}_{cl}(X)$, stands for the even, respectively odd, closed differential forms on X.

It follows that on concordance classes we get isomorphisms

$$0|1\text{-}\mathrm{EFT}^n[X] \cong \begin{cases} H^{ev}_{dR}(X) & n \text{ even} \\ H^{odd}_{dR}(X) & n \text{ odd} \end{cases}$$

where $H^{ev}_{dR}(X)$, respectively $H^{odd}_{dR}(X)$, stands for the direct sum of the even, respectively odd, de Rham cohomology groups of X.

There is a beautiful interpretation of the Chern character form of a vector bundle with connection in terms of the map from $1|1$-dimensional to $0|1$-dimensional EFTs over X, given by crossing with the standard circle, see Ha [?]. It is hence essential that the result above yields differential forms of varying degrees. However, differential forms of a specific degree n arise by forgetting the Euclidean geometry (on super points) and working with TFTs (*topological* field theories) instead. Again, there are categories $d|\delta\text{-}\mathrm{TFT}^n(X)$ of $d|\delta$-dimensional TFTs over a manifold X of degree n as well as their concordance classes $d|\delta\text{-}\mathrm{TFT}^n[X]$.

thm:0TFT **Theorem 2.** *There are natural isomorphisms of abelian groups*

$$0|1\text{-}\mathrm{TFT}^n(X) \cong \Omega^n_{cl}(X)$$

compatible with multiplication. Moreover, concordance classes lead to

$$0|1\text{-}\mathrm{TFT}^n[X] \cong H^n_{dR}(X)$$

We will show in SST [?] that this result carries over to the case of twisted topological field theories which relate to differential forms, twisted by a flat vector bundle, and the resulting twisted de Rham cohomology.

From the above theorem, it is easy to recover the entire structure of de Rham cohomology from TFTs. What's missing is the boundary map in Mayer-Vietories exact sequences for a covering of X by open sets. Equivalently, we need to express the suspension isomorphisms

$$H^n(X) \cong H^{n+1}_{cvs}(X \times \mathbb{R})$$

in terms of TFTs. Here the subscript *cvs* means *compact vertical support* (in the $\mathbb{R}$-direction). This isomorphism is given by product with a particular class $u \in H^1_{cs}(\mathbb{R})$, the *Thom class* for the trivial line bundle over $X = \mathrm{pt}$. Therefore, it suffices to express the condition of compact vertical support in terms of TFTs. However, this is easy since the first part of Theorem 2 describes the *cocyles* for de Rham cohomology in terms of TFTs and compactly supported cohomology is given by concordance classes of compactly supported cocycles. The second part of Theorem 2 alone would not be sufficient for this argument!

Similarly, it is the description of de Rham cocylces that enables us to use TFTs for building Eilenberg-MacLane spaces $K(\mathbb{R}, n)$: Consider *extended* standard k-simplices

$$\Delta_e^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^{k} t_i = 1\}$$

which are smooth manifolds (without boundary or corners). The usual face and degeneracy maps are defined on these extended simplices and hence there are simplicial sets $K_\bullet(n)$ with k-simplices $0|1\text{-TFT}^n(\Delta_e^k)$.

cor:KRn **Corollary 3.** *The geometric realization $|K_\bullet(n)|$ is an Eilenberg-MacLane space of type $K(\mathbb{R}, n)$, where $\mathbb{R}$ has the discrete topology.*

Proof. This result is well known for any ordinary cohomology theory (with arbitrary coefficients), where one replaces TFTs by the relevant cocycles for the theory. The easiest way for us to prove the result is to state the following result from [MW, Appendix]: Given any sheaf (on the big site of manifolds) $F : \mathsf{Man} \longrightarrow \mathsf{Set}$ there are natural bijections for manifolds X:

$$F[X] \cong [X, |F|]$$

The left hand side denotes concordance classes as defined above and on the right hand side $|F|$ is the geometric realization of the simplicial set $k \mapsto F(\Delta_e^k)$. Thus it suffices to show that $F := 0|1\text{-TFT}^n \cong \Omega^n_{cl}$ is a sheaf: This means that for any open covering $\{U_i\}$ of X, a closed differential form on X is the same thing as a collection of closed differential forms on U_i that agree on intersections $U_i \cap U_j$. This is clearly true if we work with all differential forms since these are sections of a vector bundle on X. It stays true for *closed* differential forms because the de Rham operator d is defined locally. □

The very last part of the above proof is our motivation for requiring the field theories in our definition to be *local*. Currently, we express this by saying that a $d|\delta$-dimensional field theory is a (symmetric monoidal) d-*functor* from a bordims d-category to a target d-category. The precise details of this definition for $d = 2$ are far from obvious and are holding up progress in this case.

This paper is organized as follows. In Section 2 we will give a quick survey of super manifolds, containing everything that's needed for our theorems above. In Section 3 we give a detailed proof that differential forms on a manifold X, with integral grading and de Rham d, are the functions on the odd tangent bundle $\pi TX \cong \mathsf{SMan}(\mathbb{R}^{0|1}, X)$ with its natural action by the super group

$$\underline{\mathrm{Diff}}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$$

Here the translational part $\mathbb{R}^{0|1}$ is infinitesimally generated by d, so $d^2 = 0$ because translations commute with each other. Moreover, the $\mathbb{Z}$-grading of $\Omega^*(X)$ comes from the dilation action of $\mathbb{R}^\times$ on $\mathbb{R}^{0|1}$ and the relation between dilations and translations shows that d must have degree one. We claim no originality for these results since they seem to be well known to several authors. We exhibit detailed arguments (including the case where X is a super manifold) in this paper because we couldn't find a reference in the literature.

In Section 4 we will review the notions of field theories of dimension $0|1$ and prove Theorems 1 and 2. In particular, we'll show in Lemma 17 that such field theories are functions on the quotient stack $\pi TX//G$ on the site SMan of super manifolds. Here G is a subgroup of $\underline{\mathrm{Diff}}(\mathrm{spt})$ *defining* the geometry on the super point $\mathrm{spt} := \mathbb{R}^{0|1}$. For a *topological* field theory, G consists of all diffeomorphisms and hence

$$0|1\text{-}\mathrm{TFT}^0(X) \cong \Omega^*(X)^{\underline{\mathrm{Diff}}(\mathrm{spt})} \cong \Omega^0_{cl}(X)$$

which is the case of Theorem 2 for degree 0. A *Euclidean* field theory is defined by setting $G := \mathbb{R}^{0|1} \rtimes \{\pm 1\}$, i.e. by allowing only translations and reflections as isometries of spt, not all dilations. It follows that

$$0|1\text{-}\mathrm{EFT}^0(X) \cong \Omega^*(X)^G \cong \Omega^{ev}_{cl}(X)$$

which is the case of Theorem 1 for degree 0. In both Theorems the degree n case is obtained by showing in Lemma 17 that functions on the quotient stack $\pi TX//G$ need to be replaced by sections of a line bundle $L^{\otimes n}$ over this stack.

2 Quick survey of super manifolds

sec:sman

We survey some basic notions of super geometry, simply because we feel that most of our readers may not be familiar with these concepts. Almost all the material is taken from the beautiful survey article on super symmetry by Deligne and Morgan, [DM].

2.1 Super algebra

Let us begin by explaining briefly what *super* means in an algebraic context, working with the ground field of real numbers. The monoidal category of *super vector spaces*, with tensor products, is by definition the same as the monoidal category of $\mathbb{Z}/2$-graded vector space, with graded tensor product. For example, a super algebra is simply a monoidal object in this category and is hence the same thing as a $\mathbb{Z}/2$-graded algebra. For example, the endomorphism ring $\mathrm{End}(V)$ of a super vector space V inherits a natural $\mathbb{Z}/2$-grading from that of V. The distinction between these notions only arises from the symmetry operators

$$\sigma : V \otimes W \cong W \otimes V$$

which are different in the two monoidal categories, yielding two very different *symmetric* monoidal categories. For super vector spaces one has

$$\sigma(v \otimes w) = (-1)^{|v| \cdot |w|} w \otimes v,$$

where $|v|$ is the $\mathbb{Z}/2$-degree of a homogenous vector $v \in V$. This basic rule is sometimes summarized as the

- **Sign rule:** Commuting two odd quantities yields a sign -1.

As a consequence, a super algebra is (super) *commutative* if for all homogenenous $a, b \in A$ we have

$$ab = (-1)^{|a||b|} ba,$$

a very different notion than a commutative $\mathbb{Z}/2$-graded algebra. Examples of commutative super algebras arising naturally are cohomology rings $H^*(X; \mathbb{R})$ of a space X, the q-dimensional torus leading to exterior algebras $\Lambda^*(\mathbb{R}^q)$. As we shall see, the generators of $\Lambda^*(\mathbb{R}^q)$ yield so-called odd coordinates on super manifolds; these anti-commute and hence are useful when trying to describe physical systems involving Fermions.

Let A be a commutative super algebra. The *derivations* of A are $\mathbb{R}$-linear maps $D \in \operatorname{End}(A)$ satisfying the Leibniz rule: [1]

$$D(a \cdot b) = Da \cdot b + (-1)^{|D||a|} a \cdot Db.$$

$\operatorname{Der} A$ is a *super Lie algebra* with respect to the bracket operation

$$[D, E] := DE - (-1)^{|D||E|} ED,$$

This means that the bracket is (super) skew symmetric

$$[D, E] + (-1)^{|D||E|}[E, D] = 0$$

and satisfies the (super) *Jacobi identity*

$$[D, [E, F]] + (-1)^{|D|(|E|+|F|)}[E, [F, D]] + (-1)^{|F|(|D|+|E|)}[F, [D, E]] = 0.$$

Note that we cyclically permuted the 3 symbols and put down the signs according to the above rule. Another way to remember the signs in the super Jacobi identity is to say that the map

$$D \mapsto (E \mapsto [D, E])$$

sends the super Lie algebra L to its algebra of derivations $\operatorname{Der} L$ (which is defined by the above sign rule).

2.2 Super manifolds

We will define super manifolds as ringed spaces following [DM]. By a morphism we will always mean a map of ringed spaces. The local model for a super manifold of dimension $p|q$ is $\mathbb{R}^p$ equipped with the sheaf $\mathcal{O}_{\mathbb{R}^{p|q}}$ of commutative super $\mathbb{R}$-algebras $U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$.

Definition 4. A *super manifold* M of dimension $p|q$ is a pair $(|M|, \mathcal{O}_M)$ consisting of a (Hausdorff and second countable) topological space $|M|$ together with a sheaf of commutative super $\mathbb{R}$-algebras $\mathcal{O}_M$ that is locally isomorphic to $(\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^{p|q}})$.

[1]Whenever we write formulas involving the degree $|.|$ of certain elements, we implicitly assume that these elements are homogenous.

A morphism (f, F) between super manifolds M, N is defined as a continuous map $f : |M| \to |N|$, together with a map F of sheafs covering f. More precisely, for every open subset $U \subseteq |N|$ there are algebra maps

$$F(U) : \mathcal{O}_N(U) \longrightarrow \mathcal{O}_M(f^{-1}(U))$$

that are compatible with the restriction maps of the two sheafs. We denote this category of super manifold by SMan.

To every super manifold M there is an associated *reduced manifold*

$$M^{red} := (|M|, \mathcal{O}_M/\mathrm{Nil})$$

obtained by dividing out the ideal of nilpotent functions. By construction, this gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of super manifolds $M^{red} \hookrightarrow M$.

Other geometric super objects can be defined in a similar way. For example, replacing $\mathbb{R}$ by $\mathbb{C}$ and C^∞ by analytic functions one obtains *complex (analytic) super manifolds.* There is also an important notion of *cs manifolds.* These are spaces equipped with sheaves of commutative super $\mathbb{C}$-algebras that locally look like $\mathcal{O}_{\mathbb{R}^{p|q}} \otimes \mathbb{C}$. One relevance of *cs* manifolds is that they appear naturally as the smooth super manifolds underlying complex analytic super manifolds. In our work, cs manifolds are essential to define the notion of a *unitary* field theory but this is not relevant for the current discussion.

Example 5. Let E be a real vector bundle of fiber dimension q over the ordinary manifold X^p. Then $(X, \Gamma(\Lambda^* E))$ is a super manifold of dimension $p|q$, denoted by πE. Bachelor's theorem says that every super manifold is isomorphic to one of this type. More precisely, let BunMan denote the category of real vector bundles over smooth manifolds, and for $M \in \mathsf{SMan}$, consider the vector bundle $J(M)$ over M^{red} with sheaf of sections $\mathcal{O}_M/\mathrm{Nil}^2$. Then the functors

$$\pi : \mathsf{BunMan} \to \mathsf{SMan} \quad \text{and} \quad J : \mathsf{SMan} \to \mathsf{BunMan}$$

come equipped with natural isomorphisms $J \circ \pi(E) \cong E$ and *non-natural* isomorphism $\pi \circ J(M) \cong M$. In other words, these functors induce a bijection on isomorphism classes of objects and inclusions on morphisms but they are not equivalences of categories because there are many more morphims in SMan than the linear bundle maps coming from BunMan.

The following proposition gives two extremely useful ways of looking at morphisms between super manifolds. We shall use the notation $C^\infty(M) := \mathcal{O}_M(M)$ for the algebra of (global) functions on a super manifold M.

prop:coords **Proposition 6.** *For $S, M \in \mathsf{SMan}$, the functor C^∞ induces natural bijections*

$$\mathsf{SMan}(S, M) \cong \mathsf{Alg}(C^\infty(M), C^\infty(S))$$

In the language of algebraic geometry one may say that 'super manifolds are affine'. If $M \subseteq \mathbb{R}^{p|q}$ is an open super submanifold (a domain*), SMan(S,M) is in bijective correspondence with those $(f_1, ..., f_p, \eta_1, ..., \eta_q)$ in $(C^\infty(S)^{ev})^p \times (C^\infty(S)^{odd})^q$ that satisfy*

$$(|f_1|(s), ..., |f_p|(s)) \in |M| \subseteq \mathbb{R}^p \text{ for all } s \in |S|.$$

The f_i, η_j are called the coordinates of $\phi \in \mathsf{SMan}(S, M)$ defined by

$$f_i = \phi^*(x_i) \quad \text{and} \quad \eta_j = \phi^*(\theta_j),$$

where $x_1, ..., x_p, \theta_q, ..., \theta_p$ are coordinates on $M \subseteq \mathbb{R}^{p|q}$. Moreover, by the first part we see that $f_i \in C^\infty(S)^{ev} = \mathsf{SMan}(S, \mathbb{R})$ and hence $|f_i| \in \mathsf{Man}(|S|, \mathbb{R})$.

The proof of the first part is based on the existence of partitions of unity for super manifolds, so it is false in analytic settings. The second part always holds and is proved in [?]. Lei

2.3 The functor of points

Since sheaves are generally difficult to work with, one often thinks of super manifolds in terms of their S-points, i.e. instead of M itself one considers the morphism sets $\mathsf{SMan}(S, M)$, where S varies over all super manifolds S. More formally, embed the category SMan of super manifolds in the category of contravariant functors from SMan to Set by

$$M \mapsto (\ S \mapsto \mathsf{SMan}(S, M)\ ).$$

This Yoneda embedding identifies super manifolds with representable contravariant functors $\mathsf{SMan} \to \mathsf{Set}$ and morphisms between super manifolds with natural transformations. Note that the last proposition makes it easy to describe the morphism sets $\mathsf{SMan}(S, M)$. We'd also like to point out that the functor of points approach is closely related to computations involving odd quantities in many physics papers.

2.4 Super Lie groups

According to the functor of points approach, a group object in SMan can be described by giving a representable contravariant functor $G : \mathsf{SMan} \to \mathsf{Set}$ together with functorial group structures on $G(S)$ for all S.

Example 7. The most important super Lie groups are as follows.

1. The additive group structure on $\mathbb{R}^{p|q}$ is given by the following composition law on $\mathsf{SMan}(S, \mathbb{R}^{p|q})$, obviously natural in S:

$$(f_1, ..., \eta_q) \times (h_1, ..., \psi_q) \mapsto (f_1 + h_1, ..., \eta_q + \psi_q).$$

2. The *super general linear group* $GL(p|q)$ is defined by

$$GL(p|q)(S) := \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S^{p|q}) \cong \mathrm{Aut}_{C^\infty(S)}(C^\infty(S)^{p|q}),$$

where $A^{p|q}$ denotes the A-module freely generated by p even and q odd generators. We need to check that this is representable. We claim that $GL(p|q)(\_)$ is represented by the open super submanifold $G \subset \mathbb{R}^{p^2+q^2|2pq}$ characterized by

$$|G| = \{\, x \in \mathbb{R}^{p^2+q^2} \mid x \in GL_p \times GL_q \,\}.$$

This follows directly from proposition prop:coords 6 using that a map between super algebras is invertible if and only if it is invertible modulo nilpotent elements.

3. Using the Berezinian, a super version of the determinant, one can define a super subgroup $SL(p|q) < GL(p|q)$.

4. Let V be a super vector space and $b : V \times V \to \mathbb{R}$ an even symmetric bilinear map. Then the *Heisenberg* super Lie group $H(V, b)$ has underlying super manifold $\mathbb{R} \times V$ and the group structure on $\mathsf{SMan}(S, \mathbb{R} \times V)$ is given by the formula

$$(f_1, v_1), (f_2, v_2) \mapsto (f_1 + f_2 + b(v_1, v_2), v_1 + v_2).$$

In particular, using the simplest (nontrivial) symmetric pairing b on $V = \mathbb{R}^{0|1}$, one can equip $\mathbb{R}^{1|1}$ with the structure of a Heisenberg group. The relevance of this super group lies in the particular structure of its super Lie algebra $\mathfrak{g}$: We shall prove below that $\mathfrak{g}$ is *freely* generated by one *odd* generator. This property also explains the appearance of $\mathbb{R}^{1|1}$ in the context of odd ODEs on super manifolds (see below). For us, the multiplication μ (restricted to $\mathbb{R}^{1|1}_{>0}$) will turn out to be important, since it describes the gluing of 'Riemannian' super intervals, see section ??. susy and K

Even though we haven't yet introduced the super Lie algebra of a super Lie group, we want to explain how super Lie groups can be understood in terms of super Lie algebras.

Theorem 8. *The following categories are equivalent:*

- *The category of* 1*-connected super Lie groups.*
- *The category of tripels* $(G_0, \mathfrak{g}, a)$*, where* G_0 *is a* 1*-connected Lie group,* $\mathfrak{g}$ *is a super Lie algebra whose even part is the Lie algebra of* G_0*, and* a *is an action of* G_0 *on* $\mathfrak{g}$ *extending the adjoint action of* G_0.
- *The category of (finite-dimensional) super Lie algebras over* $\mathbb{R}$

The first equivalence holds even without the assumption on the fundamental group. The second equivalence follows from Lie's theorem. Finite-dimensional simple complex super Lie algebras have been completely classified by Victor Kac in the 70s.

2.5 Super vector bundles

There are two reasonable ways to define (super) vector bundles over a super manifold M:

- as a (super) fiber bundle $E \to M$ with structure group $GL(p|q)$,
- as a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_M$-modules of dimension $p|q$.

These are equivalent because coordinate changes between local trivializations are given by the same data in both cases: For a fiber bundle $E \to M$, a change of trivialization over $U \subset M$ is given by a map $\varphi : U \to GL(p|q)$. However, this is nothing but an automorphism of $\mathcal{O}_U^{p|q}$ (recall the definition of $GL(p|q)$ in terms of its S-points) which is exactly the datum giving a change of local trivializations of a locally free sheaf of dimension $p|q$.

Let us now look at the basic example of a super vector bundle, the *tangent bundle* of a super manifold $M^{p|q}$. It is the sheaf of $\mathcal{O}_M$-modules $\mathfrak{X}M$ defined by

$$\mathfrak{X}M(U) := \mathrm{Der}\,\mathcal{O}_M(U).$$

$\mathfrak{X}M$ is locally free of dimension $p|q$: If $x_1, ..., \theta_q$ are local coordinates on M, then a local basis is given by $\partial_{x_1}, ..., \partial_{\theta_q}$. Note that there is also a linear fibre bundle $TM \to M$ with structure group $GL(p|q)$, where TM is a super manifold of dimension $2p|2q$. The set of sections of this projection can be identified with $(\mathfrak{X}M)^{ev}$. To capture all vectorfields on M, i.e. the global sections in $\mathfrak{X}M$, one would need to pull back TM along all S-points of M and consider sections of those pull backs.

The *cotangent bundle* of M is the sheaf of $\mathcal{O}_M$-modules $\Omega^1 M$ dual to $\mathfrak{X}M$. As in the case of usual manifolds one obtains differential forms on M by looking at the exterior algebra of $\Omega^1 M$. Furthermore, a de Rham differential d on $\Omega^* M$ can be defined. It turns out that this complex is quasi-isomorphic to the usual de Rham complex of M^{red}. In particular, the cohomology of this complex is isomorphic to the de Rham cohomology $H^*_{dR}(M^{red})$.

2.6 The super Lie algebra of a super Lie group

Now we can define the super Lie algebra $\mathfrak{g}$ of a super Lie group G. A vector field $\xi \in \mathfrak{X}G$ is called *left-invariant* if ξ is related to itself under the left-translation by all $f : S \to G$:

$$S \times G \xrightarrow{f \times \mathrm{id}} G \times G \xrightarrow{\mu} G.$$

Here we interpret ξ as a vertical vector field on $S \times G$ in the obvious way. The super Lie algebra $\mathfrak{g}$ consists of all left-invariant vector fields on G. Evaluation at $e \in G$ defines an isomorphism $\mathfrak{g} \cong T_e G$, in particular, the vector space dimension of $\mathfrak{g}$ is $p|q$.

Example 9. For the Heisenberg super group structure on $\mathbb{R}^{1|1}$, left-translation by a map $f = (f_1, f_2) : S \to \mathbb{R}^{1|1}$ is given by the formula

$$S \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1},\ (s, t, \theta) \mapsto (f_1(s) + t + f_2(s)\theta, f_2(s) + \theta)$$

Differentiation yields that this maps the vector fields ∂_t and ∂_θ to

$$\partial_t \quad \text{and} \quad -f_2(s)\partial_t + \partial_\theta.$$

Hence ∂_t is a left-invariant vector field. Solving the appropriate linear equation, one sees easily that another left-invariant vector field is given by

$$D := -\theta\partial_t + \partial_\theta \quad \text{satisfying} \quad D^2 = \frac{1}{2}[D, D] = -\partial_t.$$

Hence we see that the Lie algebra of $\mathbb{R}^{1|1}$ is *freely* generated by one odd generator D.

This is the reason why $\mathbb{R}^{1|1}$ with the Heisenberg super group structure plays a role for odd ODEs on super manifolds: An odd vector field $\xi \in (\mathfrak{X}M)^{odd}$ determines a unique map from the super Lie algebra of $\mathbb{R}^{1|1}$ to vector fields on M. This, in turn, generates the flow of ξ on M. Hence the flow property for the flow of an odd vector field on a super manifold is expressed in terms of the Heisenberg super group structure of $\mathbb{R}^{1|1}$.

All the subtleties regarding how long the flow is defined only take place in the reduced manifold M^{red} where the flow is generated by the restriction of the even vector field $[\xi, \xi]$ to M^{red}. An important case of a flow that's always defined is when $[\xi, \xi] = 0$ because the induced flow on M^{red} is constant. Then one obtains an action of $\mathbb{R}^{0|1}$ on M. Conversely, any $\mathbb{R}^{0|1}$-action leads to an operator with square zero. An important example is the *odd tangent bundle* πTM. We'll see in the next section that πTM represents the super manifold $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, M)$ and hence it has an obvious action of

$$\mathrm{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^{\times}$$

given by pre-composition. This leads to the most conceptual interpretation of the de Rham differential d (as infinitesimal generator of the translational part $\mathbb{R}^{0|1}$) and the grading (given by the $\mathbb{R}^{\times}$-action) on differential forms. In particular, the commutation relations among translations and dilations show that d has degree one and square zero!

3 Super points in M

sec:piTM

For a super manifold M, we would like to talk about the super manifold $\mathrm{SP}(M)$ of *super points* in M. By definition, this is the inner Hom from the super point spt $:= \mathbb{R}^{0|1}$ to M in the category SMan, usually denoted by $\underline{\mathsf{SMan}}(\mathrm{spt}, M)$. If such an inner Hom exists, it is defined (up to canonical isomorphism) by the property that it is a representing object for the contravariant functor

$$S \mapsto \mathsf{SMan}(\mathrm{spt} \times S, M).$$

More explicitly, one requires natural bijections

$$\mathsf{SMan}((\mathrm{spt} \times S, M) \cong \mathsf{SMan}(S, \mathrm{SP}(M)) \tag{10}$$

The following proposition implies that $\mathrm{SP}(M)$ exists in SMan.

prop:SP **Proposition 11.** *The odd tangent bundle πTM represents the super points* $\mathrm{SP}(M) := \underline{\mathsf{SMan}}(\mathrm{spt}, M)$*, where* $\mathrm{spt} := \mathbb{R}^{0|1}$*. More briefly,* $\mathrm{SP}(M) \cong \pi TM$*.*

Proof. We split the proof of the desired bijection into the following natural correspondences, where in (3) Der_f denotes derivations $C^\infty(M) \to C^\infty(S)$ w.r.t. f, in the sense that $C^\infty(S)$ is a $C^\infty(M) - C^\infty(M)$-bimodule using the algebra homomorphism f.

$$\begin{aligned}
\mathsf{SMan}((\mathrm{spt} \times S, M) &\underset{(1)}{\longleftrightarrow} \mathsf{Alg}(C^\infty(M), C^\infty(\mathrm{spt} \times S)) \\
&\underset{(2)}{\longleftrightarrow} \mathsf{Alg}(C^\infty(M), \Lambda^*(\mathbb{R}) \otimes C^\infty(S)) \\
&\underset{(3)}{\longleftrightarrow} \{(f, g) \mid f \in \mathsf{Alg}(C^\infty(M), C^\infty(S)), g \in \mathrm{Der}_f^{odd}\} \\
&\underset{(4)}{\longleftrightarrow} \{(f, g) \mid f \in \mathsf{SMan}(S, M), g \in (f^* \mathfrak{X} M)^{odd}\} \\
&\underset{(5)}{\longleftrightarrow} \mathsf{SMan}(S, \pi TM)
\end{aligned}$$

(1) follows directly from proposition prop:coords 6 and (2) just uses the definition of products of super manifolds together with $C^\infty(\mathrm{spt}) = \Lambda^*(\mathbb{R})$. To see (3), decompose $\varphi : C^\infty(M) \to \Lambda^*(\mathbb{R}) \otimes C^\infty(S) = C^\infty(S)[\theta]$ as a sum

$$\varphi = f + \theta g, \text{ with } f, g : C^\infty(M) \to C^\infty(S).$$

Here θ is the usual odd coordinate on $\mathrm{spt} = \mathbb{R}^{0|1}$. Note that f preserves the grading, whereas g reverses it. For $a, b \in C^\infty(M)$ we have $\varphi(ab) = f(ab) + \theta g(ab)$, and since φ is an algebra homomorphism this is also equal to

$$\begin{aligned}
\varphi(a)\varphi(b) &= (f(a) + \theta g(a))(f(b) + \theta g(b)) \\
&= f(a)f(b) + \theta(g(a)f(b) + (-1)^{|a|} f(a)g(b)).
\end{aligned}$$

Comparing the coefficients we conclude that f is an algebra homomorphism and that g is an odd derivation w.r.t. f. Conversely, any such pair (f, g)

defines an algebra map φ. It is clear that the bijection is natural w.r.t. super algebra maps $C^\infty(S) \to O_{S'}$.

Proposition prop:coords 6 takes care of the part involving f in (4). The statement concerning g follows from the next lemma by looking at the odd part of the global sections.

Lemma 12. *Let $(|f|, f) : S \to M$ be a map of super manifolds. Then*

$$f^*\mathfrak{X}M \cong Der_f(|f|^{-1}\mathcal{O}_M, \mathcal{O}_S) \text{ as sheaves of } \mathcal{O}_S\text{-modules.}$$

Here $Der_f(|f|^{-1}\mathcal{O}_M, \mathcal{O}_S)$ denotes the sheaf of derivations from $|f|^{-1}\mathcal{O}_M$ to $\mathcal{O}_S$ w.r.t. f.

The proof consists of putting together several standard isomorphisms of sheaves. The construction is indicated in Ma [Ma, Ch.4,§1.10].

Finally, (5) is just the definition of the super manifold πTM in terms of its S-points, see DM [DM, p.72]. This finishes the proof of the proposition. We would like to point out that proposition prop:coords 6 is not at all crucial for the proof. One can write down the equivalences in terms of maps of sheaves (instead of their restriction to global sections), the only thing that changes is that the notation becomes more complicated. □

Let us write down the above natural bijection more explicitly for super domains $M = U \subseteq \mathbb{R}^{p|q}$. In this case we can make the identification $\pi TU \cong U \times \mathbb{R}^{q|p}$. Going through the above bijections one sees that if the morphism $\varphi : S \times \mathrm{spt} \to U$ is given by coordinates

$$(x_1 + \theta\hat{x}_1, ..., x_p + \theta\hat{x}_p, \eta_1 + \theta\hat{\eta}_1, ..., \eta_q + \theta\hat{\eta}_q) \in (\mathcal{O}^{ev}_{S\times\mathbb{R}^{0|1}})^p \times (\mathcal{O}^{odd}_{S\times\mathbb{R}^{0|1}})^q,$$

then its image in $\tilde{\varphi} \in \mathsf{SMan}(S, U \times \mathbb{R}^{q|p})$ has coordinates

$$(x_1, ..., x_p, \eta_1, ..., \eta_q, \hat{\eta}_1, ..., \hat{\eta}_q, \hat{x}_1, ..., \hat{x}_p) \in (\mathcal{O}^{ev}_S)^p \times (\mathcal{O}^{odd}_S)^q \times (\mathcal{O}^{ev}_S)^q \times (\mathcal{O}^{odd}_S)^p.$$

sec:translations

3.1 The translation action of $\mathbb{R}^{0|1}$.

Using the additive group structure

$$\mu : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \longrightarrow \mathbb{R}^{0|1}, \ (\eta, \theta) \mapsto \eta + \theta$$

we make $\mathbb{R}^{0|1}$ into a super Lie group. Then for every $\eta : S \to \mathbb{R}^{0|1}$ there is a *right translation* by η

$$r_\eta : \mathbb{R}^{0|1} \times S \xrightarrow{\mathrm{id} \times \eta} \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \xrightarrow{\mu} \mathbb{R}^{0|1}$$

From this, we obtain a group action $\mathbb{R}^{0|1} \times \mathrm{SP}(M) \longrightarrow \mathrm{SP}(M)$ by precomposing by right translations: To an S-point

$$(\eta, \varphi) \in \mathsf{SMan}(S, \mathbb{R}^{0|1}) \times \mathsf{SMan}(\mathbb{R}^{0|1} \times S, M) \cong \mathsf{SMan}(S, \mathbb{R}^{0|1} \times \mathrm{SP}(M))$$

it associates the composition (using the projection $p_S : \mathbb{R}^{0|1} \times S \to S$)

$$\varphi_\eta = \varphi(r_\eta, p_S) \in \mathsf{SMan}(\mathbb{R}^{0|1} \times S, M) \cong \mathsf{SMan}(S, \mathrm{SP}(M)).$$

In local coordinates this just means that φ_η is given by replacing θ in the coordinate representation of φ by $\theta + \eta$. Translating by Proposition prop:SP 11 from $\mathrm{SP}(M)$ to πTM and working locally on a super domain, this action can be written in coordinates as

$$(\eta, x_1, ..., \eta_q, \hat{\eta}_1, ..., \hat{x}_p) \mapsto (x_1 + \eta\hat{x}_1, ..., \eta_q + \eta\hat{\eta}_q, \hat{\eta}_1, ..., \hat{x}_p).$$

Differentiating the action with respect to η we obtain the action of the generator ∂_η of the super Lie algebra of $\mathbb{R}^{0|1}$ on the algebra of functions on πTM. It is a globally defined odd vector field D with local representation

$$(x_1, ..., \eta_q, \hat{\eta}_1, ..., \hat{x}_p) \mapsto (\hat{x}_1, ..., \hat{\eta}_q, 0, ..., 0)$$

which we may write as

$$D = \sum_{i=1}^{p} \hat{x}_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \hat{\eta}_i \frac{\partial}{\partial \eta_i}.$$

Writing the coordinates as $(y_1, ..., y_{p+q}) := (x_1, ..., \eta_q)$ and letting $\partial_i := \frac{\partial}{\partial y_i}$ this takes the simple form

$$D = \sum_{i=1}^{p+q} \hat{y}_i \partial_i$$

Note that since $\partial_\eta^2 = \frac{1}{2}[\partial_\eta, \partial_\eta] = 0$ applying D twice annihilates all functions on πTM.

3.2 The de Rham complex $\Omega^* M$

For a super manifold M, the algebra of differential forms on M has two gradings, the $\mathbb{Z}/2$-parity and the (cohomological) $\mathbb{Z}$-degree. There are two conventions how to deal with this situation, we will work with 'point of view I' in [DM, p.62] that makes $\Omega^* M$ into a $\mathbb{Z}$-graded commutative super algebra and leads to an *odd* de Rham differential d. This seems to be a natural choice, since we want to relate d to the action of the odd vector field D on the commutative super algebra $C^\infty(\pi TM)$.

Let V be a module over the super commutative algebra A. Following the convention of Bernstein-Leites we define the exterior algebra on V to be

$$\Lambda_A^*(V) := \mathrm{Sym}_A(\pi V).$$

Here $\mathrm{Sym}_A(W)$ is the quotient of the tensor algebra on W by the ideal generated by all super commutators $w_1 \otimes w_2 - (-1)^{|w_1||w_2|} w_2 \otimes w_1$. The commutative super algebra $\Lambda_A^*(V)$ has the universal property that giving a super algebra map from $\Lambda_A^*(V)$ to any commutative A-super algebra B is the same as giving an A-module map $\pi V \to B$, compare [Ma, Ch.3, §2.5].

Let $\Omega^1 M := \mathrm{Hom}_{\mathcal{O}_M}(\mathfrak{X}M, \mathcal{O}_M)$ be the cotangent sheaf of M and define

$$\Omega^* M := \Lambda^*_{\mathcal{O}_M}(\Omega^1 M).$$

Clearly, $\mathcal{O}_M$ and $\pi\Omega^1 M$ are subsheaves of $\Lambda^*_{\mathcal{O}_M}(\Omega^1 M)$ in a natural way. From the universal *even* differential $d_{ev} : \mathcal{O}_M \to \Omega^1 M$ that is characterized by

$$d_{ev} f(X) = (-1)^{|f||X|} X(f) \ \text{ for all } X \in \mathfrak{X}M$$

we obtain an *odd* differential $d := \pi d_{ev}$ by composing with the *odd* parity reversal $\pi : \Omega^1 M \to \pi\Omega^1 M$. A de Rham differential on $\Omega^* M$ is an extension of d whose square is zero and which satisfies the Leibniz rule. These properties characterize such an extension and we will see below that de Rham differential indeed exists.

3.3 Differential forms on M and functions on πTM

The next step is to interpret differential forms on M as functions on πTM.

Lemma 13. *There is an embedding of sheaves of $\mathcal{O}_M$-super algebras*

$$\iota : \Omega^* M \hookrightarrow \mathcal{O}_{\pi TM}$$

that maps onto the functions that are polynomial on every fiber.

Proof. Let $x_1, ..., \eta_q$ be local coordinates on M. We have canonically associated coordinates

$$(x_1, ..., \eta_q, \hat{\eta}_1, ..., \hat{\eta}_q, \hat{x}_1, ..., \hat{x}_p)$$

on πTM. Recall that the $\hat{x}_i$'s are odd, whereas the $\hat{\eta}_j$'s are even. On the other hand, a local basis for the $\mathcal{O}_M$-module $\Omega^1 M$ is given by $dx_1, ..., d\eta_q$. According to the convention we picked for the definition of the de Rham complex, the dx_i and $d\eta_j$ have odd and even parity, resp. Hence we can define a map of super $\mathcal{O}_M$-modules $\iota_0 : \pi\Omega^1 M \longrightarrow \mathcal{O}_{\pi TM}$ by prescribing $dx_i \mapsto \hat{x}_i$ and $d\eta_j \mapsto \hat{\eta}_j$. It is not hard to check that this is independent of the coordinate system chosen.[2] According to the defining property of $\mathrm{Sym}_{\mathcal{O}_M}(\pi\Omega^1 M)$ the map ι_0 extends to a unique homomorphism of $\mathcal{O}_M$-algebras $\iota : \Omega^* M \to \mathcal{O}_{\pi TM}$. It is clear that ι is injective with image as stated above. □

The map ι is surjective if and only if M is an ordinary manifold. For example, if $M = \mathbb{R}^{0|q}$ then $\Omega^* M = \Lambda(\mathbb{R}^q)[x_1, \dots, x_q]$, the polynomial ring on q even generators x_i over the ground ring $\Lambda(\mathbb{R}^q)$. It has to be completed in the x_i-directions to obtain

$$C^\infty(\pi TM) = C^\infty(\mathbb{R}^{q|q}) = \Lambda(\mathbb{R}^q) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^q)$$

3.4 Interpreting D as the de Rham differential on $\Omega^* M$

Since D is an odd vector field on πTM we have the Leibniz rule

$$D(fg) = (Df)g + (-1)^{|f|} f(Dg) \text{ for all functions } f, g \text{ on } \pi TM.$$

Furthermore, we already know that $D^2 = 0$. Hence the restriction of D to $\Omega^* M$ deserves to be called a de Rham differential once we have shown

lem:d **Lemma 14.** *The restriction of D to $\mathcal{O}_M \subset \mathcal{O}_{\pi TM}$ is the odd differential d. More precisely, we have*

$$D = \iota d : \mathcal{O}_M \longrightarrow \mathcal{O}_{\pi TM}.$$

[2] In fact, one can see this using the (global!) vector field D considered in Section 3.1 (sec:translations): The map ι_0 is equal to the composition

$$\pi\Omega^1 M \underset{\pi^{-1}}{\longrightarrow} \Omega^1 M \hookrightarrow \Omega^1(\pi TM) \underset{\bar{D}}{\longrightarrow} \mathcal{O}_{\pi TM}, \text{ where } \bar{D}(\omega) = (-1)^{|\omega|}\omega(D).$$

Proof. It is clear from the local representation of D that the image of D is contained in $\iota(\pi\Omega^1 M) \subset \mathcal{O}_{\pi TM}$. The claim is equivalent to showing that the composition

$$\tilde{D} := \pi^{-1}\iota^{-1}D : \mathcal{O}_M \longrightarrow \Omega^1 M$$

is equal to d_{ev}, i.e. for all $f \in \mathcal{O}_M$ we have to check that

$$\tilde{D}f(X) = (-1)^{|f||X|}X(f) \text{ for all vector fields } X \in \mathfrak{X}M.$$

It suffices to prove this for (local) basis vector fields $\partial_j = \frac{\partial}{\partial y_j}$, $j = 1, ..., p+q$, where the y_i are local coordinates on M. We first compute

$$\tilde{D}f = \pi^{-1}\iota^{-1}\left(\sum_{i=1}^{p+q} \hat{y}_i \partial_i f\right) = \pi^{-1}\iota^{-1}\left(\sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)}(\partial_i f)\hat{y}_i\right)$$

Since ι and π are even respectively odd $\mathcal{O}_M$-module maps, we get

$$\tilde{D}f = \pi^{-1}\left(\sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)}(\partial_i f)dy_i\right) = \sum_{i=1}^{p+q} (-1)^{|y_i|(|f|+1)}(\partial_i f)d_{ev}y_i$$

Applying this 1-form to ∂_j and using $(d_{ev}y_i)\partial_j = (-1)^{|y_i||y_j|}\delta_{ij}$ yields

$$\tilde{D}f(\partial_j) = (-1)^{|y_j|(|f|+1)+|y_j|} = (-1)^{|y_j||f|}\partial_i(f)$$

as desired. □

If M is purely even the cohomological degree of $\alpha \in \Omega^* M$ is equal to the parity of its image in $O_{\pi TM}$ modulo 2. Hence the Leibniz rule above is exactly the (graded) Leibniz rule for differential forms, and so D is equal to the usual de Rham differential on $\Omega^* M$.[3]

3.5 The diffeomorphism super group of the super point

We used the translation action of $\mathbb{R}^{0|1}$ on itself to define an action of $\mathbb{R}^{0|1}$ on the super points $\mathrm{SP}(M) \cong \pi TM$. In fact, the whole super Lie group $\underline{\mathrm{Diff}}(\mathrm{spt})$ of diffeomorphisms of the super manifold $\mathrm{spt} = \mathbb{R}^{0|1}$ acts on $\mathrm{SP}(M)$. We briefly describe this action. By definition, $\underline{\mathrm{Diff}}(\mathrm{spt})$ is the super Lie group representing the group-valued functor

$$S \mapsto \mathrm{Diff}_S(\mathrm{spt} \times S, \mathrm{spt} \times S).$$

[3]Note that in the purely even case our definition of $\Omega^* M$ coincides with the usual one.

Here $\mathrm{Diff}_S(\mathrm{spt} \times S, \mathrm{spt} \times S)$ is the group of diffeomorphisms of $\mathrm{spt} \times S$ that are compatible with the projection to S. We have the following result which follows directly from Proposition 11 with $M = \mathrm{spt}$.

lem:aut **Lemma 15.** $\underline{\mathrm{Diff}}(\mathrm{spt}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$*, where* $\mathbb{R}^\times$ *acts on* $\mathbb{R}^{0|1}$ *by multiplication.*

Here the $\mathbb{R}^{0|1}$ part corresponds to the odd translation action we described before, whereas $t \in \mathbb{R}^\times$ acts as the even dilation $\theta \mapsto t\theta$. Using the S-point formalism and Proposition 11 one sees that there is a natural action of $\underline{\mathrm{Diff}}(\mathrm{spt})$ on πTM. In local coordinates, the action of $\mathbb{R}^\times$ on πTM is given by

$$(t, y_1, ..., y_{p+q}, \hat{y}_1, ..., \hat{y}_{p+q}) \mapsto (y_1, ..., y_q, t\hat{y}_1, ..., t\hat{y}_{p+q}).$$

Every function f on πTM which is polynomial on fibers is locally a finite sum of functions of the form

$$f = g\hat{y}_1^{i_1} ... \hat{y}_{p+q}^{i_{p+q}}, \text{ where } g \in \mathcal{O}_M.$$

It follows that the action of $t \in \mathbb{R}^\times$ on f is given by the formula

$$t(f) = t^{\sum_{k=1}^{p+q} i_k} \cdot f$$

cor:dilations **Corollary 16.** *The* $\mathbb{R}^\times$*-action on* $\Omega^* M$ *coming from dilations of the super point determines the* $\mathbb{Z}$*-degree operator and vice versa. More precisely,* $t \in \mathbb{R}^\times$ *maps* α *of cohomological degree* k *to* $t^k \cdot \alpha$*.*

4 $0|1$-dimensional field theories

sec:0TFT

The usual definition of a d-dimensional TFT over X, going back to Atiyah and Segal, is in terms of a symmetric monoidal functor E with domain a bordism category whose objects are closed $(d-1)$-manifolds, equipped with a smooth map to the manifold X. Morphisms are diffeomorphism classes of compact d-dimensional cobordisms (again equipped with a smooth map to the manifold X). The target of the functor is the category of finite dimensional vector spaces. In [ST2] we explain a precise version of this definition and add several bells and whistles, for example super symmetry and a notion of degree. We also describe what these precise definitions mean in dimension 0 and $0|1$, which can be summarized as follows (in the case of ground field $\mathbb{R}$).

There is only one (-1)-dimensional manifold, the empty set $\emptyset$, so the symmetric monoidal functor E takes it to the monoidal unit $\mathbb{R}$ in the category of $\mathbb{R}$-vector spaces. Moreover, any compact 0-manifold is the disjoint union of points, so the real number $E(\mathrm{pt})$ determines the TFT completely (and vice versa) in the absence of a target manifold X. In the presence of X, the functor has to be evaluated on every map $\mathrm{pt} \to X$ and hence TFTs over X are exactly real valued functions on X. Note that there is no smoothness or continuity requirement on these functions!

The first thing we do in [ST2] is to introduce a family version of the bordism and vector space categories, working with bundles of $(d-1)$ and d-manifolds (respectively vector spaces) over a base manifold S. More precisely, the domain and target categories of E are both categories fibred over the site Man of smooth manifolds and E is required to preserve cartesian arrows and the projections to Man. As a consequence, we obtain the bijection

$$0\text{-}\mathrm{TFT}(X) \cong C^\infty(X;\mathbb{R})$$

and so there is a single concordance class, $0\text{-}\mathrm{TFT}[X] = 0$. This means that no cohomological information can be derived from 0-dimensional TFTs but surprisingly, this changes as soon as we introduce one odd dimension!

The same discussion as above shows that a $0|1$-dimensional TFT E over X assigns a real number to every map $\mathrm{spt} \to X$ from the super point $\mathrm{spt} = \mathbb{R}^{0|1}$ to X. More precisely, we now work in families over the site SMan of super manifolds, so a (universal) example of a morphism in our bordism category would be the trivial bundle $p_1 : \mathrm{SP}(X) \times \mathrm{spt} \to \mathrm{SP}(X)$ where the total space maps to X via the evaluation map $ev : \mathrm{SP}(X) \times \mathrm{spt} \to X$, corresponding to the identity of $\mathrm{SP}(X)$. By definition, the value of an TFT E on this family (p_1, ev) over X is a function on the base $\mathrm{SP}(X)$ of the family, i.e.

$$E(p_1, ev) \in C^\infty(\mathrm{SP}(X)) \cong \Omega^*(X).$$

If the set of *all* differential forms was the final answer, we would again get a single concordance class. However, unlike the point, the *super* point spt has non-trivial diffeomorphisms that need to be taken into account. Recall from Lemma 15 that the diffeomorphism super group of spt is given by

$$D := \underline{\mathrm{Diff}}(\mathrm{spt}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times,$$

where $\mathbb{R}^\times$ is the even dilational part and $\mathbb{R}^{0|1}$ are the odd translations of spt. The diffeomorphism invariance of a *topological* field theory E over X implies that $E(p_1, ev)$ must be invariant under this action:

lem:stack **Lemma 17.** *Consider the quotient stack*

$$\mathrm{SP}(X)//D = \underline{\mathsf{SMan}}(\mathrm{spt}, X)//\underline{\mathrm{Diff}}(\mathrm{spt})$$

on the site SMan *of super manifolds. It carries a real line bundle L defined by the projection $D \twoheadrightarrow \mathbb{R}^\times$ such that its spaces of sections lead to isomorphisms*

$$0|1\text{-}\mathrm{TFT}^n(X) \cong \Gamma(L^{\otimes n})$$

In particular, $0|1$-dimensional TFTs over X of degree 0 are just functions on the moduli stack $\mathrm{SP}(X)//D$ of super points in X.

From Section sec:piTM 3, particularly Proposition prop:SP 11, Lemma lem:d 14 and Corollary cor:dilations 16, we get isomorphisms

$$\Gamma(L^{\otimes n}) \cong \Omega^n_{cl}(X)$$

which implies Theorem thm:0TFT 2. For the proof of Theorem thm:0EFT 1 we *define* the notion of a *Euclidean* field theory by giving the super point spt the geometry determined by the subgroup $G := \mathbb{R}^{0|1} \rtimes \{\pm 1\}$ of D given by translations and reflections. It then follows just like in the proof of Lemma lem:stack 17 below that

$$0|1\text{-}\mathrm{EFT}^n(X) \cong \Gamma(\mathrm{SP}(X)//G; L^{\otimes n})$$

where L is the same line bundle as above, given by the projection $G \twoheadrightarrow \{\pm 1\}$. Finally, the results of Section sec:piTM 3 again lead to the computation of these spaces of invariant sections as given in Theorem thm:0EFT 1.

Proof of Lemma lem:stack *17.* By definition of a quotient stack, an object of $\mathrm{SP}(X)//D$ over the super manifold S is given by a D-principal bundle $P \to S$, together with a D-equivariant map $f'' : P \to \mathrm{SP}(X)$. By equation eq:SP ?? this map is the same thing as map $f' : P \times \mathrm{spt} \to X$ and the D-equivariance means that this map factors through a map

$$f : Q := P \times_D \mathrm{spt} \longrightarrow X$$

This associated bundle $Q \to S$ is a general bundle of super points over S and it comes equipped with a map $f : Q \to X$. Therefore, the objects of the stack $\mathrm{SP}(X)//D$ over S are in 1-1 correspondence with such pairs (Q, f). Now these pairs are exactly the objects of the bordism category which is the domain of a $0|1$-dimensional TFT E over X, at least if we restrict our $0|1$-manifolds to be the super point spt (the others are taken care of by disjoint union). The same applies to the morphisms and hence by definition, E is a functor on $\mathrm{SP}(X)//D$ over SMan with values in the representable stack $\mathbb{R}$. It follows that

$$0|1\text{-}\mathrm{TFT}^0(X) \cong \mathsf{SMan}(\mathrm{SP}(X), \mathbb{R})^D = \Omega^{ev}(X)^D = \Omega^0_{cl}(X).$$

To understand degree n field theories, we have to go back to their definition in ST2 [ST2].

□

References

DM [DM] Pierre Deligne and John W. Morgan, *Notes on Supersymmetry (following Joseph Bernstein)*

MW [MW] Ib Madsen and Michael Weiss

Ma [Ma] Yuri Manin, *Gauge Theory and Complex Geometry*

ST [ST] Stephan Stolz and Peter Teichner, *Elliptic Objects*

ST2 [ST2] Stephan Stolz and Peter Teichner, *Survey*