FROM MINIMAL GEODESICS TO SUPER SYMMETRIC FIELD THEORIES

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In memory of Raoul Bott, friend and mentor.

Abstract. There are many models for the K-theory spectrum known today, each one having its own history and applications. The purpose of this note is to give an elementary description of eight such models (and certain completions of them) and to relate all of them by canonical maps, some of which are homeomorphisms (rather than just homotopy equivalences). Our survey begins with Raoul Bott's iterated spaces of minimal geodesics in orthogonal groups, which he used to prove his famous periodicity theorem, and includes Milnor's spaces of Clifford module structures as well as the Atyiah-Singer spaces of Fredholm operators. From these classical descriptions we move via spaces of unbounded operators and super semigroups of operators to our most recent model, which is given by certain spaces of super symmetric (1|1)-dimensional field theories. These theories were introduced by the second two authors for the purpose of generalizing them to cocycles for elliptic cohomology in terms of certain super symmetric (2|1)-dimensional field theories.

Introduction

At the first Arbeitstagung 1957 in Bonn, Alexander Grothendieck presented his version of the Riemann-Roch theorem in terms of a group (now known as *Grothendieck group*) constructed from (isomorphism classes of) algebraic vector bundles over algebraic manifolds. Some people say that he used the letter K to abbreviate 'Klassen', the German word for (isomorphism) classes. Michael Atiyah and Friedrich Hirzebruch instantly realized that the same construction can be applied to all complex vector bundles over a topological space X, yielding a commutative ring $K(X)$, where addition and multiplication come from direct sum respectively tensor product of vector bundles. For example, every complex vector bundle over the circle is trivial and hence $K(S^1) = \mathbb{Z}$. Moreover, it is also easy to see that

$$K(S^2) = \mathbb{Z}[L]/(L-1)^2,$$

is generated by Hopf's line bundle L over the 2-sphere.

At the second Arbeitstagung in 1958, Raoul Bott explained his celebrated periodicity theorem which can be expressed as the computations $K(S^{2n-1}) \cong$

$K(S^1)$ and $K(S^{2n}) \cong K(S^2)$ for all $n \in \mathbb{N}$. Bott also proved a real periodicity theorem involving the Grothendieck group of (isomorphisms classes of) real vector bundles over X. After dividing by the subgroup generated by trivial bundles (the quotient is denoted by a tilde over the K-groups) one obtains

$$\widetilde{KO}(S^n) \cong \begin{cases} \mathbb{Z} & \text{for} \quad n \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{for} \quad n \equiv 1, 2 \mod 8 \\ 0 & \text{else} \end{cases}$$

It was again Atiyah and Hirzebruch who realized that Bott's periodicity theorem could be used to define *generalized cohomology theories* $K^n(X)$ and $KO^n(X)$ that are 2- respectively 8-periodic and satisfy $K^0 = K, KO^0 = KO$. They satisfy the same Eilenberg-Steenrod axioms (functoriality, homotopy invariance and Mayer-Vietoris principle) as the ordinary cohomology groups $H^n(X)$ but if one takes X to be a point one obtains non-trivial groups for some $n \neq 0$. In fact, the above computation yields by the suspension isomorphism

$$K^{-n}(pt) \cong \widetilde{K}^{-n}(S^0) \cong \widetilde{K}^0(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

and similarly for real K-theory. Several classical problems in topology were solved using this new cohomology theory. For example, the maximal number of independent vector fields on the n-sphere was determined explicitly. A modern way to express any generalized cohomology theory is to write down a spectrum, i.e. a sequence of spaces with certain structure maps. For ordinary cohomology, these would be the Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$ and for K-theory one can use Bott's spaces of iterated minimal geodesics in orthogonal groups, see below.

The purpose of this note is to give an elementary description of eight models (and their completions) of the spaces in the K-theory spectrum and relate all of them by canonical maps, most of which are homeomorphisms (rather then just homotopy equivalences). We will exclusively work with real K-theory KO but all statements and proofs carry over to the complex case, and, with a little more care, also to Atiyah's Real K-theory.

Before we start a more precise discussion, we'll give a list of the models that will be described in this paper. Recall that a pointed CW-complex E_n is said to *represent* the functor KO^n if there are natural isomorphisms of pointed sets

$$[X, E_n] \cong KO^n(X)$$

for all CW-complexes X. By Brown's representation theorem, such E_n exist and are unique up to homotopy equivalence. The suspension isomorphism

$\widetilde{KO}^n(E_n) \cong \widetilde{KO}^{n+1}(\Sigma E_n)$ in real K-theory then takes the identity map on E_n to a map $\Sigma E_n \to E_{n+1}$ whose adjoint must be a homotopy equivalence

$$\epsilon_n : E_n \xrightarrow{\simeq} \Omega E_{n+1}.$$

The sequence $\{E_n, \epsilon_n\}_{n\in\mathbb{Z}}$ of spaces and structure maps is an Ω-spectrum representing the cohomology theory KO. The *homotopy groups* of the spectrum, $\pi_n KO := \pi_n E_0$, are then given by the connected components $\pi_0 E_{-n} \cong \pi_0 \Omega^n E_0 \cong \pi_n E_0$ which explains partially why we describe these negatively indexed spaces below.

To fix notation, let C_n be the Clifford algebra associated to the *positive* definite inner product on $\mathbb{R}^n$. It has generators $e_1, ..., e_n$ satisfying the relations

$$e_i^2 = -\mathbb{1}, \quad e_i e_j + e_j e_i = 0 \text{ for } i \neq j$$

and it turns into a C^*-algebra via $e_i^* = -e_i$. For $n \geq 0$, we define C_{-n} to be the Clifford C^*-algebra for the negative definite inner product, so the operators e_i are self-adjoint and $e_i^2 = \mathbb{1}$. We also fix a separable real Hilbert space H_n with a $*$-representation of C_{n-1} such that all (i.e. one or two) irreducible C_{n-1}-modules appear infinitely often. Note that C_n is a graded algebra such that the even part C_n^{ev} is isomorphic to C_{n-1}. Thus we get a graded C_n-module $\mathcal{H}_n := H_n \otimes_{C_n^{ev}} C_n$. We denote the grading involution by α. All gradings in this paper are $\mathbb{Z}/2$-gradings.

thm:main

Theorem 1. *The following spaces are all homotopy equivalent and represent the $(-n)$-th space in the real K-theory spectrum. The spaces in (2) to (5) are actually homeomorphic.*

1. *The Bott space $\mathcal{B}_n$. Here $\mathcal{B}_1$ is defined to be the union of all orthogonal groups and for $n > 1$ and the space $\mathcal{B}_n$ is the space of minimal geodesics in $\mathcal{B}_{n-1}$.*
2. *The Milnor space $\mathcal{M}_n^{fin}$ (resp. $\mathcal{M}_n$) of "C_{n-1}-module structures" on H_n. More precisely, for $n > 1$ these are unitary structures J on H_n, such that $J - e_{n-1}$ has finite rank (is compact) and $Je_i = -e_i J$ for $1 \leq i \leq n-2$.*
3. *The space Inf_n^{fin} (resp. Inf_n) of "infinitesimal generators", i.e. odd, self-adjoint unbounded C_n-linear operators on $\mathcal{H}_n$ with finite rank (compact) resolvent.*
4. *The configuration space Conf_n^{fin} (resp. Conf_n) of finite dimensional (ungraded) mutually perpendicular C_n-submodules V_λ of $\mathcal{H}_n$, labelled by finitely (countably) many $\lambda \in \mathbb{R}$ satisfying $V_{-\lambda} = \alpha(V_\lambda)$.*
5. *The space $\mathcal{SGO}_n^{fin}$ (resp. $\mathcal{SGO}_n$) of super semigroups of self-adjoint C_n-linear finite rank (compact) operators on $\mathcal{H}_n$.*
6. *The space $\mathcal{EFT}_n$ of super symmetric, positive Euclidean Field Theories of dimension $(1|1)$ and degree $(-n)$.*

(7) *The classifying space Q_n of (internal space) categories that arises from a certain C_n-module category by Quillen's $S^{-1}S$-construction.*
(8) *The Atiyah-Singer space $\mathcal{F}_n$ of certain skew-adjoint Fredholm operators on H_n, anti-commuting with the C_{n-1}-action.*

The above theorem only gives very rough descriptions of the spaces involved. Detailed definitions for each item (k), $k = 1, \dots, 8$, can be found in Section k below. Section k also contains the proof that the spaces in (k) are homeomorphic (resp. homotopy equivalent) to spaces appearing previously.

To our best knowledge, the homeomorphisms between the spaces in (2), (4), (5) and (7) are new, even though it was well known that the spaces are homotopy equivalent for abstract reasons (since they represent the same Ω-spectrum and have the homotopy type of a CW complex; in fact, our results imply that this is the case for all the spaces in Theorem 1). Moreover, our maps relating the spaces in (3) and (8) seem to be new and slightly easier than the original ones. One new aspect is the precise treatment of super semigroups and the spaces $\mathcal{SGO}_n$ Finally, the main new result is the equivalence of the older description of K-theory with the one in (6). This is the way that the long term project of the last two authors tries to address a geometric description of "elliptic cohomology": we are trying to make sense out of super symmetric Euclidean field theories of dimension $(2|1)$, so we are trying to raise the bosonic dimension by one.

Remark 2. The spaces in (3) to (7) are defined for all $n \in \mathbb{Z}$ and the theorem holds for all integers n. The Bott and Milnor spaces only make sense for $n \geq 1$ and the same seems to be true for the spaces in (8). This comes from the fact that the Atiyah-Singer spaces $\mathcal{F}_n$ are defined in terms of the ungraded Hilbert space H_n and for $n \leq 0$ our translation to $\mathcal{H}_n$ doesn't work well. However, this can be circumvented by never mentioning the Clifford algebra C_{n-1} in the definitions and working with the ungraded algebra C_n^{ev} instead. Then the spaces $\mathcal{F}_n$ are defined for all $n \in \mathbb{Z}$ and our theorem holds. We chose the formulation above to better connect with the reference [AS].

We now give a rough outline of where the spaces come from and how they are related. The spaces $\mathcal{B}_n$ were defined by Raoul Bott in his classic paper [B] on "the stable homotopy of the classical groups" whose periodicity theorem is the heart and soul of K-theory. This model actually predates the invention of K-theory as a generalized cohomology theory, but was used by Bott to completely calculate the coefficients of this theory.

Atiyah, Bott and Shapiro [ABS] showed the significance of Clifford algebras in K-theory and they suggested to look for a proof of the periodicity theorem using Clifford algebras. A proof along these lines was then found by Wood in [W] and also by Milnor in his beautiful book [Mi] on Morse theory. We will recall in Section 1 how one can easily compute (iterated) spaces of minimal geodesics in the orthogonal group in terms of Clifford module structures on H_n. In Section 2 we'll define the Milnor spaces $\mathcal{M}_n$ to be a certain completion of these spaces of Clifford module structures. These new spaces do not any more depend on a basis of H_n but they have the same homotopy type as $\mathcal{B}_n$ by a theorem of Palais [P].

The configuration spaces $\mathrm{Conf}_n^{\mathrm{fin}}$ are the easiest to work with because one can geometrically picture its elements well. It came as a surprise to us that these simple spaces are actually the geometric realizations $\mathcal{Q}_n$ of certain (internal space) Quillen categories. Given a configuration $\{V_\lambda\}$ in $\mathrm{Conf}_n^{\mathrm{fin}}$, we can interpret it as the eigenspaces and eigenvalues of an odd, self-adjoint, C_n-linear operator $\mathcal{D} \in \mathrm{Inf}_n^{\mathrm{fin}}$ with domain $\bigoplus_\lambda V_\lambda$ that is given by

$$\mathcal{D}(v) := \lambda \cdot v \quad \forall\, v \in V_\lambda.$$

We note that the domain of $\mathcal{D}$ is finite dimensional; in particular, $\mathcal{D}$ is *not* densely defined, a common requirement for self-adjoint operators in text books. We expand the usual definition of a self-adjoint (unbounded) operator by just requiring that the operator is self-adjoint on the closure of its domain. As a consequence, it is very natural to study completions Conf_n of the spaces $\mathrm{Conf}_n^{\mathrm{fin}}$ where there is a discrete set of labels and hence the corresponding operator $\mathcal{D} \in \mathrm{Inf}_n$ may have dense domain (and it has compact resolvent). The resulting spaces Inf_n are equipped with the generalized norm topology and the fact that one can retract the completed spaces back to their finite rank subspaces goes back to (at least) Segal [Se] but we reprove this fact here.

The operator $\mathcal{D}$ can be used as the infinitesimal generator of the super semigroup

$$(t, \theta) \mapsto e^{-t\mathcal{D}^2 + \theta\mathcal{D}}$$

of finite rank (resp. compact) operators on $\mathcal{H}_n$. These are the elements of $\mathcal{SGO}_n^{\mathrm{fin}}$ (resp. $\mathcal{SGO}_n$). Here $(t, \theta) \in \mathbb{R}_{>0}^{1|1}$ parametrize a certain super semigroup whose super Lie algebra is free on one odd generator. Such super semigroups of operators should be considered the 'fermionic' or 'odd' analogue of usual semigroups of operators.

The homeomorphism from $\mathrm{Conf}_n^{\mathrm{fin}}$ to $\mathcal{M}_n$ is basically given by applying the inverse of the Cayley transform to the operator $\mathcal{D}$. If one applies this transformation to elements in Conf_n one obtains interesting completions of the Milnor and Bott spaces.

The spaces $\mathcal{SGO}_n$ were introduced by two of us in [ST] as super semigroups of self-adjoint operators and at the time we thought of them as Euclidean field theories, without having a precise definition for these. In the meantime, we have developed a very general notion of super symmetric Euclidean field theories and we denote the resulting spaces by $\mathcal{EFT}_n$. As a consequence of part (8) of the above theorem, the homotopy type of the space stays unchanged by these modifications.

If one starts with a closed n-dimensional spin manifold M then the C_n-linear Dirac operator D_M (called *Atiyah-Singer operator* in [LM, p.140]) is an example of a (non-finite) element in Inf_n, where $\mathcal{H}_n$ are the L^2-sections of the C_n-linear spinor bundle on M. One can think of this operator as the infinitesimal generator of a super symmetric $(1|1)$-dimensional quantum field theory, with Euclidean (rather than Minkowski) signature. This is the reason why we use the terminology *Euclidean* field theory. Actually, physicists would call it super symmetric quantum mechanics on M, not a field theory, since space is 0-dimensional.

The spaces $\mathcal{F}_n$ first appeared in the article [AS] by Atiyah-Singer and are probably the most common model for K-theory. They make all the wonderful applications to analysis possible. From our point of view, the connection is easiest to make with the space Inf_n: Starting with a skew-adjoint Fredholm operator T_0 on H_n that anti-commutes with $e_1, \dots, e_{n-1}$, we can turn it into an odd, self-adjoint, C_n-linear Fredholm operator

$$T = T_0 \otimes e_n \quad \text{or equivalently} \quad T = \begin{pmatrix} 0 & T_0^* \\ T_0 & 0 \end{pmatrix}$$

on $\mathcal{H}_n \cong H_n \oplus H_n$. It is easy to see that the map $T_0 \mapsto T$ is a homeomorphism and it is important to note that the skew-symmetry of T_0 is equivalent to the relation $Te_n = e_n T$. This correspondence actually extends to the well known case $n = 0$ where one starts with all Fredholm operators on H_0 and gets all odd, self-adjoint Fredholm operators on $H_0 \oplus H_0$.

The essential spectrum of a Fredholm operator has a gap around zero and hence one can push the essential spectrum outside zero all the way into $\pm\infty$ by a homotopy. This turns a bounded operator into an unbounded one and is the basic step in the homotopy equivalence that takes a Fredholm operator T to an infinitesimal generator $\mathcal{D}$. In the analytic literature, one can sometimes find concrete formulas in terms of functional calculus (which just describes the movement of the spectrum of T) like

$$T = \frac{\mathcal{D}}{1 + \mathcal{D}^2}.$$

Such precise formulas are not important from our point of view but the following subtlety arises in the operator $\mathcal{D}$: its eigenspace at ∞, by definition the orthogonal complement of the domain of $\mathcal{D}$, is decomposed into the parts at $+\infty$ respectively $-\infty$. Such a datum is not present in general elements of Inf_n and it reflects the fact that we started with a *bounded* operator. Roughly speaking, this represents no problem up to homotopy if both these parts at $\pm\infty$ are infinite dimensional. This uses Kuiper's theorem and is the only non-elementary aspect of this paper.

Taking into account the C_n-action, this is related to the following well-known subtlety in the Atiyah-Singer spaces of Fredholm operators. If $n \not\equiv 3 \mod 4$, the spaces $\mathcal{F}_n$ are given by operators T_0 (or equivalently T) as above. However, Atiyah-Singer showed that for $n \equiv 3 \mod 4$ the space of C_{n-1}-antilinear skew-adjoint Fredholm operators on H_n has two boring, contractible components $\hat{\mathcal{F}}_n^{\pm}$ consisting of operators T_0 such that

$$e_1 \cdot e_2 \cdots e_{n-1} \cdot T_0$$

is essentially positive (respectively negative). Recall that an operator is *essentially positive* if it is positive on a closed invariant subspace of finite codimension. So in the precise version for $\mathcal{F}_n$ in [AS], the two components $\hat{\mathcal{F}}_n^{\pm}$ are disregarded. It turns out that the above functional calculus leads to a map of all C_{n-1}-antilinear skew-adjoint Fredholm operators to our spaces Inf_n but this map is a quasifibration (with contractible fibres) only on the component $\mathcal{F}_n$. Hence our spaces automatically remove the need for thinking about the above subtleties that arise from bounded operators and are interestingly only visible in the presence of special Clifford actions.

We end this introduction by explaining the easiest description (that we know) of a symmetric ring spectrum that represents K-theory. Let $\mathcal{H}_{-1}$ be a C_{-1}-module as in the paragraph preceding Theorem 1, in particular it contains a submodule $V \cong C_{-1}$ as C_{-1}-modules. Then $\mathcal{H}_{-n} := \mathcal{H}_{-1}^{\otimes n}$ has the desired properties and there are corresponding spaces $E_n := \mathrm{Inf}_{-n}^{\mathrm{fin}}$ of operators as in (3) of Theorem 1. One can also use the completed version Inf_{-n} instead. E_n contains a canonical base point, namely the operator whose domain is zero (and thus all eigenvalues are at ∞).

thm:sym **Theorem 3.** *For $n \geq 0$, the spaces E_n form a symmetric ring spectrum representing real K-theory. The relevant structures are given as follows.*

- *The symmetric group Σ_n acts by permuting the n factors of $\mathcal{H}_{-n}$.*
- *The multiplication maps $E_n \wedge E_m \longrightarrow E_{n+m}$ are given by the formula*

$$(D_n, D_m) \mapsto D_n \otimes \mathbb{1} + \mathbb{1} \otimes D_m$$

- *The Σ_n-equivariant structure maps $\mathbb{R}^n \longrightarrow E_n$ are given by sending $v \in \mathbb{R}^n$ to its Clifford action on the C_{-n}-module $V^{\otimes n}$ (and ∞ on the orthogonal complement in $\mathcal{H}_{-n}$). As $v \to \infty$, Clifford multiplication also goes to ∞ and hence the structure maps can be extended to S^n, sending the point at ∞ to the base point in E_n. These operators are odd and* self-adjoint *which explains the negative sign of $-n$.*

This result is a reformulation of a theorem of Michael Joachim [Jo], so we shall not give a proof. By using complex Hilbert spaces and Clifford algebras, all our results translate to complex K-theory. In fact, keeping track of the involution of complex conjugation, one also gets Atiyah's Real K-theory which contains both, real K-theory (via taking fixed points) and complex K-theory (by forgetting the conjugation map).

sec:Bott

1. Bott spaces of minimal geodesics

The origin of topological K-theory is Raoul Bott's classical paper [B] on "The stable homotopy of the classical groups". For a Riemannian manifold M, let $v = (P, Q, h)$ denote a 'base point in M' which is actually a pair of points $P, Q \in M$, together with a fixed homotopy class h of paths connecting P and Q. If $P = Q$ then h is just an element in $\pi_1(M, P)$. Bott considers the space M^v of minimal geodesics from P to Q in the homotopy class h. Let $|v|$ be the first positive integer which occurs as the index of some geodesic with base point v. Then Bott proved the following theorem in [B]:

thm:Bott

Theorem 4 (Bott)**.** *If M is a symmetric space, so is M^v. Moreover, the based loop space Ω_v can be built, up to homotopy, by starting with M^v and attaching cells of dimension $\geq |v|$:*

$$\Omega_v \simeq M^v \cup e^{|v|} \cup \textit{ (higher dimensional cells)} \qquad \textit{written as} \qquad M^v \xrightarrow{|v|} M$$

For example, if $M = S^n$, $n > 1$, with the round metric and P, Q are not antipodal, then there is a unique minimal geodesic from P to Q. The second shortest geodesic from P to Q reaches the point $-Q$ that has an $(n-1)$-dimensional variation of geodesics to Q and hence $|v| = n-1$. In the notation above, one gets

$$pt = M^v \xrightarrow{(n-1)} M = S^n$$

which implies that ΩS^n is $(n-2)$-connected, or equivalently, that S^n is $(n-1)$-connected, not such a great result. However, if one considers *all* geodesics, one can say much more. In fact, the indices of geodesics from P to Q are $k(n-1)$ for $k = 0, 1, 2, \ldots$. This is a case of Morse's original application of his theory to infinite dimensional manifolds: the energy functional

$$E : \Omega_v M \longrightarrow \mathbb{R}, \quad E(\gamma) := \int_0^1 |\gamma'(t)|^2 dt$$

is a Morse function with critical points the geodesics and indices given by the number of conjugate points (counted with multiplicity) along the given geodesic. Morse shows that infinite dimensionality is not an issue, because the space of paths with bounded energy has the homotopy type of a finite dimensional space, namely the piecewise geodesics (where the number of corners is related to the energy bound and the injectivity radius of M). As a consequence of our example above,

$$\Omega_v S^n \simeq S^{n-1} \cup e^{2(n-1)} \cup e^{3(n-1)} \cup \dots$$

If P and Q are antipodal points on S^n, then the energy is not a Morse function. For example, the minimal geodesics form an $(n-1)$-sphere, parametrized by the equator in S^n. However, Bott developed a theory for such cases, now known as Morse-Bott theory, where the critical points form a submanifold whose tangent space equals the null space of the Hessian of the given function, the *Morse-Bott condition*. Applied to the case at hand, we can derive the same cell decomposition of $\Omega_v S^n$ as above but this time the bottom cell consists of the minimal geodesics:

$$S^{n-1} = M^v \xrightarrow{2(n-1)} M = S^n$$

So Freudenthal's suspension theorem is a direct consequence of this result, using only the index of the *first* non-minimal geodesics (not the first two, as in the generic case studied by Morse). More generally, for any symmetric space, this is Bott's proof of Theorem thm:Bott 4 above.

His approach to study the homotopy types of the classical groups was to apply this method to compact Lie groups which are symmetric spaces in their bi-invariant metric. For example, consider $M = O(2m)$ and $P = \mathbb{1}, Q = -\mathbb{1}$. Then every geodesic γ from P to Q is of the form

$$\gamma(t) = \exp(\pi t \cdot A), \quad t \in [0,1]$$

where A is skew-adjoint. Thus we can 'diagonalize' A by an orthogonal matrix T, i.e. TAT^{-1} is a block sum B of matrices

$$\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$$

with $a_1, \dots, a_m \geq 0$ after normalization. Since $\gamma(1) = -\mathbb{1}$, we see that the a_i are odd integers. It is not hard to see that the energy of γ is given by the formula

$$E(\gamma) = 2(a_1^2 + \cdots + a_m^2)$$

so that minimal energy (or equivalently, minimal length) means that all $a_i = 1$. We note that for $m > 1$, the energy determines the homotopy class h of such a path so that we don't need to mention it for minimal energy (or length) paths (this stays true in all considerations below as well). We conclude that

$$A^2 = T^{-1} B^2 T = T^{-1}(-\mathbb{1})T = -\mathbb{1} \quad \text{and} \quad A^* = -A = A^{-1}$$

so that A is a complex structure on $\mathbb{R}^{2m}$. Just like for the standard complex structure we have

$$A = \exp((\pi/2) \cdot A) = \gamma(1/2)$$

and we obtain the following result.

prop:Bott **Proposition 5** (Bott). *The space $\mathcal{B}_2(2m)$ of minimal geodesics in $O(2m)$ with basepoint v as above is isometric to the space $\mathcal{M}_2(2m)$ of unitary structures on $\mathbb{R}^{2m}$ (consisting of $J \in O(2m)$ with $J^2 = -\mathbb{1}$). Moreover, this is a totally geodesic submanifold of $O(2m)$ and just like for antipodal points on S^n, the homeomorphism is given by sending a geodesic γ to its midpoint $\gamma(1/2)$.*

We are introducing a notation that is consistent with

$$\mathcal{B}_1(m) = O(m) = \mathcal{M}_1(m)$$

and will lead to the Bott and Milnor spaces in the limit when $m \mapsto \infty$. Now recall that $\mathcal{B}_2(2m)$ is again a symmetric space by Bott's theorem so that we can iterate the construction: Pick a complex structure J_1 and study the space $\mathcal{B}_3(4m)$ of minimal geodesics in $\mathcal{B}_2(4m)$ from J_1 to $-J_1$ (they automatically lie in a fixed homotopy class).

By a very similar discussion as above, it turns out that the midpoint map gives an isometry

$$\mathcal{B}_3(4m) \cong \mathcal{M}_3(4m) := \{J \in O(4m) \mid J^2 = -\mathbb{1},\ JJ_1 = -J_1J\}.$$

Note that the right hand side is the space of (orthogonal) quaternion structures on $\mathbb{R}^{4m}$ that are compatible with the given unitary (or orthogonal complex) structure J_1. The set of such structures form a totally geodesic submanifold of $O(4m)$. More generally, we make the following

def:Milnor **Definition 6.** Assume that $\mathbb{R}^m$ is a C_n-module for some $n > 1$. More precisely, there is a $*$-homomorphism $C_n \to \mathrm{End}(\mathbb{R}^m)$ sending e_i to J_i for $i = 1, \dots, n$. Note that the compatibility with $*$ implies that $J_i \in O(m)$. Then we define *Milnor spaces*

$$\mathcal{M}_n(m) := \{J \in O(m) \mid J^2 = -\mathbb{1}, JJ_i = -J_iJ \quad \forall i = 1, \dots, n-2\}$$

to be "the space of all C_{n-1}-structures on $\mathbb{R}^m$", compatible with the given C_{n-2}-structure. Note that these spaces can be empty.

Proposition 7 (Bott). *All $\mathcal{M}_n(m)$ are totally geodesic submanifolds of $O(m)$. The space of minimal geodesics in $\mathcal{M}_n(m)$ from J_{n-1} to $-J_{n-1}$ is isometric to $\mathcal{M}_{n+1}(m)$ via the midpoint map.*

Proof. Let's assume the first sentence and show the second assertion. Any geodesic γ from J_{n-1} to $-J_{n-1}$ is of the form

$$\gamma(t) = J_{n-1} \cdot \exp(\pi t \cdot A), \quad t \in [0, 1]$$

for some skew-adjoint matrix A. One checks that $\gamma(1/2) = J_{n-1} \cdot A$ has square $-\mathbb{1}$ and anticommutes with $J_1, \dots J_{n-2}$ if and only if γ lies in the submanifold $\mathcal{M}_n(m)$. □

def:Bott **Definition 8.** Inductively, let the *Bott spaces* $\mathcal{B}_{n+1}(m)$ be the space of minimal geodesics in $\mathcal{B}_n(m)$ from J_{n-1} to $-J_{n-1}$ (for those m where such a path exists). Then the previous discussion shows that the midpoint map gives an isometry

$$\mathcal{B}_{n+1}(m) \cong \mathcal{M}_{n+1}(m)$$

Theorem 9 (Bott)**.** *Let d_n be the minimal dimension of a C_n-module. Recall Bott's notation for the cell decomposition of the space of loops. Then*

$$\mathcal{B}_{n+1}(m) \xrightarrow{\frac{m}{d_n}-1} \mathcal{B}_n(m)$$

This implies in particular that for $m \to \infty$, the smallest dimension of a cell needed to get the loop space from the space of minimal geodesics also goes to infinity. Thus in the limit, one gets a homotopy equivalence

$$\Omega_v \mathcal{B}_n \simeq \mathcal{B}_{n+1}.$$

To make this precise, we define the spaces $\mathcal{M}_n(\infty)$, which were studied by Milnor in [Mi], as the union of all $\mathcal{M}_n(m)$ inside $O(\infty)$ (which is the union of all $O(m)$). In fact, we only take the union over those m that are divisible by d_n. With a similar definition of $\mathcal{B}_n = \mathcal{B}_n(\infty)$, the midpoint maps give homeomorphisms

$$\mathcal{B}_n(m) \approx \mathcal{M}_n(m) \quad \forall\, m = 1, 2, \ldots \qquad \text{(including } m = \infty\text{)}.$$

between these Bott and Milnor spaces. Now by Morita equivalence $\mathcal{M}_n(\infty) \approx \mathcal{M}_{n+8}(\infty)$ because C_{n+8} is a real matrix ring over C_n. As a consequence,

Corollary 10 (Bott)**.** *There are homeomorphisms and homotopy equivalences*

$$\mathcal{B}_n \approx \mathcal{B}_{n+8} \simeq \Omega^8 \mathcal{B}_n$$

and the homotopy groups of $O(\infty)$ are 8-periodic.

These groups are known as the 'stable' homotopy groups of the orthogonal group because

$$\pi_i O(m) \cong \pi_i O(\infty) \quad \forall i < m - 1.$$

sec:Milnor

2. Milnor spaces of Clifford module structures

For each $n \geq 1$, let H_n be a separable Hilbert space that is a C_{n-1}-module such that each irreducible representation of C_{n-1} appears with infinite multiplicity.

Definition 11. For $n = 1$ we define

$$\mathcal{M}_1^{\text{fin}} = \{\, A \in O(H_1) \mid A \equiv \mathbf{1} \text{ modulo finite rank operators } \}.$$

For $n \geq 2$ the (finite rank) *Milnor space* $\mathcal{M}_n^{\text{fin}}$ is the space of orthogonal operators J on H_n satisfying

- $J^2 = -\mathbb{1}$, or, equivalently, J is skew-adjoint,
- J anti-commutes with $e_1, \ldots, e_{n-2}$,
- $J - e_{n-1}$ has finite rank.

If we replace finite rank operators by compact operators in the above definition, we get the Milnor spaces $\mathcal{M}_n$ for $n \geq 1$.

The main result from the previous section is that the Bott spaces $\mathcal{B}_n$ are homeomorphic to the filtered union (with the direct limit topology)

$$\mathcal{M}_n(\infty) = \bigcup_{k=1}^{\infty} \mathcal{M}_n(k \cdot d_n)$$

i.e. they can be calculated in terms of spaces of Clifford algebra structures on $\mathbb{R}^\infty$, see Definition def:Milnor 6. Let M be the unique irreducible C_{n-1}-module if n is not divisible by 4, respectively the sum of the two irreducible C_{n-1}-modules if n is divisible by 4. We may choose embeddings

$$H_n(m) := \bigoplus_{k=1}^{m} M \subset H_n \quad \forall\, m = 1, 2, \ldots \quad \text{(including } m = \infty\text{)}.$$

This is just the choice of an orthonormal basis in the case $n = 1$. We get an embedding of $H_n(\infty)$ ($= \mathbb{R}^\infty$ for $n = 1$) into H_n and an induced inclusion of $\mathcal{M}_n(\infty)$ into $\mathcal{M}_n$.

thm:Palais **Theorem 12.** *For all $n \geq 1$, this inclusion is a homotopy equivalence*

$$\mathcal{M}_n(\infty) \xrightarrow{\simeq} \mathcal{M}_n.$$

It will follow from Proposition finite vs non-finite 29 that the inclusions $\mathcal{M}_n^{\text{fin}} \hookrightarrow \mathcal{M}_n$ are homotopy equivalences (and hence so are the inclusions $\mathcal{M}_n(\infty) \hookrightarrow \mathcal{M}_n^{\text{fin}}$). In fact, in the definition of the Milnor spaces, one can use any space of operators in between finite rank and compact operators (with the norm topology) to make this result true.

Proof. We shall use Palais' Theorem (A) from P [P] which states the following. Let E be a Banach space and $\pi(m)$ continuous projection operators onto finite dimensional subspaces $E(m) \subset E(m+1)$ which tend strongly to the identity as $m \to \infty$. Then for any open subspace $\mathcal{O}$ of E, the inclusion map

$$\mathcal{O}(\infty) \hookrightarrow \mathcal{O}$$

is a homotopy equivalence. Here $\mathcal{O}(\infty)$ is the direct limit of all $\mathcal{O}(m) := \mathcal{O} \cap E(m)$. In our setting, we have an extra parameter $n \geq 2$, where we leave the easiest case $n = 1$ to the reader. We define

$$E_n := \{A \in \mathcal{K}(H_n) \mid A^* = -A, Ae_i = -e_iA \text{ for } i = 1, \ldots, n-2\}$$

where $\mathcal{K}(H_n)$ is the Banach space of compact operators on H_n with the norm topology. Consider the C_{n-1}-linear orthogonal projections $H_n \twoheadrightarrow H_n(m)$ which by pre- and post composition induce projections

$$\pi(m) : E_n \twoheadrightarrow E_n(m) := E_n \cap \mathcal{K}(H_n(m)).$$

We may assume that H_n is the closure of $H_n(\infty)$ and hence that the $\pi(m)$ tend strongly to the identity. For the open subset in Palais' theorem we choose

$$\mathcal{O}_n := \{ A \in E_n \mid A + e_{n-1} \text{ is invertible } \}$$

and $\mathcal{O}_n(m) := \mathcal{O}_n \cap E_n(m)$. Then the map $A \mapsto A + e_{n-1}$ gives homeomorphisms of $\mathcal{O}_n$, respectively $\mathcal{O}_n(m)$, with

$$\mathcal{G}_n := \{ B \in GL(H_n) \mid B - e_{n-1} \in E_n \}$$

respectively $\mathcal{G}_n(m) := \mathcal{G}_n \cap GL(H_n(m))$. Palais' theorem says that the inclusion induces a homotopy equivalence

$$\mathcal{G}_n(\infty) \overset{\simeq}{\hookrightarrow} \mathcal{G}_n.$$

To prove our Theorem thm:Palais 12, we need to replace the general linear groups by the orthogonal groups in all of the above. This can be done by the polar decomposition of the invertible skew-adjoint operators B above. We may write

$$B = U \cdot P = P \cdot U$$

where $P := \sqrt{B^*B} = \sqrt{BB^*}$ is positive and U is an orthogonal operator. Note that U is actually also skew-adjoint:

$$U^* = (BP^{-1})^* = P^{-1}B^* = P^{-1}(-B) = -U$$

Moreover, B anticommutes with e_i for $i = 1, \dots, n-2$ and therefore P commutes with these e_i and hence U anticommutes with them again. Finally, it is easy to check that $U - e_{n-1}$ is a compact operator and hence U has the same properties as B and therefore lies in $\mathcal{M}_n \subset \mathcal{G}_n$. This implies that here is a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_n(\infty) & \xrightarrow{\simeq} & \mathcal{G}_n \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{M}_n(\infty) & \rightarrow & \mathcal{M}_n \end{array}$$

where the vertical maps are given by polar decomposition $B \mapsto U$. Since these are well known to be the homotopy inverse to the inclusion maps, the proof of Theorem thm:Palais 12 is completed.

□

sec:Inf

3. Infinitesimal Generators

In this section we will review some basic facts about self-adjoint (unbounded) operators, reminding the reader of a nice topology on this space. Let H be a separable complex Hilbert space. Denote by Inf the set of all self-adjoint operators on H with compact resolvent. Note that we do not require an element $\mathcal{D} \in \mathrm{Inf}$ to be densely defined. By 'self-adjoint' we mean that $\mathcal{D}$ defines a self-adjoint operator on the closure of its domain. The compact resolvent condition means that the spectrum of $\mathcal{D}$ consists of eigenvalues of finite multiplicity that do not have an accumulation point in $\mathbb{R}$. Hence, if the domain of $\mathcal{D}$ is infinite dimensional, the operator $\mathcal{D}$ on $\overline{\mathrm{dom}(\mathcal{D})}$ is necessarily unbounded. Because of this, we will think of $\mathrm{dom}(\mathcal{D})^\perp$ as the eigenspace of $\mathcal{D}$ associated with the 'eigenvalue' ∞.

Functional calculus gives a bijection, see e. g. [HG],

$$\mathrm{Inf} \longleftrightarrow \mathrm{Hom}(C_0(\mathbb{R}), \mathcal{K})$$

where the right hand side is the space of all C^*-homomorphisms from (complex valued) continuous functions on $\mathbb{R}$ that vanish at ∞ to the compact operators $\mathcal{K}$ on H. Note that both of these C^*-algebras do not have a unit. Below, we will also deal with C^*-algebras that do have a unit and in this case Hom will denote those C^*-homomorphisms that preserve the unit.

We define the space $\widetilde{\mathrm{Inf}}$ to be just as above, except that we do *not* require the spectrum to be discrete (and the eigenspaces can be infinite dimensional). The "Cayley transform" is defined for such operators by functional calculus using the Möbius transformation

$$c(x) := \frac{x+i}{x-i}$$

which takes $\mathbb{R} \cup \{\infty\}$ to the unit circle S^1. It defines the mapping from the very left to the very right in the following theorem.

thm:Infb

Theorem 13. *There are bijections*

$$\widetilde{\mathrm{Inf}} \overset{a}{\longleftrightarrow} \mathrm{Hom}(C_0(\mathbb{R}), \mathcal{B}) \overset{c}{\cong} \mathrm{Hom}(C(S^1), \mathcal{B}) \overset{b}{\longleftrightarrow} \mathcal{U}$$

where $\mathcal{B}$ and $\mathcal{U}$ are the bounded respectively unitary operators on H. Moreover, the bijection b on the right, given by functional calculus, is a homeomorphism from the pointwise norm topology on $\mathrm{Hom}(C(S^1), \mathcal{B})$ *to the operator norm topology on $\mathcal{U}$.*

Definition 14. We give $\widetilde{\mathrm{Inf}}$ the topology coming from the above bijections. This is sometimes referred to as *generalized norm topology* because of Lemma 16 below.

Remark 15. Just like Inf has an interpretation in terms of configuration spaces, by using the pattern of eigenvalues and their eigenspaces, the space $\widetilde{\mathrm{Inf}}$ can be interpreted as the space of all projection valued measures on $\mathbb{R}$, see [RS, Thm.VIII.6]. The fact that the operators must not be densely defined is reflected in the fact that the projection corresponding to all of $\mathbb{R}$ is not necessarily the identity but projects onto the domain. Thus the result becomes cleaner than in [RS] where the map b is not onto.

Theorem 13 is well known, we just need to collect various bits and pieces of the argument, for example from Rudin [R] or Reed-Simon [RS]. These authors only define the adjoint of a *densely defined* operator D because otherwise the adjoint is not determined by the formula

$$\langle Dv, w\rangle = \langle v, D^* w\rangle$$

In particular, self-adjoint operators are assumed to be have dense domain. As a consequence, [R, Thm.13.19] proves that the Cayley transform gives an inclusion of all *densely defined* self-adjoint operators onto the space of unitary operators without eigenvalue 1. If one allows non-dense domains, i.e. eigenvalue ∞ (defining the adjoint also to be ∞ on that subspace), then the Cayley transform takes the eigenspace of ∞ to the eigenspace of 1 and therefore becomes onto all unitary operators, i.e. gives the desired bijection $\widetilde{\mathrm{Inf}} \leftrightarrow \mathcal{U}$.

Proof of Theorem 13. The map a is given by functional calculus which is well defined on self-adjoint operators that are densely defined. Since the functions f vanish at ∞ one can extend this for all $D \in \widetilde{\mathrm{Inf}}$ by defining $f(D)$ to be zero on the orthogonal complement of the domain of D. For the second map, note that $C(S^1)$ is obtained from $C_0(\mathbb{R})$ by adding a unit $\mathbb{1}$ (and using the above Möbius transformation c). We get an isomorphism between the two spaces of C^*-homomorphisms since we require that $\mathbb{1}$ maps to $\mathbb{1}$ (if the algebras have units). Finally, the map b is given by evaluating a homomorphism at the identity map $z : S^1 \to S^1$. It is clear that the composition from left to right is therefore the Cayley transform and hence a bijection. Recall that by Fourier decomposition, there is an isomorphism of complex C^*-algebras

$$C(S^1) \cong C^*(\mathbb{Z})$$

where $\mathbb{Z}$ is the infinite cyclic group, freely generated by an element z (which corresponds to the above identity z on S^1). It follows that $C^*(\mathbb{Z})$ is free as a C^*-algebra on one unitary element z and hence C^*-homomorphisms out of it are just unitary elements in the target. Moreover, the bijection is given by evaluating functions on this unitary z which is our map b above.

To show that b is a homeomorphism, we need to show that a sequence φ_n of C^*-homomorphisms converges if and only if $\varphi_n(z)$ converges (in norm).

By definition, the φ_n converge if $\varphi_n(f)$ converges (in norm) for all f, so one direction is obvious. For the other, assume that $u_n := \varphi_n(z)$ converges to $u \in \mathcal{U}$ and note that $\varphi_n(f) = f(\varphi_n(z)) = f(u_n)$. We want to show that $f(u_n)$ converges to $f(u)$ and we claim that this is easy to check in the case when f is a Laurent polynomial. For the general case, pick $\epsilon > 0$ and choose a Laurent polynomial $p = p(z)$ such that $\|f - p\|_{sup} < \epsilon/3$ and an $N >> 0$ such that $\|p(u_n) - p(u)\| < \epsilon/3$ for $n > N$. Then for $n > N$ we have

$$\|f(u_n) - f(u)\| = \|(f(u_n) - p(u_n)) + (p(u_n) - p(u)) + (p(u) - f(u))\| < \epsilon$$

and hence $f(u_n)$ converges to $f(u)$. This argument is very similar to the one in [RS, Thm.VIII.20(a)]. □

lem:norm **Lemma 16.** *The Cayley transform on bounded operators*

$$\mathcal{B}^{sa} \subset \widetilde{\mathrm{Inf}} \longleftrightarrow \mathcal{U}$$

is an open embedding, i.e. the generalized norm topology on $\widetilde{\mathrm{Inf}}$ *extends the operator norm topology on* $\mathcal{B}^{sa}$*, the bounded self-adjoint operators on* H*.*

Again this result is well known, see for example [RS, Thm.VIII.18]. Reed and Simon use the resolvent instead of the Cayley transform but this is just a different choice of Möbius transformation, using $x \mapsto (x+i)^{-1}$ instead of c. This has the effect that the image of $\mathbb{R}$ is not the unit circle but a circle of radius $1/2$ inside the unit circle. Therefore, one does not get unitary operators but there is certainly no difference for the induced topology. Unfortunately, in the above Theorem VIII.18, Reed and Simon assume an additional property on the sequence considered, namely that it is uniformly bounded. It turns out, however, that this assumption is unnecessary which is an easy consequence of Theorem VIII.23(b) in [RS].

Remark 17. It is interesting to recall from [R, Thm.13.19] that the Cayley transform can also be applied to *symmetric* operators, i.e. those that are formally self-adjoint and with $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D^*)$. The result is an isometry U with

$$\mathrm{Dom}(U) = \mathrm{Range}(D + i \cdot \mathbb{1}) \quad \text{and} \quad \mathrm{Range}(U) = \mathrm{Range}(D - i \cdot \mathbb{1})$$

D is closed if and only if U is closed and D is self-adjoint if and only if U is unitary. Using the Cayley transform and its inverse one sees that the self-adjoint extensions of D are in 1-1 correspondence with unitary isomorphisms between the orthogonal complements of $\mathrm{Dom}(U)$ and $\mathrm{Range}(U)$. In particular, self-adjoint extensions exist if and only if these complements have the same dimensions, usually referred to as the *deficiency indices*.

An example to keep in mind is the infamous right shift which is an isometry with deficiency indices 0 and 1. Thus its inverse Cayley transform has no self-adjoint extension.

Let $\mathrm{Inf}^{\mathrm{fin}}$ be the space of all self-adjoint operators on H, with finite spectrum and multiplicity (not necessarily densely defined).

prop:infk **Proposition 18.** *The Cayley transform induces the following bijections*

$$\mathrm{Inf} \overset{a}{\longleftrightarrow} \mathrm{Hom}(C_0(\mathbb{R}), \mathcal{K}) \overset{c}{\cong} \mathrm{Hom}(C(S^1), \mathcal{K} + \mathbb{C}\cdot \mathbb{1}) \overset{b}{\longleftrightarrow} \mathcal{U} \cap (\mathcal{K} + \mathbb{C}\cdot \mathbb{1})$$

$$\mathrm{Inf}^{fin} \overset{a}{\longleftrightarrow} \mathrm{Hom}(C_0(\mathbb{R}), \mathcal{FR}) \overset{c}{\cong} \mathrm{Hom}(C(S^1), \mathcal{FR}+\mathbb{C}\cdot \mathbb{1}) \overset{b}{\longleftrightarrow} \mathcal{U}\cap(\mathcal{FR}+\mathbb{C}\cdot \mathbb{1})$$

where $\mathcal{K}$ and $\mathcal{FR}$ are the compact respectively finite rank operators on H. Moreover, the bijections b on the right, given by functional calculus, are homeomorphisms from the pointwise norm topology on the spaces of C^*-homomorphisms to the operator norm topology on $\mathcal{U}$.

Proof. The Cayley transforms give the bijections from the very left to the very right because one can read off the conditions of being compact resp. finite rank from the spectrum and multiplicities of the operators. These conditions are mapped into each other by definition of the spaces. The fact that the maps b are homeomorphisms is proved exactly as in Theorem thm:infb 13 □

We now have complete control over the topology on our various spaces. The largest space $\widetilde{\mathrm{Inf}}$ is homeomorphic to $\mathcal{U}$ and hence contractible by Kuiper's theorem, whereas the subspaces $\mathrm{Inf}^{\mathrm{fin}}$ and Inf are homotopy equivalent (see Proposition finite vs non-finite 29) and have a very interesting topology.

We shall now add some bells and whistles, like grading, real structure and Clifford action to make these spaces even more interesting. In a first step, assume that our complex Hilbert space H has a real structure, i.e. that

$$H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

for some real Hilbert space $H_{\mathbb{R}}$. If we think of the real structure (a.k.a. complex conjugation) on H as a grading involution α (which has the property that the even and odd parts are isomorphic) then the above Proposition prop:infk 18 leads to the following result.

prop:infodd **Proposition 19.** *The Cayley transform induces homeomorphisms*

$$\mathrm{Inf}_{odd}(H) \overset{c}{\approx} O(H_{\mathbb{R}})\cap(\mathcal{K}+\mathbb{C}\cdot \mathbb{1}) \quad \text{and} \quad \mathrm{Inf}^{fin}_{odd}(H) \overset{c}{\approx} O(H_{\mathbb{R}})\cap(\mathcal{FR}+\mathbb{C}\cdot \mathbb{1})$$

Here $\mathrm{Inf}_{odd}(H)$ *denotes the subspace of* odd *operators in* $\mathrm{Inf}(H)$ *(which are still $\mathbb{C}$-linear) and $O(H_{\mathbb{R}})$ is the usual orthogonal group, thought of as the subgroup of real operators in the unitary group $\mathcal{U}(H)$.*

Proof. Since both sides have the subspace topology, it suffices to show that the Cayley transform is a bijection between the spaces given in the Proposition. An operator D in $\mathrm{Inf}(H)$ is odd if and only if $D^{\alpha} := \alpha D \alpha = -D$. Since our grading involution α is $\mathbb{C}$-antilinear, we also have $i^\alpha = -i$ and therefore[1]

$$\overline{c(D)} = c(D)^\alpha = \left(\frac{D+i}{D-i}\right)^\alpha = \frac{-D-i}{-D+i} = \frac{D+i}{D-i} = c(D)$$

Since the operators in $\mathcal{U}(H)$ that commute with complex conjugation are clearly those in the real orthogonal group $O(H_\mathbb{R})$, we get $c(D) \in O(H_\mathbb{R})$ and as before, $c(D) - \mathbb{1}$ is compact (resp. finite rank). Conversely, a similar calculation shows that if $c(D)$ is real then $D^\alpha = -D$. □

Let $\mathcal{H}_n$ be as in the introduction, a graded real Hilbert space with a $*$-action of the real Clifford algebra C_n. For example, the above discussion is the case $n = 1$ if we define $\mathcal{H}_1 = H$ with the grading and $C_1 = \mathbb{C}$-action as above. Note that in this case one can think of $\mathbb{C}$-linear operators, say in $\mathcal{K}(H)$, as $\mathbb{R}$-linear operators that commute with the C_1-action. This motivates the following definition.

def:Kn and FRn

Definition 20. We denote by $\mathcal{K}_n$ (resp. $\mathcal{FR}_n$) the space of all C_n-linear self-adjoint compact (resp. finite rank) operators on $\mathcal{H}_n$, and by Inf_n (resp. $\mathrm{Inf}_n^{\mathrm{fin}}$) the subspace of Inf (resp. $\mathrm{Inf}^{\mathrm{fin}}$) that consists of all C_n-linear and odd operators.

prop:infodd

In order to extend Proposition 19 to Inf_n we have to identify the image of these spaces under the Cayley transform in $O(H_n)$. We now assume the following model for our graded Hilbert space $\mathcal{H}_n$. For $n \geq 1$, let H_n be a real Hilbert space that is a C_n^{ev}-module and consider the graded C_n-module

$$\mathcal{H}_n := H_n \otimes_{C_n^{ev}} C_n \cong H_n \otimes_\mathbb{R} \mathbb{C}.$$

The last isomorphism should be interpreted as saying that the complex structure on $\mathcal{H}_n$ is given by the last basis element $e_n \in C_n$ and that the grading can be thought of as corresponding to complex conjugation, just like in our previous discussion.

Conf-Milnor

Proposition 21. *For all $n \geq 1$ there are homeomorphisms*

$$\mathrm{Inf}_n^{fin} \approx \mathcal{M}_n^{fin} \quad \textit{and} \quad \mathrm{Inf}_n \approx \mathcal{M}_n.$$

For $n = 1$ they are given by the Cayley transform $D \mapsto c(D)$ and for $n > 1$ by $D \mapsto e_{n-1}c(D)$.

Proof. We will show that there is a homeomorphism $\mathrm{Inf}_n \approx \mathcal{M}_n$ which restricts to the desired homeomorphism on $\mathrm{Inf}_n^{\mathrm{fin}}$. The case $n = 1$ was discussed above because in this case we have by definition

$$O(H_1) \cap (\mathbb{1} + \mathcal{K}) = \mathcal{M}_1 \quad \text{and} \quad O(H_1) \cap (\mathbb{1} + \mathcal{FR}) = \mathcal{M}_1^{\mathrm{fin}}.$$

[1] The notation $(D \pm i)^{-1}$ best interpreted on each pair of eigenspaces $V_\lambda \oplus V_{-\lambda}$ of D separately; there it definitely makes sense and that is all we care about here.

Now, let $n \geq 2$. Recall that the complex structure on $\mathcal{H}_n$ is given by e_n, hence the relation $De_n = e_n D$ gives the $\mathbb{C}$-linearity of $c(D)$. We claim that the relations $De_i = e_i D$ for the remaining $n-1$ generators e_i of C_n imply that the generators e_i of C_{n-1} satisfy

$$e_i c(D) = c(D)^{-1} e_i.$$

To see this, note that we have the relations

$$e_i(D \pm i) = (D \mp i)e_i \text{ and } e_i(D \pm i)^{-1} = (D \mp i)^{-1} e_i$$

which together yield

$$e_i c(D) = e_i(D-i)(D+i)^{-1} = (D+i)(D-i)^{-1} e_i = c(D)^{-1} e_i.$$

Note that all these are operators on $\mathcal{H}_n$ but that our *odd* operator D gives an action of $c(D)$ on H_n. We assert that the same relation holds for this operator $c(D)$ on H_n. First, note that since $c(D)$ is $\mathbb{C}$-linear, i.e. it commutes with e_n, we have $e_n e_i c(D) = c(D)^{-1} e_n e_i$. Next, one checks that under the isomorphism $C_n^{ev} \cong C_{n-1}$ the action of $e_n e_i \in C_n$, $i = 1, ..., n-1$, corresponds to the automorphism $e_i \otimes \mathrm{id}$ of $H_n \otimes \mathbb{C}$. This together with the relation we computed for $c(D) \otimes \mathrm{id}$ implies $e_i c(D) = c(D)^{-1} e_i$ for $i = 1, \dots, n-1$.

Hence we see that the Cayley transform c gives a homeomorphism

$$\mathrm{Inf}_n \approx \{A \in O(H_n) \mid A \equiv \mathbf{1} \bmod \mathcal{K}(H_n) \text{ and } e_i A = A^{-1} e_i \text{ for } i = 1, ..., n-1\}$$

The space on the right-hand side is not quite $\mathcal{M}_n$ yet. However, we claim that it can be identified with $\mathcal{M}_n$ by associating to an operator A the complex structure

$$J := e_{n-1} A \in \mathcal{M}_n.$$

It is clear that $J \equiv e_{n-1} \bmod \mathcal{K}(H_n)$. Furthermore, J is indeed a complex structure:

$$J^2 = e_{n-1} A e_{n-1} A = e_{n-1} A A^{-1} e_{n-1} = -\mathbb{1}.$$

It remains to check that J anti-commutes with the generators of C_{n-2}. The following computation shows this claim using $\tilde{e}_i = (e_{n-1} e_i)$:

$$(e_{n-1} e_i)(e_{n-1} A) = (e_{n-1} e_i)(A^{-1} e_{n-1}) = e_{n-1} A e_i e_{n-1} = -(e_{n-1} A)(e_{n-1} e_i)$$

where we have interpreted H_n as a module over C_{n-2} via

$$C_{n-2} \xrightarrow{\cong} C_{n-1}^{ev}, \ \tilde{e}_i \mapsto e_{n-1} e_i,$$

with $\tilde{e}_1, ..., \tilde{e}_{n-2}$ denoting the standard generators of C_{n-2}. □

For later use, we define the real graded C^*-algebra $\mathcal{S}$ to be given by *real valued* functions in $C_0(\mathbb{R})$ with trival $*$ and with grading involution induced by $x \mapsto -x$ (leading to the usual decomposition into even and odd functions).

To motivate the use of self-adjoint operators, we make the following easy observation that comes from the above case $n = 1$.

Lemma 22. *Restriction to self-adjoint elements defines homeomorphisms*

$$\mathrm{Hom}_{gr}(C_0(\mathbb{R}), \mathcal{K}(H)) \longleftrightarrow \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}^{sa}(H))$$

where H is a complex Hilbert space with grading involution as above and Hom_{gr} *denotes grading preserving $*$-homomorphisms. The analogous statement holds for $\mathcal{FR}$ in place of $\mathcal{K}$.*

Proof. Recall that $\mathcal{S}$ are just the real valued functions in $C_0(\mathbb{R})$ (a.k.a. the self-adjoint elements in this complex C^*-algebra) and that the grading involutions agree. Moreover, there is an isomorphism

$$\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C} \cong C_0(\mathbb{R})$$

The same statements apply to $\mathcal{K}$ (resp. $\mathcal{FR}$) and therefore the complexification map gives an inverse to the restriction map in the lemma. □

The same argument as above then leads to the following result.

Inf and Hom **Proposition 23.** *Functional calculus induces the homeomorphisms*

$$\mathrm{Inf}_n \approx \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}_n) \quad \textit{and} \quad \mathrm{Inf}_n^{fin} \approx \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{FR}_n).$$

sec:Conf 4. Configuration Spaces

The unbounded operators of the previous section can be visualized as configurations on the real line: an operator $D \in \mathrm{Inf}$ is completely determined by its eigenvalues and eigenspaces and hence by the map V that associates to $\lambda \in \mathbb{R}$ the subspace $V(\lambda)$ on which $D = \lambda$. We call V a 'configuration on $\mathbb{R}$', since $V(\lambda)$ may be thought of as a label attached at $\lambda \in \mathbb{R}$. sec:AS Since slightly different spaces of configurations will appear in Section 8, we give a general definition that also covers the case considered there.

Let Λ be a topological space equipped with an involution s and $\mathcal{H}$ a separable graded Hilbert space with grading involution α. A *configuration on Λ indexed by orthogonal subspaces of $\mathcal{H}$* is a map V from Λ to the set of closed (ungraded) subspaces of $\mathcal{H}$ such that

- the subspaces $V(\lambda)$ are pairwise orthogonal
- $\mathcal{H}$ is the Hilbert sum of the $V(\lambda)$'s
- V is compatible with s and α, i.e. $V(s(\lambda)) = \alpha(V(\lambda))$ for all $\lambda \in \Lambda$.

Recall that closed subspaces of $\mathcal{H}$ correspond precisely to continuous self-adjoint projection operators on $\mathcal{H}$. Hence we may interpret V as a map

$$V : X \longrightarrow \mathrm{Proj}(\mathcal{H}) \subset B(\mathcal{H}).$$

To save space, we write $V_\lambda := V(\lambda)$. Define $\mathrm{supp}(V) := \overline{\{\, \lambda \in \Lambda \mid V_\lambda \neq 0 \,\}}$.

Definition 24. The *space* $\mathrm{Conf}(\Lambda;\mathcal{H})$ *of configurations on* Λ *indexed by orthogonal subspaces of* $\mathcal{H}$ is the set of all configurations $V : \Lambda \to \mathrm{Proj}(\mathcal{H})$ equipped with the topology generated by the subbasis consisting of the sets

$$\mathcal{B}(U, L) := \{\, V \in \mathrm{Conf}(\Lambda;\mathcal{H}) \mid V_U := \sum_{\lambda\in U} V_\lambda \in L,\ \mathrm{supp}(V) \cap \partial U = \emptyset \,\},$$

where U and L range over all open subsets $U \subset \Lambda$ and $L \subset \mathrm{Proj}(\mathcal{H})$.

We will need the following variations. Let $\Theta \subset \Lambda$ be a subspace that is preserved under the involution s. Define $\mathrm{Conf}(\Lambda,\Theta;\mathcal{H}) \subset \mathrm{Conf}(\Lambda;\mathcal{H})$ to be the subspace of configurations V such that V_λ has *finite rank* for all $\lambda \in \Theta^c := \Lambda \setminus \Theta$ and such that the subset of all $\lambda \in \Theta^c$ with $V(\lambda) \neq 0$ is *discrete* in Θ^c.[2] Replacing the discreteness condition by requiring that there should be only *finitely many* $\lambda \in \Theta^c$ with $V_\lambda \neq 0$ we obtain the space $\mathrm{Conf}^{\mathrm{fin}}(\Lambda,\Theta;\mathcal{H})$ of configurations that are 'finite away from Θ'. Finally, if C is an $\mathbb{R}$-algebra and $\mathcal{H}$ is a C-module, we can replace subspaces of $\mathcal{H}$ by C-submodules in order to obtain spaces $\mathrm{Conf}_C(\Lambda,\Theta;\mathcal{H})$. If C is graded then we assume that $\mathcal{H}$ is a graded C-module, but the subspaces $V(\lambda)$ are still ungraded; only those for which $s(\lambda) = \lambda$ are *graded* modules over C. Our main examples will be the Clifford algebras $C = C_n$.

Examples 25. Consider the one-point compactification $\bar{\mathbb{R}}$ of $\mathbb{R}$ equipped with the involution $s(x) := -x$. Define

$$\mathrm{Conf}_n := \mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n),$$

where $\mathcal{H}_n$ is the graded C_n-module from the previous section. We will see in Proposition 27 that Conf_n gives a different model for the space Inf_n of unbounded operators introduced above. The homeomorphism $\mathrm{Inf}_n \to \mathrm{Conf}_n$ is given by mapping $\mathcal{D} \in \mathrm{Inf}_n$ to the configuration defined by associating to $\lambda \in \mathbb{R}$ the λ-eigenspace V_λ of $\mathcal{D}$. Here we let $V_\infty := \mathrm{dom}(\mathcal{D})^\perp$. Since $\mathcal{D}$ has compact resolvent, the set of $\lambda \in \mathbb{R}$ with $V_\lambda \neq 0$ is indeed discret in $\mathbb{R}$ and each eigenspace V_λ, $\lambda \in \mathbb{R}$, is finite-dimensional. The relation $V(s(\lambda)) = \alpha(V(\lambda))$ corresponds to $\mathcal{D}$ being odd.

In order to get a better feeling for the topology on Conf_n, let us describe a neighborhood basis for each configuration in Conf_n. This will also be useful for the proof that the map $\mathrm{Inf}_n \to \mathrm{Conf}_n$ is a homeomorphism.

We begin by pointing out that the topology on Conf_n is generated by the sets $\mathcal{B}(U, L)$, where $U \subset \mathbb{R}$ is bounded. To see this, note that, by definition of Conf_n, $\infty \in \mathrm{supp}(V)$ for all $V \in \mathrm{Conf}_n$. Hence $\mathcal{B}(U, L) = \emptyset$ whenever $\infty \in \partial U$ so that the case of unbounded $U \subset \mathbb{R}$ is irrelevant. Furthermore, if

[2] This terminology will be convenient for our purposes. However, we should point out that with our notation $\mathrm{Conf}(\Lambda;\mathcal{H}) = \mathrm{Conf}(\Lambda,\Lambda;\mathcal{H})$ and *not* $\mathrm{Conf}(\Lambda;\mathcal{H}) = \mathrm{Conf}(\Lambda,\emptyset;\mathcal{H})$.

$\infty \in U$ we can use $\mathcal{B}(U, L) = \mathcal{B}(U^c, 1 - L)$ to describe $\mathcal{B}(U, L)$ in terms of $U^c := \bar{\mathbb{R}} \setminus U$. Thus it is sufficient to consider $\mathcal{B}(U, L)$ for $U \subset \mathbb{R}$ bounded.

def:V **Definition 26.** Let $V \in \mathrm{Conf}_n$ and let K be a (large) positive real number such that $V_K = 0$. Let $B_K(0)$ be the ball of radius K around 0 and denote by $\lambda_1, ..., \lambda_{l_K}$ the numbers in $B_K(0)$ such that $V_{\lambda_i} \neq 0$. Let $\delta > 0$ and $\varepsilon > 0$ be (small) real numbers; we may choose δ so small that $B_\delta(\lambda_i) \cap B_\delta(\lambda_j) = \emptyset$ for $i \neq j$. Denote by $V_{K,\delta,\varepsilon}$ the set of all configurations W such that $\|V_{B_\delta(\lambda_i)} - W_{B_\delta(\lambda_i)}\| < \varepsilon$ for all i and such that $W_\lambda = 0$ for all $\lambda \in B_K(0)$ that do not lie in one of the balls $B_\delta(\lambda_i)$.

Thus, an element $W \in V_{K,\delta,\varepsilon}$ almost looks like V on $B_K(0)$: the only thing that can happen is that a label V_λ 'splits' into labels W_{λ_j} with $|\lambda - \lambda_j|$ small ($< \delta$) and $\sum_j W_{\lambda_j}$ close to V_λ ($< \varepsilon$). The $V_{K,\delta,\varepsilon}$ form indeed a neighborhood basis of V: assume $V \in \bigcap_{k=1}^n \mathcal{B}(U_k, L_k)$, with $U_k \subset \mathbb{R}$ bounded. Choose K as above with $\bigcup_{k=1}^n U_k \subset B_K(0)$. Picking $\delta > 0$ so small that $B_\delta(\lambda_i) \subset \bigcap_{k=1}^n U_k$ for all i it follows easily using the triangle inequality that for $\varepsilon > 0$ sufficiently small $V_{K,\delta,\varepsilon} \subset \bigcap_{i=1}^n \mathcal{B}(U_i, L_i)$.

In particular, we see that the topology on Conf_n controls configurations well on compact subsets of $\mathbb{R}$ but not near infinity. The discussion also shows that Conf_n is first countable since we may choose K, δ, and ε in $\mathbb{Q}$.

Given any $V \in \mathrm{Conf}_n = \mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n)$ and any function $f \in \mathcal{S}$, we can define a C_n-linear operator $f(V)$ on $\mathcal{H}_n$ by requiring that $f(V)$ has eigenvalue $f(\lambda)$ exactly on V_λ. This operator is always compact and it is of finite rank if and only if $V \in \mathrm{Conf}_n^{\mathrm{fin}} := \mathrm{Conf}_{C_n}^{\mathrm{fin}}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n) \subset \mathrm{Conf}_n$, the subspace of configuration that are finite away from $\{\infty\}$.

triangle lemma **Proposition 27.** *Functional calculus $F(V)(f) := f(V)$ gives homeomorphims*

$$F : \mathrm{Conf}_n \xrightarrow{\approx} \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}_n) \quad \textit{and} \quad \mathrm{Conf}_n^{fin} \xrightarrow{\approx} \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{FR}_n).$$

Combining this result with Proposition Inf and Hom *23 we obtain as a corollary*

$$\mathrm{Inf}_n \approx \mathrm{Conf}_n \quad \textit{and} \quad \mathrm{Inf}_n^{fin} \approx \mathrm{Conf}_n^{fin}.$$

Proof. It is clear that F is a bijection because the map that identifies operators with the eigenspaces and eigenvalues is obviously a bijection and it is the composition of F with a homeomorphism. Since Conf_n is first countable, we can check the continuity of F on sequences. To do so, assume $V_n \to V$ and fix $f \in \mathcal{S}$. We have to prove $f(V_n) \to f(V)$. Given $\varepsilon > 0$, choose $K > 0$ such that $|f(x)| < \varepsilon$ if $|x| > K$. Since the continuous map f is automatically uniformly continuous on compact sets, we can find a $\delta > 0$ such that for all $x \in B_K(0)$ we have $|f(x) - f(y)| < \varepsilon$ provided $|x - y| < \delta$.

The assumption $V_n \to V$ tells us that $V_n \in V_{K,\delta,\varepsilon}$ for large n. The claim now follows from the following estimate that holds for all $W \in V_{K,\delta,\varepsilon}$:

$$\begin{aligned} \|f(V) - f(W)\| &= \| \sum_{\lambda\in\mathbb{R}} f(\lambda)V_\lambda - \sum_{\mu\in\mathbb{R}} f(\mu)W_\mu \| \\ &\le \| \sum_{\lambda\in B_K(0),\, V_\lambda\neq 0} \left(f(\lambda)V_\lambda - \sum_{\mu\in B_\delta(\lambda)} f(\mu)W_\mu \right) \| + 2\varepsilon \\ &\le \#\{\lambda \in B_K(0) \mid V_\lambda \neq 0\} \cdot \left(\max_{\lambda\in B_K(0)} f(\lambda)\cdot\varepsilon + \varepsilon \right) + 2\varepsilon \\ &\le C\cdot\varepsilon, \end{aligned}$$

where the constant C only depends on f and V. The first inequality follows by re-arranging the terms and using the triangle inequality together with $|f(x)| < \varepsilon$ for $|x| > K$. The second inequality follows from

$$\begin{aligned} \|f(\lambda)V_\lambda - \sum_{\mu\in B_\delta(\lambda)} f(\mu)W_\mu\| &\le \|f(\lambda)(V_\lambda - W_{B_\delta(\lambda)})\| + \|\sum_{\mu\in B_\delta(\lambda)} (f(\lambda) - f(\mu))W_\mu\| \\ &\le \max_{\lambda\in B_K(0)} f(\lambda)\cdot\varepsilon + \varepsilon. \end{aligned}$$

The space $\mathrm{Hom}_{gr}(S, \mathcal{K}_n)$ is also first countable. This follows since $f(V_n) \to f(V)$ for all $f \in \mathcal{S}$ if and only if this is the case for $f(x) = e^{-x^2}$ and $f(x) = xe^{-x^2}$ (see the proof of Lemma 42). Thus we can check the continuity of F^{-1} on sequences as well. Assume $f(V_n) \to f(V)$ for all $f \in \mathcal{S}$ and $V \in \mathcal{B}(U, L)$. We have to show $V_n \in \mathcal{B}(U, L)$ for n sufficiently large. More explicitly: $\mathrm{supp}(V_n)\cap\partial U = \emptyset$ for n large and $\lim_{n\to\infty} \|(V_n)_U - V_U\| = 0$.

Note that for an accumulation point $\gamma \in \mathbb{R}$ of the set $\bigcup_n \mathrm{supp}(V_n)$ we must have $V_\gamma \neq 0$, because otherwise we would also have $\|f(V_n) - f(V)\| \ge \frac{1}{2}$ for infinitely many n if we choose f to be a bump function with $f(\gamma) = 1$ that is concentrated near γ. This together with $\mathrm{supp}(V) \cap \partial U = \emptyset$ implies that there is a neighborhood $\nu(\partial U)$ of ∂U such that $(V_n)_\lambda \neq 0$ for $\lambda \in \nu(\partial U)$ occurs only for finitely many n. In particular, $\mathrm{supp}(V_n)\cap\partial U = \emptyset$ for n large. Now, choose $f \in \mathcal{S}$ such that $f|_{\mathbb{R}\setminus U} = 0$ and $f|_{U\setminus\nu(\partial U)} = 1$. By construction, $f(V_n) = \chi_U(V_n)$ for n large, where χ_U denotes the indicator function for U. The same identity holds for V and hence we can conclude

$$\lim_{n\to\infty} \|(V_n)_U - V_U\| = \lim_{n\to\infty} \|\chi_U(V_n) - \chi_U(V)\| = \lim_{n\to\infty} \|f(V_n) - f(V)\| = 0.$$

This completes the proof. □

Remark 28. A continuous map $f : \Lambda \to \Lambda'$ that commutes with the involutions on Λ and Λ' induces

$$f_* : \mathrm{Conf}(\Lambda; \mathcal{H}) \longrightarrow \mathrm{Conf}(\Lambda'; \mathcal{H}),\ (f_*(V))_{\lambda'} := \sum_{\lambda\in f^{-1}(\lambda')} V_\lambda.$$

We show that the map f_* is continuous under the assumption that the space Λ' is normal. Let $V \in f_*^{-1}(\mathcal{B}(U, L))$. From the definition of $\mathcal{B}(U, L)$ we find $\mathrm{supp}(f_* V) \cap \partial U = \emptyset$. Since Λ' is normal, there is an open neighborhood N_V of $\mathrm{supp}(f_* V)$ such that $\overline{N_V} \cap \partial U = \emptyset$. Unravelling the definitions one finds

$$V \in \mathcal{B}(f^{-1}(U), L) \cap \mathcal{B}(f^{-1}(N_V), \{\mathrm{id}_{\mathcal{H}}\}) \subset f_*^{-1}(\mathcal{B}(U, L))$$

so that $f_*^{-1}(\mathcal{B}(U, L))$ is a neighborhood of V. Thus $f_*^{-1}(\mathcal{B}(U, L))$ is open.

The additional properties that we required in the definition of the spaces $\mathrm{Conf}(\Lambda, \Theta; \mathcal{H})$ are not stable under pushforward. In order to get an induced map we have to require f to be 'nice'. For example, if $f : (\Lambda, \Theta) \to (\Lambda', \Theta')$ is a proper map between locally compact Hausdorff spaces, we get an induced map $f_* : \mathrm{Conf}(\Lambda, \Theta; \mathcal{H}) \to \mathrm{Conf}(\Lambda', \Theta'; \mathcal{H})$.

A sightly more complicated argument along the same lines can be used to show that a homotopy $h : \Lambda \times [0, 1] \to \Lambda'$ induces a homotopy $H : \mathrm{Conf}(\Lambda, \mathcal{H}) \times [0, 1] \to \mathrm{Conf}(\Lambda', \mathcal{H})$, at least if Λ is compact.

The following result implies that the 'finite' and 'non-finite' versions of the spaces considered in the previous chapters are homotopy equivalent.

finite vs non-finite **Proposition 29.** *The inclusion* $\mathrm{Conf}_n^{fin} \hookrightarrow \mathrm{Conf}_n$ *is a homotopy equivalence.*

Proof. Consider the family of maps $h_t : \bar{\mathbb{R}} \longrightarrow \bar{\mathbb{R}}$ defined by

$$h_t(x) := \begin{cases} \frac{x}{1 - t|x|} & \text{if } x \in (-\frac{1}{t}, \frac{1}{t}) \\ \infty & \text{else.} \end{cases}$$

These induce a homotopy

$$H_t := (h_t)_* : \mathrm{Conf}_n \longrightarrow \mathrm{Conf}_n$$

from the identity on Conf_n to H_1. Note that the image of H_1 equals $\mathrm{Conf}_n^{\mathrm{fin}}$. Thus, we see that the inclusion $\mathrm{Conf}_n^{\mathrm{fin}} \hookrightarrow \mathrm{Conf}_n$ is a homotopy equivalence with homotopy inverse $H_1 : \mathrm{Conf}_n \to \mathrm{Conf}_n^{\mathrm{fin}}$. □

colim Confk **Remark 30.** We will later consider the space $\mathrm{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)}$, where $\mathrm{Conf}_n^{(k)}$ is the subspace of configurations $V \in \mathrm{Conf}_n$ such that $V_\lambda \neq 0$ for at most k numbers λ with $0 < \lambda < \infty$. As a set, $\mathrm{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)}$ is just $\mathrm{Conf}_n^{\mathrm{fin}}$, but the topology has more open sets than the topology of $\mathrm{Conf}_n^{\mathrm{fin}}$. We claim that

$$\mathrm{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)} \hookrightarrow \mathrm{Conf}_n$$

is also a homotopy equivalence. The same homotopy as in the proof of Proposition 29 (finite vs non-finite) can be used. This works because the map $H_1 : \mathrm{Conf}_n \longrightarrow \mathrm{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)}$ is still continuous. This follows from the observation that

$$C : \mathrm{Conf}_n \longrightarrow \mathbb{N}, \; V \mapsto C(V) := \sum_{-1 \le \lambda \le 1} \dim V_\lambda$$

is locally bounded so that for every $V \in \mathrm{Conf}_n$ we can find an open neighborhood N of V such that $H_1(N) \subset \mathrm{Conf}_n^{(k)}$ for some k (since H_1 moves all labels outside $(-1, 1)$ to ∞). Since continuity can be checked locally and since on $\mathrm{Conf}_n^{(k)}$ the topology induced from Conf_n and the colimit topology coincide, it follows that $H_1 : \mathrm{Conf}_n \to \mathrm{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)}$ is continuous.

In particular, we see that the identity map $\mathrm{id} : \mathrm{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)} \to \mathrm{Conf}_n^{\mathrm{fin}}$ is a homotopy equivalence.

The same argument applies in the case of the filtration X_k of $\mathrm{Conf}_n^{\mathrm{fin}}$ given by the dimension of a configuration: if $X_k \subset \mathrm{Conf}_n^{\mathrm{fin}}$ is the subspace of configurations V with $\dim(V) := \dim_{C_n}(\oplus_{\lambda\in\mathbb{R}} V_\lambda) \leq 2k$, then the identity map $\mathrm{colim}_{k\to\infty} X_k \to \mathrm{Conf}_n^{\mathrm{fin}}$ is a homotopy equivalence.

sec:SGO

5. Super semigroups of operators

In this section we will define super semigroups of operators (SGOs) using as little super mathematics as possible. We will only need basic definitions and results from the theory of supermanifolds, as can be found in Chapter 2 of [DM]. Super manifolds are particular ringed spaces, i.e. topological spaces together with a sheaf of rings, and morphisms are maps of ringed spaces. The local model for a supermanifold of dimension $(p|q)$ is Euclidean space $\mathbb{R}^p$ equipped with the sheaf of commutative super $\mathbb{R}$-algebras

$$U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q).$$

This ringed space is the supermanifold $\mathbb{R}^{p|q}$.

Definition 31. A *supermanifold* M of dimension $(p|q)$ is a pair $(|M|, O_M)$ consisting of a (Hausdorff and second countable) topological space $|M|$ together with a sheaf of commutative super $\mathbb{R}$-algebras O_M that is locally isomorphic to $\mathbb{R}^{p|q}$.

To every supermanifold M there is an associated *reduced manifold*

$$M^{red} := (|M|, O_M/\mathrm{nil})$$

obtained by dividing out nilpotent functions. By construction, this gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of supermanifolds $M^{red} \hookrightarrow M$. For example, $(\mathbb{R}^{p|q})^{red} = \mathbb{R}^p$.

The main invariant of a supermanifold M is its ring of functions $C^\infty(M)$, defined as the global sections of the sheaf O_M. For example, $C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda^*(\mathbb{R}^q)$. It turns out that the maps between supermanifolds M and N are just given by grading preserving algebra homomorphisms between the rings of functions:

$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}_{Alg}(C^\infty(N), C^\infty(M))$$

Example 32. Let $E \to M$ be a real vector bundle of fiber dimension q over the smooth manifold M^p. Then $(M, \Gamma(\Lambda^* E))$ is a supermanifold of dimension $(p|q)$. Bachelor's theorem says that every supermanifold is isomorphic (but not canonically) to one of this type. This result does not hold in analytic categories, and it shows that in the smooth category, supermanifolds are only interesting if one takes their morphisms seriously and doesn't just consider isomorphism classes.

f:super moduli space

Definition 33. Define the 'twisted' super Lie group structure on $\mathbb{R}^{1|1}$ by

$$m : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1},\ (t,\theta),(s,\eta) \mapsto (t+s+\theta\eta, \theta+\eta).$$

This super Lie group plays a special role in super geometry, the reason being the particular structure of its super Lie algebra: $\mathrm{Lie}(\mathbb{R}^{1|1}) \cong \mathbb{R}[D]$ is *the* super Lie algebra generated freely by one *odd* generator D. *Thus, $\mathbb{R}^{1|1}$ may be considered the odd analogue of the Lie group* $\mathbb{R}$. For example, integrating an odd vector field on a supermanifold M leads to a flow $M \times \mathbb{R}^{1|1} \to M$, and formulating the flow property involves the 'twisted' group structure.

From the definition of m it is clear that the open sub supermanifold $\mathbb{R}^{1|1}_{>0}$ defined by the inclusion $\mathbb{R}_{>0} \subset \mathbb{R}$ inherits the structure of a super semigroup.[3] Now we can already guess what a SGO should be: just as an ordinary semigroup of operators is a homomorphism from $\mathbb{R}_{>0}$ to an algebra of operators, a super semigroup of operators will be a homomorphism from the super semigroup $\mathbb{R}^{1|1}_{>0}$ to a ($\mathbb{Z}_2$-graded) operator algebra. In order to make sense of such a homomorphism, we will consider the latter to be a generalized super semigroup using the 'functor of points' formalism (see [DM], §§2.8-2.9). Note that we, implicitly, already used the 'functor of points' language when writing down the group law m. The formula above tells us what the product of two elements in the group $\mathrm{Hom}(S, \mathbb{R}^{1|1})$ is. Since the rule holds functorially for all supermanifolds S, this defines the map m by the Yoneda lemma.

Finally, we would like to remark that the structure of $\mathrm{Lie}(\mathbb{R}^{1|1})$ and the existence of an *odd* infinitesimal generator $\mathcal{D}$ for a SGO Φ that we will prove below are closely related: $\mathcal{D}$ is nothing but the image of D under the derivative of Φ. However, making this precise requires some work (note that Φ maps to an infinite-dimensional space!). We will avoid such problems altogether: the super Lie algebras do not appear in our argument.

Generalized supermanifolds and super Lie groups. We will use the following, somewhat primitive, extension of the notion of supermanifolds:

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[3]A *super (Lie) semigroup* is a supermanifold M together with an associative multiplication $M \times M \to M$. In terms of the functor of points language: the morphism sets $\mathrm{Hom}(S, M)$ carry semigroup structures, functorially in S.

Definition 34. A *generalized supermanifold* M is a contravariant functor from supermanifolds to sets.[4] Similarly, if M takes values in the category of (semi)groups, we call it a *generalized super (semi)group*. Morphisms in all these categories are natural transformations.

Examples 35. (1) The Yoneda lemma implies that supermanifolds are embedded as a full subcategory in generalized supermanifolds by associating to a supermanifold M the functor

$$S \mapsto M(S) := \mathrm{Hom}(S, M).$$

The analogous statement holds for super (semi)groups. For example, we will consider $\mathbb{R}^{1|1}_{>0}$ as a generalized super semigroup by identifying it with the contravariant functor

$$S \mapsto \mathrm{Hom}(S, \mathbb{R}^{1|1}_{>0})$$

from supermanifolds to semigroups.
(2) Every $\mathbb{Z}_2$-graded real Banach space $B = B_0 \oplus B_1$ may be considered as a generalized supermanifold as follows. We define the value of the functor B on a super domain $U = (|U|, C^\infty(\_)[\theta_1, ..., \theta_q]) \subset \mathbb{R}^{p|q}$ to be

$$B(U) := (C^\infty(|U|, B)[\theta_1, ..., \theta_q])^{ev}.$$

The superscript *ev* indicates that we pick out the even elements, so that an element $f \in B(U)$ is of the form

$$f = \sum_I f_I \theta^I$$

where $I \subset \{1, ..., q\}$ and $\theta^I := \prod_{j\in I} \theta_j$ and each f_I is a smooth map $|U| \to B_{|I|}$. For a map $\varphi : U' \to U$ between super domains, the map $B(\varphi)$ is defined using the formal Taylor expansion, just as in the case of usual supermanifolds. This functor on super domains may be extended to the whole category of supermanifolds by gluing.
(3) If B is a $\mathbb{Z}_2$-graded Banach algebra, $B(U)$ is an algebra and thus B is a generalized super semigroup. Again, B may be extended to all supermanifolds by gluing.

Remark 36. Giving a morphism from an ordinary supermanifold T to a generalized supermanifold B amounts to prescribing the image of the universal element $\mathrm{id} \in \mathrm{Hom}(T, T)$ in $B(T)$. Hence $B(T)$ is exactly the set of morphisms from T to B.

[4] We use this simple notion here in order to avoid dealing with infinite-dimensional supermanifolds.

Now assume that, in addition, T and B carry super (semi)group structures. A map $\Phi : T \to B$ is a homomorphism if

$$\begin{array}{ccc} \mathrm{Hom}(S,T) \times \mathrm{Hom}(S,T) & \longrightarrow & \mathrm{Hom}(S,T) \\ \downarrow{\scriptstyle \Phi\times\Phi} & & \downarrow{\scriptstyle \Phi} \\ B(S) \times B(S) & \longrightarrow & B(S). \end{array}$$

commutes for all supermanifolds S. Again, it suffices to check the commutativity for the universal element

$$pr_1 \times pr_2 \in \mathrm{Hom}(T \times T, T) \times \mathrm{Hom}(T \times T, T).$$

Definition 37. Let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded Hilbert space, and denote by $B(\mathcal{H})$ the Banach algebra of bounded operators on $\mathcal{H}$ equipped with the $\mathbb{Z}_2$-grading inherited from $\mathcal{H}$.
(1) A *super semigroup of operators* on $\mathcal{H}$ is a morphism of generalized super semigroups

$$\Phi : \mathbb{R}^{1|1}_{>0} \longrightarrow B(\mathcal{H}).$$

As explained in the previous remark, Φ is of the form $A + \theta B$, where

$$A : \mathbb{R}_{>0} \to B^{ev}(\mathcal{H}) \text{ and } B : \mathbb{R}_{>0} \to B^{odd}(\mathcal{H})$$

are smooth maps. The homomorphism property amounts to certain relations between A and B (cf. the proof of Proposition 39).
(2) If $K \subset B(\mathcal{H})$ is a subset, we say Φ is a *super semigroup of operators with values in* K if the images of A and B are contained in K.
(3) If $\mathcal{H}$ is a module over the Clifford algebra C_n, we say Φ is C_n*-linear* if it takes values in C_n-linear operators.

Examples 38. SGOs arise in a natural way from Dirac operators. We give two examples of that type and then extract their characteristic properties to describe a more general class of examples. The verification of the SGO properties for these more general examples also includes the case of Dirac operators.
(1) Let $\mathcal{D}$ be the Dirac operator on a closed spin manifold X. There is a corresponding SGO on the Hilbert space of L^2-sections of the spinor bundle S over X. It is given by the super semigroup of operators

$$\mathbb{R}^{1|1}_{>0} \longrightarrow B(L^2(S)),\ (t,\theta) \mapsto e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2} \ (= e^{-t\mathcal{D}^2 + \theta\mathcal{D}})$$

and takes values in the compact, self-adjoint operators $K^{sa}(L^2(S)) \subset B(L^2(S))$.
(2) If $\dim X = n$, one can consider the C_n-linear spinor bundle and the associated C_n-linear Dirac operator (see [LM], chapter 2, §7). Using the same formula as in the previous example one obtains a C_n-linear SGO.
(3) Now, let $\mathcal{H}$ be any $\mathbb{Z}_2$-graded Hilbert space. For any closed subspace $V_\infty \subset \mathcal{H}$ invariant under the grading involution and any odd, self-adjoint

operator $\mathcal{D}$ on $V_\infty^\perp$ with compact resolvent, there is a unique super semigroup of self-adjoint, compact operators $\Phi = A + \theta B$ defined (using functional calculus) by

$$A(t) = e^{-t\mathcal{D}^2} \text{ and } B(t) = \mathcal{D}e^{-t\mathcal{D}^2} \text{ on } V_\infty^\perp$$

and $A(t) = B(t) = 0$ on V_∞. The first thing to check is that the maps A and B are indeed smooth; this follows easily using the fact that the map $\mathbb{R}_{>0} \to C_0(\mathbb{R})$, $t \mapsto e^{-tx^2}$, is smooth. Since $\mathcal{D}$ is self-adjoint, the same holds for A and B. Finally, we have to show that Φ is a homomorphism. Let t, θ, s, η be the usual coordinates on $\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0}$. It suffices to consider the universal element $pr_1 \times pr_2 = (t, \theta) \times (s, \eta)$. The computation, which, of course, heavily uses that odd coordinates θ and η square to zero, goes as follows (cf. [ST], page 38):

$$\begin{aligned}
& \Phi(t,\theta)\Phi(s,\eta) \\
= \; & (e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2})(e^{-s\mathcal{D}^2} + \eta\mathcal{D}e^{-s\mathcal{D}^2}) \\
= \; & e^{-t\mathcal{D}^2}e^{-s\mathcal{D}^2} + e^{-t\mathcal{D}^2}\eta\mathcal{D}e^{-s\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}e^{-s\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}\eta\mathcal{D}e^{-s\mathcal{D}^2} \\
= \; & e^{-(t+s)\mathcal{D}^2} + (\theta+\eta)\mathcal{D}e^{-(t+s)\mathcal{D}^2} + \theta\mathcal{D}\eta\mathcal{D}e^{-(t+s)\mathcal{D}^2} \\
= \; & (1 - \theta\eta\mathcal{D}^2)e^{-(t+s)\mathcal{D}^2} + (\theta+\eta)\mathcal{D}e^{-(t+s)\mathcal{D}^2} \\
= \; & e^{-(t+s+\theta\eta)\mathcal{D}^2} + (\theta+\eta)\mathcal{D}e^{-(t+s+\theta\eta)\mathcal{D}^2} \\
= \; & \Phi(t+s+\theta\eta, \theta+\eta)
\end{aligned}$$

The second to last equality uses the typical Taylor expansion in super geometry.

We call $\mathcal{D}$ the *infinitesimal generator* of Φ. We will see presently that every super semigroup of self-adjoint, compact operators has a unique infinitesimal generator and is hence one of our examples. Note that if V_∞ is a C_n-submodule and if $\mathcal{D}$ is C_n-linear, then A and B will also be C_n-linear.

Next, we will construct infinitesimal generators for super semigroups of operators. We restrict ourselves to the compact, self-adjoint case, which makes the proof an easy application of the spectral theorem for compact, self-adjoint operators. However, invoking the usual theory of semigroups of operators it should not be too difficult to prove the result for more general SGOs.

infgen **Proposition 39.** *Every super semigroup Φ of compact, self-adjoint operators on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ has a unique infinitesimal generator $\mathcal{D}$ as in Example* 3 *above and is hence of the form*

$$\Phi(t,\theta) = e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}.$$

If Φ is C_n-linear, so is $\mathcal{D}$.

We need the following technical lemma:

tech **Lemma 40.** *Let $A, B : \mathbb{R}_{>0} \to \mathcal{K}^{sa}(H)$ be smooth families of self-adjoint, compact operators on the Hilbert space H, and assume that the following relations hold for all $s, t > 0$:*

$$A(s+t) = A(s)A(t) \tag{1}$$

$$B(s+t) = A(s)B(t) = B(s)A(t) \tag{2}$$

$$A'(s+t) = -B(s)B(t). \tag{3}$$

Then H decomposes uniquely into orthogonal subspaces H_λ, $\lambda \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, such that on H_λ

$$A(t) = e^{-t\lambda^2} \text{ and } B(t) = \lambda e^{-t\lambda^2}$$

(where we set $e^{-\infty} = 0$, $\infty \cdot e^{-\infty} = 0$). For $\lambda \in \mathbb{R}$, the dimension of H_λ is finite. Furthermore, the subset of λ in $\mathbb{R}$ with $H_\lambda \neq 0$ is discrete.

Proof. The identities (1) – (3) above show that all operators $A(s)$, $B(t)$ commute. We apply the spectral theorem for self-adjoint, compact operators to obtain a decomposition of H into simultaneous eigenspaces H_λ of the $A(s)$ and $B(t)$; the label λ takes values in $\bar{\mathbb{R}}$ and will be explained presently. We define functions $A_\lambda, B_\lambda : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$A(t)x = A_\lambda(t)x \text{ and } B(t)x = B_\lambda(t)x \text{ for all } x \in H_\lambda.$$

Clearly, A_λ and B_λ are smooth and satisfy the same relations as A and B.

From (1) we see that A_λ is non-negative, and (3) shows $A'_\lambda \leq 0$, hence A_λ is decreasing. On the other hand, (1) implies $A_\lambda(\frac{1}{n}) = \sqrt[n]{A_\lambda(1)}$, so that

$$A_\lambda(0) := \lim_{t\to 0} A_\lambda(t) \text{ exists and equals 0 or 1.}$$

In the first case we conclude $A_\lambda \equiv 0$ and thus also $B_\lambda \equiv 0$; the label of the corresponding subspace is $\lambda = \infty$. In the second case, we have $A_\lambda(1) \neq 0$ and using (1) again we compute

$$A'_\lambda(s) = \frac{A_\lambda(1)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(s+t) - A_\lambda(s)}{t} = \frac{A_\lambda(s)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(1+t) - A_\lambda(1)}{t} = -\lambda^2 A_\lambda(s),$$

where $\lambda^2 := -A'_\lambda(1)/A_\lambda(1)$ defines the label λ up to choice of a sign. By uniqueness of solutions of ODEs, we must have

$$A_\lambda(t) = e^{-t\lambda^2}.$$

Finally, (3) gives

$$B_\lambda(t) = \lambda e^{-t\lambda^2},$$

picking the appropriate sign for λ. □

Proof of Proposition 39. Let $\Phi = A + \theta B$ be a super semigroup of compact, self-adjoint operators. As before, we consider $U = \mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0}$ with coordinates t, θ, s, η. For the universal element $pr_1 \times pr_2 = (t, \theta) \times (s, \eta)$ the homomorphism property of Φ gives that

$$\begin{aligned} \Phi(t + s + \theta\eta, \theta + \eta) &= A(t + s + \theta\eta) + (\theta + \eta)B(t + s + \theta\eta) \\ &= A(t + s) + A'(t + s)\theta\eta + (\theta + \eta)(B(t + s) + B'(t + s)\theta\eta) \\ &= A(t + s) + \theta B(t + s) + \eta B(t + s) + \theta\eta A'(t + s) \end{aligned}$$

equals

$$\begin{aligned} \Phi(t, \theta)\Phi(s, \eta) &= (A(t) + \theta B(t))(A(s) + \eta B(s)) \\ &= A(t)A(s) + \theta B(t)A(s) + \eta A(t)B(s) - \theta\eta B(t)B(s). \end{aligned}$$

Comparing the coefficients[5] yields exactly the relations in Lemma 40. Using the corresponding decompostion of $\mathcal{H}$ into subspaces $\mathcal{H}_\lambda$ we define the operator $\mathcal{D}$ by letting $\mathcal{D} = \lambda$ on $\mathcal{H}_\lambda$. From the construction it is clear that $\mathcal{D}$ is the desired infinitesimal generator. Since A is even and B is odd, it follows that $\mathcal{D}$ is an odd operator. If Φ is C_n-linear, so is $\mathcal{D}$. □

We can finally give the definition promised in part (5) of Theorem 1.

Definition 41. Let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded Hilbert space and $C \subset B(\mathcal{H})$ a subspace of the algebra of bounded operators on $\mathcal{H}$. We denote by $\mathcal{SGO}(C)$ the set of super semigroups of operators with values in C and in particular

$$\mathcal{SGO}_n := \mathcal{SGO}(\mathcal{K}_n) \quad \text{and} \quad \mathcal{SGO}_n^{\text{fin}} := \mathcal{SGO}(\mathcal{FR}_n)$$

(the subspaces $\mathcal{K}_n$ and $\mathcal{FR}_n$ were defined in Definition 20). We endow $\mathcal{SGO}(C)$ with the topology of uniform convergence on compact subsets, i.e.

$$\Phi_n = A_n + \theta B_n \longrightarrow \Phi = A + \theta B$$

if and only if for all compact $K \subset \mathbb{R}_{>0}$ we have

$$A_n(t) \longrightarrow A(t) \text{ and } B_n(t) \longrightarrow B(t) \text{ uniformly on } K$$

with respect to the operator norm on $B(\mathcal{H})$.

We will now relate the spaces $\mathcal{SGO}_n$ with our configuration spaces Conf_n. We have a triangle (and an analogous one for finite rank operators)

$$\begin{array}{ccc} \mathcal{SGO}_n & \xleftarrow{R} & \text{Hom}_{gr}(\mathcal{S}, \mathcal{K}_n) \\ & {\scriptstyle I}\searrow \quad \nearrow{\scriptstyle F} & \\ & \text{Conf}_n, & \end{array}$$

[5]Just to make the formal aspect of this computation clearer, we would like to point out that the considered identity is an equation in the algebra $K^{sa}(\mathcal{H})(\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0}) = C^\infty(\mathbb{R}_{>0} \times \mathbb{R}_{>0}, K^{sa}(\mathcal{H}))[\theta, \eta]^{ev}$.

where I maps a super semigroup of operators to its infinitesimal generator, F is given by functional calculus,

$$F(\mathcal{D})(f) := f(\mathcal{D}),$$

and R is given by

$$R(\varphi) := \varphi(e^{-tx^2}) + \theta\varphi(xe^{-tx^2}).$$

triangle2

Proposition 42. *The maps I, F, and R are homeomorphims, and similarly for finite rank operators.*

Proof. From the previous discussion it is clear that the composition of the three arrows is the identity no matter where in the triangle we start. We already know from Proposition triangle lemma 27 that F is a homeomorphism. We complete the proof by showing that R is a homeomorphism.

The continuity of R^{-1} follows from the following assertion. We claim that we have convergences of operators

$$f(\mathcal{D}_n) \longrightarrow f(\mathcal{D}) \text{ for all } f \in C_0(\mathbb{R})$$

if and only if the following two sequences converge:

$$e^{-\mathcal{D}_n^2} \longrightarrow e^{-\mathcal{D}^2} \text{ and } \mathcal{D}_n e^{-\mathcal{D}_n^2} \longrightarrow \mathcal{D}e^{-\mathcal{D}^2}.$$

The first obviously implies the second condition. To see the converse, note that the assumption implies that $f(\mathcal{D}_n) \to f(\mathcal{D})$ for all f that can be written as a polynomial in the functions e^{-x^2} and xe^{-x^2}. Furthermore, the Stone-Weierstraß Theorem implies that e^{-x^2} and xe^{-x^2} generate $C_0(\mathbb{R})$ as a C^*-algebra so that the set of such f is dense in $C_0(\mathbb{R})$. Using that $\|f(\mathcal{D})\| \le \|f\|$ for all $\mathcal{D}$ and the triangle inequality we can deduce that $f(\mathcal{D}_n) \to f(\mathcal{D})$ holds for all $f \in C_0(\mathbb{R})$.

The continuity of R amounts to showing that if $f(\mathcal{D}_n) \to f(\mathcal{D})$ for all f, then $e^{-t\mathcal{D}_n^2} \to e^{-t\mathcal{D}^2}$ and $\mathcal{D}_n e^{-t\mathcal{D}_n^2} \to \mathcal{D}e^{-t\mathcal{D}^2}$ *uniformly* for all t in a compact subset $K \subset \mathbb{R}_{>0}$. As before, we can use $\|f(\mathcal{D})\| \le \|f\|$ and the triangle inequality to see that for a given $\varepsilon > 0$ we can find N such that we do not only have $\|f(\mathcal{D}_n) - f(\mathcal{D})\| \le \varepsilon$ for all $n \ge N$, but that this estimate also holds for all g in a small neighborhood of f. This together with the compactness of K and the continuity of the maps $t \mapsto e^{-tx^2}$ and $t \mapsto xe^{-tx^2}$ implies the claim. □

Remark 43. The arguments in the last parts of the proof can be used to show that we could also have equipped $\mathcal{SGO}_n$ with the topology that controls *all* derivatives of a super semigroup map Φ and still would have obtained the same topological space. We find this interesting, because this is the topology that one usually considers on spaces of smooth maps.

Super semigroups of operators and Euclidian field theories. In the context of super symmetric Euclidian field theories of dimension (1|1) slight variations of the spaces $\mathcal{SGO}_n$ appear. We conclude this chapter by describing these spaces and showing that they have the same homotopy type as the spaces $\mathcal{SGO}_n$. More information about the axiomatic definition of super symmetric EFTs will be given in the next chapter.

Definition 44. Let $\mathrm{TC}(\mathcal{H}_n)$ be the Banach algebra of trace class operators on $\mathcal{H}_n$ (equipped with the nuclear norm). Define $\mathcal{SGO}_n^{\mathrm{TC}}$ to be the space of super semigroup homomorphisms $\mathbb{R}^{1|1}_{>0} \to \mathrm{TC}(\mathcal{H}_n)$ with values in self-adjoint C_n-linear operators. As before, we consider the topology given by uniform convergence on compact subsets $K \subset \mathbb{R}_{>0}$. In the same way, we define $\mathcal{SGO}_n^{\mathrm{HS}}$, where $\mathrm{TC}(\mathcal{H}_n)$ is replaced by $\mathrm{HS}(\mathcal{H}_n)$, the Banach algebra of Hilbert-Schmidt operators on $\mathcal{H}_n$.

Examples 45. An SGO defined by an infinitesimal generator $\mathcal{D}$ lies in $\mathcal{SGO}^{\mathrm{TC}}$ resp. $\mathcal{SGO}^{\mathrm{HS}}$ if the eigenvalues of $\mathcal{D}$ converge to infinity sufficiently fast. This is, for example, the case for Dirac operators on closed spin manifolds, cf. [LM], Chapter 3, §5.

trace class

Proposition 46. *The injections*

$$\mathcal{SGO}_n^{\mathrm{TC}} \hookrightarrow \mathcal{SGO}_n \quad \text{and} \quad \mathcal{SGO}_n^{\mathrm{HS}} \hookrightarrow \mathcal{SGO}_n$$

are homotopy equivalences.

Proof. We give the proof for $\mathcal{SGO}_n^{\mathrm{TC}}$. The argument for $\mathcal{SGO}_n^{\mathrm{HS}}$ is similar. The basic observation we use is that on operators of rank at most k the operator norm and the nuclear norm are equivalent. More precisely,

$$\|T\| \leq \|T\|_{nuc} \leq k \cdot \|T\|$$

if $\mathrm{rank}(T) \leq k$. Namely, if $T = \sum_{i=1}^k s_i \langle ., e_i \rangle f_i$ with $s_1 \geq \cdots \geq s_k \geq 0$ for orthonormal systems e_i and f_i, then $\|T\| = s_1$ and $\|T\|_{nuc} = \sum_{i=1}^k s_i \leq k s_1$. Denote by $\mathcal{S}^{(k)} \subset \mathcal{SGO}_n^{\mathrm{TC}}$ the subspace of SGOs whose infinitesimal generator has domain of dimension $\leq 2k$. By our basic observation, the topology on $\mathcal{S}^{(k)}$ is the same as the one we get by considering it as a subspace of $\mathcal{SGO}_n$. Using the identification $I : \mathcal{SGO}_n \approx \mathrm{Conf}_n$ and Remark 30 we see that $\mathcal{S} := \mathrm{colim}_{k\to\infty} \mathcal{S}^{(k)} \simeq \mathcal{SGO}_n$. Hence, if we can show that $\iota : \mathcal{S} \hookrightarrow \mathcal{SGO}_n^{\mathrm{TC}}$ is a homotopy equivalence, then the same is true for $\mathcal{SGO}_n^{\mathrm{TC}} \hookrightarrow \mathcal{SGO}_n$. Define homotopies $\tilde{H}$ and H^{TC} by the commutative diagram

$$\begin{array}{ccccc}
\mathcal{SGO}_n^{\mathrm{TC}} \times [0,1] & \hookrightarrow & \mathcal{SGO}_n \times [0,1] & \xrightarrow{\approx} & \mathrm{Conf}_n \times [0,1] \\
\downarrow H^{\mathrm{TC}} & & \downarrow \tilde{H} & & \downarrow H \\
\mathcal{SGO}_n^{\mathrm{TC}} & \hookrightarrow & \mathcal{SGO}_n & \xrightarrow{\approx} & \mathrm{Conf}_n,
\end{array}$$

finite vs non-finite

where H is the homotopy used in the proof of Proposition 29. It is easy to see that the map $\iota^{-1} \circ H_1^{\mathrm{TC}} : \mathcal{SGO}_n^{\mathrm{TC}} \to S$ is continuous. We claim that it is homotopy inverse for ι. The argument works exactly as in Remark 30. The only thing to check is that the homotopy H^{TC} is continuous (note that now we are using the topology that comes from the nuclear norm). For $t \neq 0$, the continuity follows from the continuity of $\tilde{H}$ and the fact that H^{TC} locally maps to $\mathcal{S}^{(k)}$, for some k, as long as $t \neq 0$.

colim Confk

Let us now consider the case $t = 0$. Assume $\Phi_m \to \Phi$ and $t_m \to 0$. For $C \subset \mathbb{R}$, we denote by Φ^C the SGO defined by the infinitesimal generator obtained from $I(\Phi)$ by omitting all labels in $\mathbb{R} \setminus C$. Let $K \subset \mathbb{R}_{>0}$ compact, $\varepsilon > 0$. Write $\|\Phi\| := \sup_{s\in K}(\|A(s)\|_{nuc} + \|B(s)\|_{nuc})$. Since $\Phi \in \mathcal{SGO}^{\mathrm{TC}}$, we can choose a big symmetric interval $C = [-\kappa, \kappa] \subset \mathbb{R}$ with $I(\Phi)_\kappa = 0$ such that $\|\Phi - \Phi^C\| < \varepsilon$. Note that $\Phi_m \to \Phi$ also implies $\Phi_m^C \to \Phi^C$. The continuity of $\tilde{H}$ implies that the restriction of H^{TC} to $\mathcal{S}^{(k)} \times [0,1]$ is continuous for all k. In particular, we have $H^{TC}(\Phi_m^C, t_m) \to \Phi^C$. Now, choose $N \in \mathbb{N}$ such that for all $m > N$

$$\|\Phi - \Phi_m\| < \varepsilon, \quad \|\Phi^C - \Phi_m^C\| < \varepsilon, \quad \text{and} \quad \|H^{\mathrm{TC}}(\Phi_m^C, t_m) - \Phi^C\| < \varepsilon.$$

It then follows that for all $m > N$

$$\begin{aligned} \||\Phi - H^{\mathrm{TC}}(\Phi_m, t_m)\|| &\leq \|\Phi - \Phi^C\| + \|\Phi^C - H^{\mathrm{TC}}(\Phi_m^C, t_m)\| + \|H^{\mathrm{TC}}(\Phi_m^{\mathbb{R}\setminus C}, t_m)\| \\ &\leq 2\varepsilon + \|\Phi_m - \Phi\| + \|\Phi - \Phi^C\| + \|\Phi^C - \Phi_m^C\| \\ &\leq 5\varepsilon \end{aligned}$$

where we used that $\|H^{\mathrm{TC}}(\Phi_m^{\mathbb{R}\setminus C}, t_m)\| \leq \|\Phi_m^{\mathbb{R}\setminus C}\| = \|\Phi_m - \Phi_m^C\|$. This inequality follows directly from the definition of H^{TC} (provided we choose $\kappa \geq \min_{s\in K} 1/\sqrt{2s}$), which we can certainly do). Hence, $H^{\mathrm{TC}}(\Phi_m, t_m) \to \Phi$ in $\mathcal{SGO}_n^{\mathrm{TC}}$ so that H^{TC} is continuous. □

ST

Remark 47. In [ST] the notation $\mathcal{EFT}_n$ was used for the space $\mathcal{SGO}_n^{\mathrm{HS}}$. In this paper we want to reserve the notation $\mathcal{EFT}$ for the "correct" notion of Euclidean field theories: we give a geometric definition of super symmetric Euclidian field theories of dimension (1|1) in the next chapter. It turns out that the resulting spaces $\mathcal{EFT}_n$ are homeomorphic to $\mathcal{SGO}_n^{\mathrm{TC}}$.

sec:EFT

6. 1-dimensional super symmetric Euclidean field theories

In this section we outline how the spaces $\mathcal{SGO}^{TC}$ of super semigroups of self-adjoint trace-class operators, studied in the previous section, are related to the space of super symmetric (1|1)-dimensional (positive) Euclidean field theories. Here we shall only treat the case $n = 0$, so there is no Clifford algebra C_n in the game. If $n \neq 0$, we would have to explain twisted Euclidean field theories that are defined in terms of certain 2-functors between 2-categories. This would lead to far afield in this paper and in the end, it would only lead to a different way that the Clifford algebras C_n arise. We also work with complex vector spaces for simplicity, thus getting a model for complex K-theory.

We first recall from [ST2] our (current) definition of super symmetric Euclidean field theories. This can be done in any dimension d but later only $d = 1$ will be relevant here. In addition to the degree n, we shall also ignore in this exposition the fact that in dimension d one should really work with d-categories and d-functors (but this is o.k. for $d = 1$) and we do not work over a target manifold X, i.e. X is a point in this discussion. All of these neglections still make it possible to get a space in the homotopy type of $BU \times \mathbb{Z}$. However, it is essential that we work in the super symmetric context, otherwise we would be left with a contractible space instead.

The symmetric monoidal category of Fréchet spaces. We start with a review of some basic facts on Fréchet spaces, see for example [K]. These are complete locally convex vector spaces which are metrizable. The last condition is equivalent to having a countable neighborhood basis of zero, a rather modest condition, satisfied for all separable spaces. More concretely, Fréchet spaces are the complete topological vector spaces whose topology is defined by an increasing sequence of semi-norms

$$\rho_1 \leq \rho_2 \leq \dots$$

Typical examples are Banach spaces (defined by a single norm) or spaces of continuous functions $C^0(X)$, X a locally compact topological space that is countable at infinity. Then X is a union of an increasing sequence of compact sets K_n and the ρ_n are given by the supremum norm on K_n. In fact, every Fréchet space is a linear subspace of such an example. For a smooth manifold M, one can also consider $C^\infty(M)$, where the Sobolev norms qualify for the ρ_s. A Fréchet space is Banach if and only if its strong dual is a Fréchet space. For a Banach space, the dual space has again a complete norm and that gives the strong topology. For a Fréchet space there are many topologies on the dual space but we shall not explain them because there is no definition that always ends up with a Fréchet space!

There are also many topologies on the algebraic tensor product of two topological vector spaces V and W. We will only study the *projective* topology which is the finest topology on $V \otimes_{alg} W$ for which the canonical bilinear map

$$V \times W \longrightarrow V \otimes_{alg} W$$

is continuous. This means that a linear map $V \otimes_{alg} W \to G$ is continuous if and only if the corresponding bilinear map $V \times W \to G$ is continuous (in both variables).

Lemma 48. *If V, W are Fréchet spaces then the completion $V \otimes W$ of the algebraic tensor product in the projective topology is again a Fréchet space,* [K, II, 178]. *This tensor product is associative and commutative in the sense that* SF*, the category of Fréchet spaces, becomes a symmetric monoidal category under this* projective tensor product.

We note that for Fréchet spaces, the projective and the *inductive* topology on the algebraic tensor product agree. The latter is defined using *seperately* continuous bilinear maps on $V \times W$ instead of maps that are continuous in both variables.

Examples 49. If V is Fréchet then $C^\infty(M; V) \cong C^\infty(M) \otimes V$. In particular,

$$C^\infty(M \times N) \cong C^\infty(M) \otimes C^\infty(N)$$

for any smooth manifolds M and N. In fact, in the graded category these statements continue to hold for super manifolds.

If V, W happen to be Hilbert spaces then $\overline{V} \otimes W$ can be identified with the space of trace-class operators from V to W by the canonical map.

The last example can be generalized as follows. Let V' be the continuous dual of a topological vector space. Then there is a bilinear map

$$\psi : V \times W \longrightarrow \mathcal{L}(V', W)$$

into the vector space of linear operators. If V, W are Fréchet then this extends to a linear map Ψ on the completion $V \otimes W$. In general, this map is *not* injective and V has the *approximation property* if Ψ is injective for all W. The image of ψ are finite rank operators and the image of Ψ are the so called *nuclear operators*. For $V = W$ they have a well-defined trace (given by evaluation) if and only if V has the approximation property. Not even all Banach spaces have this property but an important class of examples which do have it are Fréchet spaces with a *basis*, [K, II, 250]. This is a sequence of vectors $\{e_n\}$ in V such that any element $v \in V$ can be uniquely written as

$$v = \sum_{n=1}^{\infty} a_n \cdot e_n \quad , a_n \in \mathbb{C},$$

where the partial sums converge to v in the Fréchet topology. For example, $1, z, z^2, \dots$ is a basis for the Fréchet space of holomorphic maps on the open unit disk. Also $C^0(S^1)$, $C^\infty(S^1)$ and $L^p(S^1)$ have a basis for $1 \leq p < \infty$. All bounded measurable functions $L^\infty([0, 1])$ cannot have a basis since they are dual to L^1-functions and hence not separable. It follows (except for $p = \infty$ which needs a seperate argument) that for any locally compact Radon measure space (R, μ) the Banach space $L^p(R, \mu)$ has the approximation property for $1 \leq p \leq \infty$ and so do $C^0(X)$ for locally compact topological spaces X and $C^\infty(M)$ for smooth manifolds.

The following lemma will be used in the main theorem of this section.

lem:Frechet **Lemma 50.** *Let V be a Fréchet space and $h : \bar{V} \times V \to \mathbb{C}$ a continuous bilinear map that is hermitian, positive definite and induces an (algebraic) isomorphism $\bar{V} \to V'$. Then (V, h) is a Hilbert space (whose norm topology agrees with the original Fréchet topology).*

Before we give the proof of this lemma, we point out the case $V = C^\infty(M)$ for a compact Riemannian manifold M. The inclusion $C^\infty(M) \hookrightarrow L^2(M)$ is continuous but not open. It induces a positive definite inner product h on V which satisfies all the conditions except that $\bar{V} \to V'$ is only $1-1$ and not onto. So the following proof of the lemma doesn't seem totally obvious to us.

Proof. It is clear that the composition

$$V \xrightarrow{\Delta} V \times V \xrightarrow{h} \mathbb{C}$$

is continuous and therefore the identity is a continuous map $V \to H$, where V denotes the original Fréchet space and H denotes V with the norm topology given by $\|v\|^2 := h(v, v)$. We shall now show that the identity is also a continuous map $\bar{H} \to \bar{V}$ which is all that needs to be shown.

Following [K, I, 263] the Fréchet topology on V is equal to the Mackey topology V_k given by the dual pair (V, V', st), where st is the standard pairing. As for any topology compatible with this dual pair, st induces an (algebraic) isomorphism $V \cong (V_k')'$.

The Mackey topology only depends on the algebraic data of a dual pair and since $h : \bar{V} \to V'$ is an isomorphism, V_k is also the Mackey topology of the dual pair $(V, \bar{V}, h)$. Now take complex conjugates to see that $\bar{V}$ has the Mackey topology $\bar{V}_k$ for the dual pair $(\bar{V}, V, h)$. Note moreover that h gives an algebraic isomorphism

$$(\bar{V}, V, h) \cong (V_k', (V_k')', st)$$

All together, this says that $h : \bar{V} \to V_k'$ is an isomorphism of Fréchet spaces. On the H side, we see similarly that h induces a map $\bar{H} \to H_k'$ where the latter topology comes from the dual pair $(H_k', (H_k')', st)$. But since the canonical map $H \to (H_k')'$ is continuous, one still gets a continuous map $j_h : \bar{H} \to H_k'$ if one equips the latter with the topology from the dual pair (H_k', H, st).

Now start with the continuous map $\mathrm{id} : V \to H$. By [K, I, 238], it is also continuous with respect to the weak topologies V_s respectively H_s given by the standard dual pairs. Therefore, its adjoint is again a continuous map $\mathrm{id}' : H_s' \to V_s'$. These dual spaces come from Fréchet spaces $H' \cong \bar{H}$ respectively $V_k' \cong \bar{V}$ and their standard dual pairs (H', H) respectively (V_k', V_k). Hence by [K, I, 262] this map $\mathrm{id}' : H_k' \to V_k'$ is also continuous in the Mackey topologies.

Finally, we can put this information together to see that the identity map $\mathrm{id}: \bar{H} \to \bar{V}$ factors as the composition of the continuous maps

$$\bar{H} \xrightarrow{j_h} H'_k \xrightarrow{\mathrm{id}'} V'_k \xrightarrow{h^{-1}} \bar{V}$$

This finishes the proof of our lemma. □

Super Hilbert spaces. In the previous sections of this paper we were dealing with ($\mathbb{Z}/2-$) graded (complex) Hilbert spaces $\mathcal{H}$. To be precise, we'll now write $\mathcal{H} = (V, \alpha, h)$ where V is the underlying topological vector space, α is the grading, and $h : \bar{V} \times V \to \mathbb{C}$ is the (positive definite) inner product. It will turn out that Euclidean field theories yield a slightly different notion, namely the following.

Definition 51. A *super Hilbert space* is a triple (V, α, μ) where (V, ϵ) is a graded vector space and $\mu : \bar{V} \times V \to \mathbb{C}$ is a hermitian inner product with the following symmetry property:

$$\overline{\mu(v, w)} = (-1)^{|v||w|}\mu(w, v) \quad \forall v, w \in V.$$

Here $|v| \in \mathbb{Z}/2$ denotes the degree defined by α. Writing $v = v_0 + v_1$ for the decomposition of $v \in V$ into the even and odd parts, it follows that

$$\mu(v_0, v_0) \in \mathbb{R} \quad \text{and} \quad \mu(v_1, v_1) \in i \cdot \mathbb{R}.$$

As a "positivity" condition we require that the former is a positive real number for $v_0 \neq 0$ and the latter is i times a negative real number for $v_1 \neq 0$.

Then $|\mu(v, v)|^2 = \mu(v_0, v_0)^2 + \mu(v_1, v_1)^2$ gives a norm on V which we require to be complete. We will sometimes refer to μ as a *super inner product* on V.

Proposition 52. *The category of super Hilbert spaces (and continuous linear operators) is equivalent to the category of graded Hilbert space (and continuous linear operators) via the map*

$$(V, \alpha, \mu) \mapsto (V, \alpha, h), \text{ with } h(v_0, w_0) := \mu(v_0, w_0) \text{ and } h(v_1, w_1) := i \cdot \mu(v_1, w_1)$$

Moreover, the topologies on V defined by the norms $|\mu(v, v)|$ respectively $|h(v, v)| = |h(v_0, v_0)| + |h(v_1, v_1)| = |\mu(v_0, v_0)| + |\mu(v_1, v_1)|$ are equal and the two inner products define the same adjoint operators for all even *morphisms.*

The proof of this proposition is straight forward and we leave it to the reader. It is interesting to note the for odd operators, there result two different notion of "self-adjoint" with respect to μ respectively h. It is the latter notion that was used in Section 3 for the notion of infinitesimal generators $\mathcal{D} \in \mathrm{Inf}$. Because of the last sentence in the above proposition, this confusion does not arise for even operators, in particular for the semigroups generated by $\mathcal{D}$ as in Section 5.

Euclidean field theories. There are three categories in the following definition, the easiest being the category SMAN of complex super (or *cs*) manifolds. Furthermore, we introduce a certain super Riemannian bordism category SRB_d and the category SF of families of super Fréchet spaces. More precisely, an object in SF is a sheaf of $\mathbb{Z}/2$-graded, separable Fréchet modules over a cs manifold S. Roughly speaking, an object in SRB_d is a fibre bundle $Y \to S$ in SMAN whose fibres are closed Riemannian $(d|1)$-dimensional cs manifolds. This notion is defined in [ST2] for $d = 0, 1, 2$ and will be explained for $d = 1$ below. There are symmetric monoidal structures on these three categories, coming from the basic structures of

- cartesian product of cs manifolds (the parameter spaces),
- disjoint union of Riemannian super manifolds (the fibres),
- graded tensor product of sheafs of Fréchet modules, completed in the *projective* topology.

Note that these are compatible in the sense that for two cs manifold S_i

$$C^\infty(S_1 \times S_2) \cong C^\infty(S_1) \otimes C^\infty(S_2)$$

is a (graded) projective tensor product.

def:QFT **Definition 53.** A d-dimensional super symmetric Euclidean field theory is a functor E that fits into the following commutative diagram

$$\begin{array}{ccc} \mathrm{SRB}_d & \xrightarrow{\quad E \quad} & \mathrm{SF} \\ & \searrow \quad \swarrow & \\ & \mathrm{SMAN} & \end{array}$$

The functor E is required to preserve pullbacks, the grading and symmetric monoidal structure. (The two forgetful functors to SMAN are Grothendieck fibrations that preserve grading and symmetric monoidal structure.)

All three categories come equipped with an involution induced by complex conjugation. A Euclidean field theory E is called *unitary* if it also preserves these involutions.

A *grading* on a category is a natural transformation from the identity functor to itself that squares to the identity (recall that all gradings in this paper are $\mathbb{Z}/2$-gradings). For example, a graded vector space has a canonical involution given by the (even) grading operator. This induces a grading on the *category* of graded vector spaces. The gradings on the above three categories all come from this simple observation, namely that a grading gives extra *structure* as well as extra *symmetry* on the resulting objects.

Instead of requiring E to be grading preserving, we initially tried to *induce* the grading structure on the vector space $E(Y)$ associated to a closed $(d|1)$-manifold Y from the grading symmetry on $C^\infty(Y)$. This lead us to a category of *ungraded* modules over graded commutative rings $C^\infty(Y)$. However, we were unable to equip this category with a symmetric monoidal structure, compatible with the usual tensor product of graded rings (forced on us from the cartesian product of super manifold).

Remark 54. Since we work with complex super manifolds, there is a *complex* algebra of smooth functions. Taking the conjugate complex structure gives a new cs manifold and this is the involution on SRB_d used to define unitarity. It turns out that this operation induces orientation (or spin structure) reversal on the underlying reduced manifold, see Definition 55.

Using the involutions of complex conjugation in a particular way, we believe that we can get a model for Atiyah's Real K-theory out of the above notions.

Germs of super Riemannian manifolds. Ignoring for now families over S and odd directions, we consider Riemannian spin d-manifold Y^d, without boundary, but with a finite number of ends $e(Y)$. These are decomposed into

$$e(Y) = e^b(Y) \amalg e^g(Y),$$

the *boundary-like ends* $e^b(Y)$ and the *germ-like ends* $e^g(Y)$. We can form the metric completion $\hat{Y}$ in the direction of $e^b(Y)$ and we require that $\hat{Y}$ is a topological manifold with boundary $\partial\hat{Y}$ such that the canonical map induces a bijection $\pi_0(\partial\hat{Y}) \xrightarrow{\approx} e^b(Y)$.

For set theoretic reasons we assume that all cs manifolds are embedded in some fixed $\mathbb{R}^{\infty|\infty}$. Therefore, it makes good sense to talk about *germs* of neighborhoods of $\partial\hat{Y}$ in $\hat{Y}$. These are equivalence classes of manifolds Y as above under the relation generated by inclusions (preserving the boundary $\partial\hat{Y}$). These germs give the objects in SRB_d if one adds S-families and odd directions, as explained below. The picture for two distinct objects in SRB_1 is as follows:

Here the boundary-like ends are drawn as • and the germ nature of the neighborhood of • is expressed by the dotted lines. Finally, the orientations are indicated by an arrow. The precise definition of the *super point* is given in Definition 55, for more details see [ST2]. We note that we are workink with complex super (cs) manifolds and hence the functions on the local model $\mathbb{R}^{p|q}$ are the *complex* graded algebras

$$C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \mathbb{C}[\eta_1, \dots, \eta_q].$$

In particular, let us write y and η for the even respectively odd coordinate function on $\mathbb{R}^{1|1}$ and ∂_y, ∂_η for the corresponding vector fields. Then the standard super Riemannian structure on this cs manifold is given by the odd vector field

eq:D
$$D := \partial_\eta - i\eta\partial_y. \tag{4}$$

It squares to $D^2 = -i\partial_y$ and so it reduces to a complex vector field $(D^2)_{red}$ on $(\mathbb{R}^{1|1})_{red} = \mathbb{R}$ which is nowhere vanishing and whose complex conjugate equals $-(D^2)_{red}$. Since $i(D^2)_{red}$ is the vector field ∂_y, we see that this super Riemannian structure on $\mathbb{R}^{1|1}_{cs}$ induces the standard Riemannian metric and the standard spin structure on $\mathbb{R}$. Moreover, changing the complex structure on the algebra of functions changes multiplication by i to multiplication by $-i$ and hence ∂_y to $-\partial_y$: the orientation has been reversed!

def:spt

Definition 55. A super Riemannian structure on a $(1|1)$-dimensional cs manifold is an odd vector field D, defined up to sign, which is locally given by equation eq:D 4 above.

The *super point* spt is given by the germ of submanifolds $(0,\epsilon) \times \mathbb{R}^{0|1}$ of $(\mathbb{R}^{1|1}, D)$, where $\epsilon > 0$ tends to 0. By definition, the reversed super point $\overline{\text{spt}}$ only differs from spt by taking the complex conjugate $\mathbb{C}$-action on the algebra of functions $\mathbb{C}^\infty(\text{spt})$. It is not hard to check that the map $\varphi(y,\eta) := (-y, eta)$ induces and isomorphism from $\overline{\text{spt}}$ with the germ of submanifolds $(-\epsilon, 0) \times \mathbb{R}^{0|1}$ of $(\mathbb{R}^{1|1}, D)$, where $\epsilon > 0$ tends to 0.

An S-family of such manifolds is a bundle map $Y \to S$ of cs manifolds, with $(1|1)$-dimensional fibres and a vertical odd vector field D, defined up so sign, such that there are local coordinates on Y in which D is given by equation eq:D 4. Here D is called *vertical* if it annihilates all functions on Σ that factor through S.

Germs of such S-families $Y \to S$ are the objects in SRB_1 if the reduced part Y_{red} satisfies fibrewise the conditions on its ends explained above.

Since a Euclidean field theory E preserves pullbacks, its value on the trivial bundle $S \times \text{spt} \to S$ is determined by its value on $\text{spt} \to *$. Furthermore, an object in SRB_1 that is an S-family of super points is locally of the form $\pi_1 : S_0 \times \text{spt} \to S_0$. Since the value of E on this family is a sheaf of modules over S, it is determined by these local data, and hence again by $E(\text{spt})$. Finally, E preserves disjoint unions and takes the involution of complex conjugation (aka orientation reversal) to complex conjugation. We hence get the following

cor:objects

Corollary 56. *A 1-dimensional unitary Euclidean field theory E is determined on objects by the Fréchet space $E(\text{spt})$.*

Super Riemannian Bordisms. Morphisms from Y_{in} to Y_{out} in SRB_d, again ignoring S-families and odd directions for now, are isometry classes of Riemannian spin bordisms from Y_{in} to Y_{out}; here a *Riemannian spin bordism* is a diagram of isometric spin embeddings

$$Y_{out} \overset{\iota_{out}}{\hookrightarrow} \Sigma \overset{\iota_{in}}{\hookleftarrow} Y_{in}$$

where Σ is again a Riemannian spin d-manifold, without boundary, but with a finite number of ends $e(\Sigma)$. These are decomposed into

$$e(\Sigma) = e^{out}(\Sigma) \amalg e^{in}(\Sigma),$$

the *incoming and outgoing* ends. We take the metric completion $\hat{\Sigma}$ in the direction of $e^{out}(\Sigma)$ and we require that $\hat{\Sigma}$ is a topological manifold with boundary $\partial\hat{\Sigma}$ such that the canonical map induces a bijection

$$\pi_0(\partial\hat{\Sigma}) \overset{\approx}{\to} e^{out}(\Sigma).$$

The requirements on the embeddings $\iota_k : Y_k \hookrightarrow \Sigma$, satisfied for some representative of the germ of Y_k, are as follows:

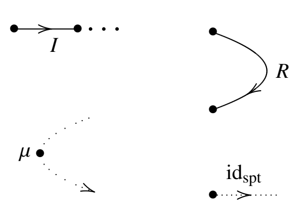
- ι_{in} is proper on the germ-like ends and induces a bijection
$$e^g(Y_{in}) \overset{\approx}{\to} e^{in}(\Sigma).$$
- ι_{out} is proper on the boundary-like ends and the unique extension to metric completions induces a homeomorphism
$$\partial\hat{Y}_{out} \overset{\approx}{\to} \partial\hat{\Sigma}.$$

The composition of bordism is given by gluing along open subsets in the obvious way. The properness conditions on the ι_k make sure that the resulting space is Hausdorff.

Remark 57. There are no disjointness requirements on ι_k, for example they could both be surjective isometries. We call a bordism *thick* if $\Sigma \setminus \iota_{in}(Y_{in})$ is d-dimensional, and otherwise *thin*. The invertible morphisms in SRB_d are necessarily thin and what one usually thinks of as a 'bordism' is the topological manifold $\hat{\Sigma} \setminus \iota_{in}(Y_{in})$. It is always compact but d-dimensional exactly in the thick case.

Here are some pictures of morphisms in SRB_1:

eq:morphisms (5)



We read all these bordisms from *right to left*, for example

$$I : \mathrm{spt} \to \mathrm{spt} \quad \text{and} \quad R : \emptyset \to \mathrm{spt} \amalg \overline{\mathrm{spt}}.$$

Here the disjoint unions are read from *top to bottom* and we note that R only exists if the length of the interval is nonzero, otherwise there is no room for the embedding ι_{out}. On the other hand, I exists for all length: the length zero morphism is the identity of spt.

The boundary-like ends of Y_k are again drawn as $\bullet$, with the understanding that on the right of it there is a germ of a neighborhood given by ι_k. At the incoming end we have drawn some dots near the $\bullet$ to reflect ι_{in} but at the outgoing end we do not emphasize the embedding ι_{out}. A surprising morphism might be μ above:

$$\mu : \overline{\mathrm{spt}} \amalg \mathrm{spt} \to \emptyset$$

It is a thin bordism because after removing the incoming object, only a single point, drawn as $\bullet$, remains. This point is the image of the *two points* $\partial\hat{Y}_{in}$, under the canonical extension of ι_{in} to the metric completion $\hat{Y}$. We did not require that the embedding ι_{in} extends to an *embedding* of $\hat{Y}$, partially to have μ in the category. It induces the adjunction transformation referred to in Remark 59 below.

rem:spin **Remark 58.** These pictures do not reflect the spin structures and certainly not the odd directions. They do, however, show how the involution acts on SRB_d: just reverse the arrow on the pictures.

rem:adjunction **Remark 59.** Originally we thought that intervals, i.e. morphisms from spt to spt, are all one needs to understand 1-manifolds: just pick a triangulation that decomposes any 1-manifold into intervals. However, there are also morphisms $\overline{\mathrm{spt}} \amalg \mathrm{spt} \to \emptyset$ and there is no way to recover them just from intervals. To deal with this problem, we introduced the 'adjunction transformations' on the bordism categories and on SF in our original approach [ST]. We required the functor E to preserve these extra structures, leading to a more awkward definition.

However, if one defines the category SRB_d as above, these additional properties are automatically satisfied. We now prefer this approach which is carefully explained in [ST2] because it revives the concept of triangulating a 1-manifold in a precise fashion: in Theorem 63 below think of μ as associated to a vertex and R as associated to an edge.

def:morphisms **Definition 60.** A morphism $\Sigma \in \mathrm{Mor}_{\mathrm{SRB}_1}(Y_{in}, Y_{out})$ is an S-family of super Riemannian cs manifolds of dimension $(1|1)$ (as in Definition 55) together with isometric embeddings $\iota_k : Y_k \hookrightarrow \Sigma$ over S whose reduced parts satisfy fibrewise the condition on the ends explained above.

The main source of examples of morphisms is as follows: Start with a function

$$f \in C^\infty(S) \cong \mathrm{SMAN}(S, \mathbb{R}^{1|1}).$$

Then we can use the super group structure m on $\mathbb{R}^{1|1}$ given in Definition 33 to define the *translation* map

$$T_f : S \times \mathbb{R}^{1|1} \xrightarrow{\mathrm{id}\times f\times id} S \times \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \xrightarrow{\mathrm{id}\times m} S \times \mathbb{R}^{1|1}$$

Note that T_f lies over the identity of S and that it is fibrewise an isometry: If we equip $\mathbb{R}^{1|1}$ with the geometric structure from equation 4 then its isometry group is $(\mathbb{R}^{1|1}, m)$ which acts on itself by right multiplication.

We would like to use T_f to define an S-family of morphisms from spt to spt. Recall that spt $\subset \mathbb{R}^{1|1}$ is an isometric submanifold whose reduced part is a germ $(0,\epsilon) \subset \mathbb{R}$ as $\epsilon \to 0$. Let us assume that

$$f_{red} : S_{red} \to \mathbb{R}^{1|1}_{red} = \mathbb{R} \quad \text{satisfies} \quad f_{red} \geq 0.$$

Then we can take $\Sigma := S \times \mathbb{R}^{1|1}_{>0}$ with ι_{out} the standard inclusion of $S \times$ spt into Σ and ι_{in} given by T_f, restricted to $S \times$spt. The properness assumptions on ι_k are easy to check if we take as germ representative of spt the whole positive line $(0, \infty)$ (or we make Σ smaller according to the size of ϵ).

Definition 61. For any cs manifold S, write $f \in C^\infty(S)$ as $f = t + \theta$ where t and θ are even respectively odd functions on S. Assume that $f_{red} = t_{red} \geq 0$. Then the above morphism is called the *super interval of length* (t, θ), written as

$$I_{t,\theta} \in \mathrm{Mor}_{\mathrm{SRB}_1}(S \times \mathrm{spt}, S \times \mathrm{spt}).$$

It lies over the identity map of S in SMAN.

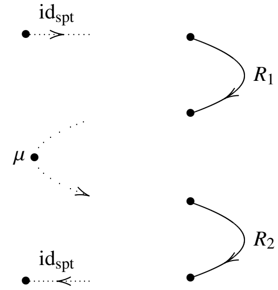
If $t_{red} > 0$ there is a close cousin also lying over the identity of S, the *right elbow of length* (t, θ), written as

$$R_{t,\theta} \in \mathrm{Mor}_{\mathrm{SRB}_1}(\emptyset, S \times (\mathrm{spt} \amalg \overline{\mathrm{spt}})).$$

It is defined like $I_{t,\theta}$ with two exceptions: Firstly, we need to take the submanifold of Σ which on the reduced part is given by those $(s, y) \in S_{red} \times \mathbb{R}$ that satisfy $0 < y < t_{red}(s)$. Secondly, there is no ι_{in} but ι_{out} is the disjoint union of the standard embedding of $S \times$ spt and the translated embedding T_f applied to $S \times \overline{\mathrm{spt}}$, see Definition 55. One also needs to make the germ representatives of spt and $\overline{\mathrm{spt}}$ small enough so that ι_{out} is an embedding.

Generators and Relations for SRB_1**.** For any base S, there is an interesting structure on the morphism set $\mathrm{Mor}_{\mathrm{SRB}_1}(\emptyset, S \times (\mathrm{spt} \amalg \overline{\mathrm{spt}}))$. It makes essential use of the morphism μ, defined over any S by pulling back from $*$

and it is given by the following picture defining $R_2 \circ R_1$:



We shall need the following result that is contained, in a slightly different version, in the first authors PhD thesis.

thm:Henning **Theorem 62.** *For $S := \mathbb{R}^{1|1}_{>0}$ there is a universal family of morphisms $R \to S$ in SRB_1 from $\emptyset$ to $\mathrm{spt} \amalg \overline{\mathrm{spt}}$ (over S). Moreover, the above composition gives an isomorphism of super semi groups*

$$\mathrm{Mor}_{\mathrm{SRB}_1}(\emptyset, \mathrm{spt} \amalg \overline{\mathrm{spt}}) \cong \mathbb{R}^{1|1}_{>0}$$

where the composition on the right hand side was given in Definition 33. Finally, this family has no automorphisms.

Using this universal family, we get a morphism $R_{t,\theta}$ from $\emptyset$ to $\mathrm{spt} \amalg \overline{\mathrm{spt}}$ for any parameter space S: Think of (t, θ) as a map of cs manifolds $S \to \mathbb{R}^{1|1}_{>0}$ and pull back the family R. Using picture 5, with $I = \mathrm{id}_{\mathrm{spt}}$, we also get a morphism $I_{t,\theta}$ from spt to spt for any parameter space S, together with a map $S \to \mathbb{R}^{1|1}_{\geq 0}$. Note that for $t > 0$ this is built only from μ and $R_{t,\theta}$ (and for $t = 0$ it is the identity).

thm:SRB **Theorem 63.** *As a symmetric monoidal involutive category over* SMAN, *the morphisms in* SRB_1 *are generated by μ and R (as in Theorem 62 above). Here μ can be defined over $S = *$ and R over $S := \mathbb{R}^{1|1}_{>0}$.*

The relations between the generators are as follows.

(1) *R satisfies the gluing law from Theorem 62:*

$$R_{t_2,\theta_2} \circ R_{t_1,\theta_1} = R_{t_1+t_2+\theta_1\theta_2, \theta_1+\theta_2}$$

(2) *The limit of $I_{t,\theta}$ is the identity as $t \to 0$.*

We now would like to get a topological space out of the above notion of a 1-dimensional unitary super symmetric Euclidean field theory E. By Corollary 56, E is completely determined on objects by the graded Fréchet space $E(\mathrm{spt})$.

Remark 64. We believe that the unitarity condition does not change the homotopy type of the space but the unitary case is the one that's closely related to the spaces in the previous section. We also believe that the positivity assumption can be given up without changing the homotopy type. However, one clearly has to decide beforehand about the signature of the Euclidean field theory, otherwise it would give an additional invariant of the set of connected components. It seems most natural to work with ambient hermitian spaces H of signature zero, instead of the positive definite case.

Proof. It is not hard to see that the unitarity of E implies that $E(I_{t,\theta})$ is a super semigroup of *self-adjoint* operators. □

sec:Quillen

7. Quillen categories and their classifying spaces

Fix a $\mathbb{Z}/2$-graded real C_n-module $\mathcal{H}_n$ as in the introduction. Then we can define the following category object $\mathfrak{C}_n$ in TOP, the category of (compactly generated) topological spaces. Libman [L] calls such category objects *internal space categories* and we follow his example. All this means is that the object set and the morphism set of $\mathfrak{C}_n$ are equipped with a (compactly generated) topology and the structure maps are continuous. This category is a topological version of Quillen's $S^{-1}S$-construction, see [G].

The objects of $\mathfrak{C}_n$ are finite dimensional graded C_n-submodules of $\mathcal{H}_n$. A morphism from W_1 to W_2 exists if and only if $W_1 \subseteq W_2$ and in this case

$$\mathrm{Mor}_{\mathfrak{C}_n}(W_1, W_2) := \{\, R \in O(W_2 - W_1) \mid R^* = R = R^{-1},\ R^\alpha = -R \,\}.$$

Here and in the following we use the notation $W_2 - W_1$ for the orthogonal complement of W_1 in W_2. The operators R are odd, orthogonal involutions on this complement.

As in Section 4 we think of subspaces of $\mathcal{H}_n$ as orthogonal projection operators, hence identifying the set of objects $\mathrm{Obj}_{\mathfrak{C}_n}$ of $\mathfrak{C}_n$ with a subspace of $B(\mathcal{H}_n)$. In order to topologize the set of morphisms $\mathrm{Mor}_{\mathfrak{C}_n}$ of $\mathfrak{C}_n$ we identify it with the set of triples (W_1, W_2, A), where $W_1, W_2 \in \mathrm{Obj}_{\mathfrak{C}_n}$ and $A \in B(\mathcal{H}_n)$ such that $W_2 - W_1$ is an invariant subspace for A, the kernel of A is $(W_2 - W_1)^\perp$ and A defines an odd, orthogonal involution on $W_2 - W_1$. We make $\mathrm{Mor}_{\mathfrak{C}_n}$ into a topological space, in fact, a metric space, by considering it as a subspace of the product of three copies of $B(\mathcal{H}_n)$.

The usual construction gives a simplicial space, the nerve $N\mathfrak{C}_n$ of the internal space category $\mathfrak{C}_n$, whose geometric realization

$$Q_n := B\mathfrak{C}_n := |N\mathfrak{C}_n|$$

is called the classifying space of $\mathfrak{C}_n$. This classifying space is directly related to configuration spaces:

thm:Quillen

Theorem 65. *There is a bijective continuous map $\tilde{G}_n : Q_n \xrightarrow{\approx} \mathrm{Conf}_n^{fin}$ that is also a homotopy equivalence.*

Proof. It suffices to consider the case $n = 0$. In this case, we suppress the index n altogether. In order to get the claim for general n, apply the $n = 0$ case to $\mathcal{H}_n$ (merely considered as a $\mathbb{Z}_2$-graded real Hilbert space) to obtain a bijective continuous map $\tilde{G} : B\mathfrak{C} \approx \mathrm{Conf}^{\mathrm{fin}}$. It is clear from the definition of $\tilde{G}$ (see below) that the classifiying space $B\mathfrak{C}_n \subset B\mathfrak{C}$ of the subcategory $\mathfrak{C}_n$ of $\mathfrak{C}$ corresponds precisely to $\mathrm{Conf}^{\mathrm{fin}}_n \subset \mathrm{Conf}^{\mathrm{fin}}$ under $\tilde{G}$. Since the homotopy used to prove that $\tilde{G}$ is a homotopy equivalence preserves the subspace $\mathrm{Conf}^{\mathrm{fin}}_n$, the claim follows for general n.

Recall that the k-simplices $x \in N_k\mathfrak{C}$ of the nerve of our internal space category $\mathfrak{C}$ are chains of $\mathbb{Z}/2$-graded finite dimensional subspaces

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k \subset \mathcal{H}$$

together with odd, orthogonal involutions R_i on $W_i - W_{i-1}$ for $i = 1, \dots, k$. We abbreviate this to $x = (W_i, R_i)$. The classifying space $B\mathfrak{C}$ is the quotient space

$$(\coprod_{k\geq 0} N_k\mathfrak{C} \times \Delta^k) \;/\; (\beta^*(x), t) \sim (x, \beta_*(t)) \quad \forall\, \beta : [m] \to [n]$$

In our context, it is convenient to replace the usual standard simplex with

$$\Delta^k := \{\, t = (t_1, \dots, t_k) \in \bar{\mathbb{R}}^k \;\mid\; 0 \leq t_1 \leq \cdots \leq t_k \leq \infty \,\}.$$

The face map $d_i : [k-1] \to [k]$ induces the map $(d_i)_* : \Delta^{k-1} \hookrightarrow \Delta^k$, given by repeating t_i, for $i = 1, \dots, k-1$. Moreover, $(d_0)_*$ adds a first coordinate equal to 0 and $(d_k)_*$ adds a last coordinate equal to ∞. For $i = 0, \dots, k-1$, the degeneracy maps $s_i : [k] \to [k-1]$ induce $(s_i)_* : \Delta^k \twoheadrightarrow \Delta^{k-1}$, given by skipping t_{i+1}.

Now, every $(x, t) \in N_kC \times \Delta^k$ defines a configuration $G(x, t) \in \mathrm{Conf}^{\mathrm{fin}}$ as follows. The label $G(x, t)_0$ at zero is W_i, where i is the largest index with $t_i = 0$. For $0 < \lambda < \infty$, $G(x, t)_{\pm\lambda}$ is the sum of the ± 1 eigenspaces of all operators R_i with indices i with $t_i = \lambda$. The label $G(x, t)_\infty$ is the orthogonal complement of all the other $G(x, t)_\lambda$'s. We claim that the map

$$\tilde{G} : B\mathfrak{C} \to \mathrm{Conf}^{\mathrm{fin}},\ \tilde{G}[x, t] := G(x, t)$$

is well-defined. We have to check that for all (x, t) and face and degeneracy maps β we have $G(\beta^*(x), t) = G(x, \beta_*(t))$.

We start with the face maps. In these cases we write $t = (t_1, \dots, t_{k-1})$. If $\beta = d_0 : [k-1] \to [k]$ then $\beta^*(x)$ is the chain of subspaces where W_0 and R_1 have been removed (and the indices of the other W_i and R_i are shifted to the left). Since $\beta_*(t) = (0, t_1, \dots, t_{k-1})$ it is clear from the definition of G that the labels of $G(\beta^*(x), t)$ and $G(x, \beta_*(t))$ coincide.

For $i = 1, \dots, k-1$ and $\beta = d_i$, the chain $\beta^*(x)$ is obtained by composing the morphisms R_i and R_{i+1}. This means that on $W_{i+1} - W_{i-1}$ we get an orthogonal sum of these two operators. Since the the ± 1 eigenspaces of this orthogonal sum equals the direct sum of the ± 1 eigenspaces of R_i and R_{i+1} and since $\beta_*(t)$ just repeats t_i, it is clear that $G(\beta^*(x), t)_{t_i} = G(x, \beta_*(t))_{t_i}$. All the other labels are clearly unchanged so that $G(\beta^*(x), t) = G(x, \beta_*(t))$.

If $\beta = d_k : [k-1] \to [k]$ then $\beta^*(x)$ is the chain of subspaces where W_k and R_k have been removed. However, since $\beta_*(t) = (t_1, \ldots, t_{k-1}, \infty)$, we have $G(\beta^*(x), t)_\lambda = G(x, \beta_*(t))_\lambda$ for all $\lambda \in \mathbb{R}$ and so $G(\beta^*(x), t) = G(x, \beta_*(t))$.

For a deneracy map $\beta = s_i : [k] \to [k-1]$, where $i = 0, \ldots, k-1$, the argument is even easier. Then for a $(k-1)$-simplex x, we get a chain $\beta^*(x)$ of length k by inserting the identity at the i-th subspace. This operation does not alter the operators R_j (the identity corresponds to $R = 0$ on a 0-space), it only shifts the indices $> i$ to the right. Similarly, $\beta_*(t_1, \ldots, t_k) = (t_1, \ldots, \hat{t}_i, \ldots, t_k)$, so that again a shifting of indices $> i$ to the right occurs and $G(\beta^*(x), t) = G(x, \beta_*(t))$ follows.

Hence the map $\tilde{G}$ is well-defined. It is bijective, since it has an inverse $\mathrm{Conf}^{\mathrm{fin}} \to B\mathfrak{C}$ defined by mapping a configuration $\{V_\lambda\}$ with exactly k non-trivial labels $V_{\lambda_1}, \ldots, V_{\lambda_k}$ with $0 < \lambda_i < \infty$ to the equivalence class $[x, t]$, where x is defined by the chain $W_0 \subseteq \cdots \subseteq W_k$, where $W_0 = V_0$ and

$$W_i := W_{i-1} \oplus V_{\lambda_i} \oplus V_{-\lambda_i}$$

for $i > 0$ and the operator R_i on $V_{\lambda_i} \oplus V_{-\lambda_i}$ is defined to be the one with the ± 1 eigenspaces $V_{\pm\lambda_i}$. It is clear that this defines an inverse for $\tilde{G}$.

It is easy to see that $\tilde{G}$ is continuous: the description of the neighborhood basis for Conf in Definition 26 and the definition of the topology on $\mathrm{Mor}_{\mathfrak{C}}$ imply that G is continuous. Hence the same is true for $\tilde{G}$.

It remains to show that $\tilde{G}$ is a homotopy equivalence. Let $\mathrm{Conf}^{(k)} \subset \mathrm{Conf}^{\mathrm{fin}}$ be the subspace defined in Remark 30 and denote by $B\mathfrak{C}^{(k)}$ the image of $N_k C \times \Delta^k$ in $B\mathfrak{C}$. According to Lemma 66 below, $\tilde{G}$ restricts to a homeomorphism

$$\tilde{G}^{(k)} : B\mathfrak{C}^{(k)} \longrightarrow \mathrm{Conf}^{(k)}$$

for all $k \geq 0$. This together with the fact that $\mathrm{id} : \mathrm{colim}_{k\to\infty} \mathrm{Conf}^{(k)} \to \mathrm{Conf}^{\mathrm{fin}}$ (see Remark 30) implies that $\tilde{G}$ is a homotopy equivalence. This completes the proof of Theorem 65. □

Lemma 66. *For all $k \geq 0$, the map*

$$\tilde{G}^{(k)} : B\mathfrak{C}^{(k)} \longrightarrow \mathrm{Conf}^{(k)}$$

is a homeomorphism.

Proof. Since $\tilde{G}$ is continuous, so is $\tilde{G}^{(k)}$. The proof that $\tilde{G}^{(k)}$ is open is based on the following fact.

Fact: *There is a function $\varepsilon_k : (0, \frac{1}{2}) \to \mathbb{R}_{>0}$ satisfying $\varepsilon_k(\delta) \to 0$ as $\delta \to 0$ with the following property. If $l \leq k$ and $V, W_1, W_2, \ldots, W_l$, are finite-dimensional subspaces of a Hilbert space H such that $\|P_V - P_{\oplus_{i=1}^l W_i}\| < \delta < \frac{1}{2}$*

and the W_i's are mutually orthogonal, then there exists a splitting of V into mutually orthogonal subspaces $V_1, ..., V_k$ such that $\|P_{V_i} - P_{W_i}\| < \varepsilon_k(\delta)$ for all i. The idea of the proof is simple. The subspaces $\tilde{W}_i := P_V(W_i) \subset V$ give a decomposition of V and the $\tilde{W}_i$ are almost orthogonal. Then use Gram-Schimdt to orthogonalize the $\tilde{W}_i$. Standard estimates plus induction on k yield the desired function ε_k (e.g. $\varepsilon_k(\delta) = 3^{k-1}\delta^{2^{-(k-1)}}$ works).

Now, given a neighborhood N of $[x,t] \in B\mathfrak{C}^{(k)}$ we have to find $K > 0$ and $\delta > 0$ such that $V_{K,\delta,\delta} \cap \mathrm{Conf}^{(k)} \subset \tilde{G}^{(k)}(N)$, where $V := \tilde{G}^{(k)}[x,t]$ and $V_{K,\delta,\delta}$ is the neighborhood of V described in Definition 26. We may choose K to be any number such that $(-K-1, K+1)$ contains all non-trivial labels $\lambda \in \mathbb{R}$ of V. In order to find δ, observe that the preimage of N in $\coprod_{i=0}^k N_i\mathfrak{C} \times \Delta^i$ contains a full ε-neighborhood of the equivalence class $[x,t]$ (recall that $N_i\mathfrak{C}\times\Delta^i$ is a metric space for all i) for some $\varepsilon > 0$. The existence of ε follows from the Lebesgue covering lemma and the fact that $[x,t]$ is compact.

Now choose $\delta > 0$ such that $\varepsilon_k(\delta) < \varepsilon$. In order to show that $V_{K,\delta,\delta} \cap \mathrm{Conf}^{(k)} \subset \tilde{G}^{(k)}(N)$, consider any $W \in V_{K,\delta,\delta} \cap \mathrm{Conf}^{(k)}$. Let j be minimal with $W \in \mathrm{Conf}^{(j)}$. Using the Fact it is easy to see that there is $(y,s) \in [x,t] \cap N_j\mathfrak{C} \times \Delta^j$ that lies within ε-range of the unique preimage of W in $N_j\mathfrak{C} \times \Delta^j$ under G. In other words, (y,s) lies within ε distance from $[x,t]$. Hence the preimage of W under $\tilde{G}^{(k)}$ lies in N, as desired.

This completes the proof that $\tilde{G}^{(k)}$ is open. □

sec:AS

8. Spaces of Fredholm operators

In this chapter we relate the spaces Conf_n to the spaces of skew-adjoint Fredholm operators considered by Atiyah and Singer in [AS].

Fredholm operators. Recall that a *Fredholm operator* is a bounded operator whose kernel and cokernel are finite dimensional. Let $\mathrm{Fred}(H) \subset B(H)$ be the subspace of Fredholm operators on the infinite dimensional separable real Hilbert space H. Denote by $C(H) := B(H)/K(H)$ the C^*-algebra of bounded operators modulo compact operators (a.k.a. *Calkin algebra*) and by $\pi : B(H) \to C(H)$ the projection. Then $\mathrm{Fred}(H)$ is the preimage of the group of units in $C(H)$ under π, i.e. we have

$$T \in B(H) \text{ is Fredholm } \iff \pi(T) \in C(H) \text{ is invertible.}$$

We will need the following facts about the spectrum $\sigma(T)$ of a self-adjoint bounded operator T. Let $\sigma_{\mathrm{ess}}(T) := \sigma(\pi(T))$ be the *essential spectrum* of T, i.e. the spectrum of $\pi(T)$ in $C(H)$. Then there is a decomposition

$$\sigma(T) = \sigma_{\mathrm{ess}}(T) \amalg \sigma_{\mathrm{discrete}}(T),$$

where $\sigma_{\mathrm{discrete}}(T)$ consists precisely of the isolated points in $\sigma(T)$ such that the corresponding eigenspace has finite dimension. From the definition of the essential spectrum it is clear that for a Fredholm operator T

$$(*) \qquad \sigma_{\mathrm{ess}}(T) \cap (-\varepsilon(T), \varepsilon(T)) = \emptyset \text{ for } \varepsilon(T) := \|\pi(T)^{-1}\|^{-1}_{C(H)},$$

where $\|.\|_{C(H)}$ is the C^*-norm on the Calkin algebra. In other words: the essential spectrum of T has a gap of size $\varepsilon(T)$ around 0. Note that the map $\varepsilon : \mathrm{Fred}(H) \to \mathbb{R}_{>0}$ is continuous.

K-theory and Fredholm operators. The most important invariant of a Fredholm operator T is its *index*

$$\mathrm{index}(T) := \dim(\ker T) - \dim(\operatorname{coker} T).$$

It turns out that the index is invariant under deformations, i.e. it is a locally constant function on $\mathrm{Fred}(H)$. In fact, it defines an isomorphism

$$\pi_0 \,\mathrm{Fred}(H) \xrightarrow{\cong} \mathbb{Z}, \ [T] \mapsto \mathrm{index}(T).$$

This is a special case of the well-known result that $\mathrm{Fred}(H)$ is a classifying space for the real K-theory functor KO^0. More explicitly, for compact spaces X there are natural isomorphisms

$$KO^0(X) \cong [X, \mathrm{Fred}(H)].$$

This isomorphism is defined as follows. Consider $[f] \in [X, \mathrm{Fred}(H)]$. Changing f by a homotopy one can achieve that the dimensions of the kernel and the cokernel of $f(x)$ are locally constant. This implies that they define vector bundles $\ker f$ and $\operatorname{coker} f$ over X. The image of $[f]$ is defined to be

$$[\ker f] - [\operatorname{coker} f] \in KO^0(X).$$

For $X = pt$ this reduces to the above isomorphism

$$\pi_0 \,\mathrm{Fred}(H) \cong [pt, \mathrm{Fred}(H)] \xrightarrow{\cong} KO^0(pt) \cong \mathbb{Z}.$$

Atiyah and Singer showed that the other spaces in the Ω-spectrum representing real K-theory can also be realized as spaces of Fredholm operators.

The Atiyah-Singer spaces $\mathcal{F}_n$. Let $n \geq 1$ and let H_n be a real Hilbert space with an action of C_{n-1}, just as before. Define

$$\widetilde{\mathcal{F}_n} := \{\, T_0 \in \mathrm{Fred}(H_n) \mid T_0^* = -T_0 \text{ and } T_0 e_i = -e_i T_0 \text{ for } i = 1 \ldots, n-1 \,\}.$$

Furthermore, let $\mathcal{F}_n := \widetilde{\mathcal{F}_n}$ if $n \not\equiv 3\ (4)$. In the case $n \equiv 3\ (4)$ define $\mathcal{F}_n \subset \widetilde{\mathcal{F}_n}$ to be the subspace of operators T_0 satisfying the following additional condition (AS): the essential spectrum of the self-adjoint operator $e_1 \cdots e_{n-1} T_0$ contains positive *and* negative values (we say that $e_1 \cdots e_{n-1} T_0$ is neither essentially positive nor negative). Atiyah and Singer introduce this condition, because it turns out that for $n \equiv 3\ (4)$ the space $\widetilde{\mathcal{F}_n}$ has three connected components, two of which are contractible. However, for the relation with

K-theory only the third component, whose elements are characterized by the above requirement on the essential spectrum of $e_1 \dots e_{n-1} T_0$, is interesting. In fact, the main result of [AS] is that for all $n \geq 1$ the space $\mathcal{F}_n$ represents the functor KO^{-n}. We shall reprove this result in terms of our configuration spaces.

The elements in $\widetilde{\mathcal{F}}_n$ can also be interpreted as odd operators on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}_n = H_n \otimes_{C_n^{ev}} C_n$. If we define

$$\widetilde{\mathcal{F}}_n^{gr} := \{\, T \in \mathrm{Fred}(\mathcal{H}_n) \mid T \text{ is odd, } C_n\text{-linear, and self-adjoint} \,\}$$

we can identify $\widetilde{\mathcal{F}}_n$ and $\widetilde{\mathcal{F}}_n^{gr}$ using the homeomorphism

$$\psi_{\otimes e_n} : \widetilde{\mathcal{F}}_n \xrightarrow{\approx} \widetilde{\mathcal{F}}_n^{gr}, \; T_0 \mapsto T := T_0 \otimes e_n.$$

The operator T has the matrix representation

$$T = \begin{pmatrix} 0 & T_0^* \\ T_0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H}_n \cong H_n \oplus H_n$. It is important to note that the skew-symmetry of T_0 is equivalent to the relation $Te_n = e_n T$.

In the next lemma, we will show that $\widetilde{\mathcal{F}}_n^{gr}$ is homotopy equivalent to a configuration space. Let $\widetilde{\mathbb{R}} := [-\infty, \infty]$ be the two-point compactification of $\mathbb{R}$ equipped with the involution $s(x) := -x$.

defretract

Lemma 67. *The subspace $\mathcal{A} \subset \widetilde{\mathcal{F}}_n^{gr}$ of all operators T with $\|T\| = 1$ and $\varepsilon(T) = 1$, see (*) for the definition of $\varepsilon(T)$, is a strong deformation retract of $\widetilde{\mathcal{F}}_n^{gr}$. Furthermore, $\mathcal{A}$ is homeomorphic to the configuration space $\mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ as defined in Chapter 4.*

Proof. Define a homotopy $H : \widetilde{\mathcal{F}}_n^{gr} \times [0,1] \to \widetilde{\mathcal{F}}_n^{gr}$ by

$$(T, t) \mapsto H_t(T) := (t + (1-t)\|T\|) \cdot \phi\Big((t\varepsilon(T)^{-1} + (1-t)\|T\|^{-1}) \cdot T\Big).$$

Here $\phi : \mathbb{R} \to [-1, 1]$ is defined by $\phi|_{[-1,1]} = \mathrm{id}$, $\phi|_{[1,\infty)} \equiv 1$ and $\phi|_{(\infty,-1]} \equiv -1$, and $\phi(\_)$ denotes the functional calculus with ϕ. The continuity of H follows from the continuity of $\|\,\|$ and ε and from the usual continuity properties of functional calculus, see [RS], Theorem VIII.20. Also, C_n-linearity and parity of T are preserved under functional calculus. Furthermore,

$$H_0 = \mathrm{id}_{\widetilde{\mathcal{F}}_n^{gr}}, \; H_t = \mathrm{id}_{\mathcal{A}} \text{ for all } t, \text{ and } H_1(\widetilde{\mathcal{F}}_n^{gr}) \subset \mathcal{A}.$$

Hence $\mathcal{A}$ is a strong deformation retract of $\widetilde{\mathcal{F}}_n^{gr}$.

Now, for all $T \in \mathcal{A}$ we have $\sigma(T) \subset [-1, 1]$ and all $\lambda \in \sigma(T) \cap (-1, 1)$ are eigenvalues of finite multiplicity. The spectral theorem for self-adjoint operators implies that the eigenspaces $V(T)_\lambda$ of T are pairwise orthogonal and span all of $\mathcal{H}_n$. Since T is odd, $V(T)_{-\lambda} = \alpha(V(T)_\lambda)$, where α is the grading involution on $\mathcal{H}_n$. We thus obtain a map

$$\mathcal{A} \to \mathrm{Conf}_{C_n}([-1,1],\{\pm 1\};\mathcal{H}_n),\ T \mapsto V(T)$$

by associating to T the configuration $\lambda \mapsto V(T)_\lambda$ on $[-1,1]$. Here the involution on $[-1,1]$ is $x \mapsto -x$. It is easy to see that this map is a homeomorphism. Finally, $[-1,1] \xrightarrow{\approx} \widetilde{\mathbb{R}}$, $x \mapsto \frac{x}{1-|x|}$ induces a homeomorphism of configuration spaces $\mathrm{Conf}_{C_n}([-1,1],\{\pm 1\};\mathcal{H}_n) \approx \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}},\{\pm\infty\};\mathcal{H}_n)$, thus completing the proof of the second statement in the lemma. □

Now we can formulate the relationship between the Atiyah-Singer spaces $\mathcal{F}_n$ and our configuration spaces Conf_n. Consider the map $p : \widetilde{\mathbb{R}} \to \bar{\mathbb{R}}$ that is the identity on $\mathbb{R}$ and that maps $\pm\infty$ to ∞. It induces a continuous map

$$p_* : \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}},\{\pm\infty\};\mathcal{H}_n) \longrightarrow \mathrm{Conf}_{C_n}(\bar{\mathbb{R}},\{\infty\};\mathcal{H}_n) = \mathrm{Conf}_n\,.$$

Let H be the homotopy equivalence defined as the composition

$$H : \widetilde{\mathcal{F}}_n \xrightarrow{\approx} \widetilde{\mathcal{F}}_n^{gr} \xrightarrow{\simeq} \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}},\{\pm\infty\};\mathcal{H}_n).$$

The main result of this section is:

EFT and F_n **Theorem 68.** *For all $n \geq 1$, p_*H restricts to a homotopy equivalence*

$$p_*H|_{\mathcal{F}_n} : \mathcal{F}_n \xrightarrow{\simeq} \mathrm{Conf}_n\,.$$

Proof. Since H is a homotopy equivalence, the same is true for $H|_{\mathcal{F}_n}$. It remains to show that the restriction of p_* to the path component $H(\mathcal{F}_n)$ of $\mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}},\{\pm\infty\};\mathcal{H}_n)$ is a homotopy equivalence. In order to do this, it will be convenient to work with subspaces consisting of certain 'finite' elements. More precisely, if we define

$$\mathrm{Conf}'_n := H(\mathcal{F}_n) \cap \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}},\{\pm\infty\};\mathcal{H}_n).$$

then we have a commutative diagram

$$\begin{array}{ccc} H(\mathcal{F}_n) & \xrightarrow{p_*} & \mathrm{Conf}_n \\ \simeq\uparrow & & \uparrow\simeq \\ \mathrm{Conf}'_n & \xrightarrow{p_*} & \mathrm{Conf}^{\mathrm{fin}}_n \end{array}$$

whose vertical arrows are homotopy equivalences (for the right arrow this was done in Proposition 29; the same argument works for the arrow on the left). Hence $p_*|_{H(\mathcal{F}_n)}$ is a weak homotopy equivalence exactly if this is the case for $p_* : \mathrm{Conf}'_n \to \mathrm{Conf}^{\mathrm{fin}}_n$. This will be proved in Theorem 71 below. It follows that $p_*H|_{\mathcal{F}_n}$ is a weak homotopy equivalence. Since $\mathcal{F}_n$ and Conf_n both have the homotopy type of a CW-complex, the map $p_*H|_{\mathcal{F}_n}$ is a homotopy equivalence, cf. [Mi2]. □

As a first step towards Theorem 71, let us give a characterization of the configurations contained the subspace Conf'_n. Since the map H is surjective, we have $\mathrm{Conf}'_n = \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ for $n \not\equiv 3\ (4)$ (recall that $\widetilde{\mathcal{F}_n} = \mathcal{F}_n$ in this case).

The interesting case is $n \equiv 3\ (4)$. Our task is to understand what the Atiyah-Singer condition (AS) means for the corresponding configurations.

Before we proceed, we need to collect some representation theoretic facts about graded Clifford modules. Recall from [LM] that for $n \not\equiv 3(4)$ the Clifford algebra C_n is simple, whereas it is the product of two simple algebras in the other cases. Therefore, there is a unique irreducible C_n-module (and a unique graded irreducible C_{n+1}-module) for $n \not\equiv 3\ (4)$ otherwise there are exactly two such modules.

rem:AS-graded **Remark 69.** Let $n \equiv 3(4)$ and recall from [LM, Chapter I, Prop. 5.9] that in this case the action of the *volume element* $e := e_1 \cdots e_n \in C_n$ distinguishes the two distinct (ungraded) irreducible C_n-modules. (Since e is a central orthogonal involution, it acts as $\pm\,\mathrm{id}$ on these modules.)

Now, let $T \in \widetilde{\mathcal{F}}^{gr}_n$. The diagonal entries of the even operator eT are given by $e_1 \cdots e_{n-1} T_0 : H_n \to H_n$ and therefore the Atiyah-Singer condition (AS) that the operator $e_1 \cdots e_{n-1} T_0 : H_n \to H_n$ is neither essentially positive nor negative is equivalent to the same condition on $eT : \mathcal{H}_n \to \mathcal{H}_n$.

Denote by $\mathcal{A}^{\mathrm{fin}} \subset \mathcal{A}$ the subspace of operators with finite spectrum.

lem:AS-graded **Lemma 70.** *Assume $n \equiv 3\ (4)$. Let $T \in \mathcal{A}^{fin}$ and denote by $W_\pm$ the (± 1)-eigenspaces of T.*

(1) *eT is essentially positive (resp. negative) if and only if the volume element e has a finite dimensional (-1)-eigenspace on W_+ (resp. W_-).*
(2) *The (AS) condition is equivalent to the (± 1)-eigenspaces of e, restricted to W_+, both being infinite dimensional.*
(3) *W_+ is a C_n-module and the (AS) condition is equivalent to W_+ containing both irreducible C_n-modules infinitely often.*

Proof. For part (1) we observe that $eT = Te$ and hence we can find simultanous eigenspace decompositions for these self-adjoint operators. Note that the eigenvalues (± 1) are the only possible accumulation points in the spectrum of T and hence such eigenspace decompositions exist. Since $e^2 = 1$, the operator e has spectrum inside $\{\pm 1\}$.

A vector $v \in \mathcal{H}_n$ is in an essentially positive eigenspace of eT if and only if either $v \in W_+$ and $e(v) = +1$, or $v \in W_-$ and $e(v) = -1$. Since T is odd, its spectrum is symmetric and, in particular, the grading involution α takes W_+ to W_-. Furthermore, α anti-commutes with e and hence

$$e_{|W_+} = e \circ \alpha_{|W_-} = -\alpha \circ e_{|W_-}$$

so that α takes the $(+1)$-eigenspace of $e_{|W_+}$ to the (-1)-eigenspace of $e_{|W_-}$. In particular, these vector spaces have the same dimension. This finishes the proof of part (1) as well as part (2).
To prove part (3) notice that T is C_n-linear and therefore W_+ is a C_n-module (which is not graded since α takes it to W_-). The claim follows from the well known algebraic fact stated at the beginning of the lemma. □

Note that Conf'_n is exactly the image of $\mathcal{A}^{\mathrm{fin}} \cap \mathcal{F}_n^{gr}$ under the identification $\mathcal{A} \approx \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ from Lemma defretract 67. Moreover, the (± 1)-eigenspaces $W_\pm$ of an operator $T \in \mathcal{A}^{\mathrm{fin}}$ turn into the $(\pm\infty)$-eigenspaces $W_{\pm\infty}$ of the corresponding configuration $W \in \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$.

Hence part (3) of Lemma lem:AS-graded 70 tells us that for $n \equiv 3\ (4)$ the subspace $\mathrm{Conf}'_n \subset \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ consists precisely of the configurations $\{W_\lambda\}$ such that $W_{\pm\infty}$ contains both irreducible C_n-modules infinitely often.

We will now see why this is a very natural condition in terms of our configuration spaces:

thm:quasifib **Theorem 71.** *The restriction*

$$p := p_*|_{\mathrm{Conf}'_n} : \mathrm{Conf}'_n \longrightarrow \mathrm{Conf}^{fin}_n$$

is a quasi-fibration with contractible fibers (see Definition def:quasifib *74).*

Remark 72. In the case $n \equiv 3\ (4)$ the space Conf'_n is the unique connected component of $\mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ that is not contractible. On the two remaining components the map p_* is *not* a quasi-fibration as well shall see below (the fibres have distinct homotopy groups).

lem:M **Lemma 73.** *Let M be a graded C_n-module.*

(1) *If M contains all (one or two) irreducible C_n-modules infinitely often then the C_n-action on M extends to a graded C_{n+1}-action.*
(2) *Let M_0 be a graded irreducible C_n-module. Then there is a graded vector space R, the multiplicity space, such that M is isomorphic to the graded tensor product $M_0 \otimes R$. In the case $n \not\equiv 3\ (4)$ we may, and shall, assume that R is concentrated in even degree.*
(3) *With this notation, the grading preserving Clifford linear orthogonal group $O_{C_n}(M)$ is isomorphic to*

$$O_{C_n}(M) \cong O(R) \cong O(R^{ev}) \times O(R^{odd})$$

In particular, this group is contractible (by Kuiper's theorem) if and only if the multiplicity spaces R^{ev} and R^{odd} are both either zero or infinite dimensional. This is equivalent to M containing either only one type of graded irreducible C_n-module, or containing both infinitely often.

Proof. There are two cases to consider for proving (1): If only one graded irreducible C_n-module exists, then take any irreducible C_{n+1}-module M_0 and restrict it to C_n. It is clear that over C_n, M must be given by infinitely many copies of M_0.

Let's say there are two graded irreducible C_n-modules M_0, M_1 and hence n is divisible by 4. By assumption, M is the sum of infinitely many copies of $M_0 \perp M_1$. It then suffices to show that $M_0 \perp M_1$ has an C_{n+1}-action. We first claim that $M_1 \cong M_0^{op}$, i.e. M_1 is obtained from M_0 by flipping the grading. Using the characterization of ungraded C_{n-1}-modules given in Lemma lem:AS-graded 70, it suffices to show that the volume element $\tilde{e}_1 \cdots \tilde{e}_{n-1}$ acts with different sign on M_0^{ev} and M_0^{odd}. Here $\tilde{e}_i = e_i e_n$ are the usual generators of $C_{n-1} \cong C_n^{ev}$. Since n is divisible by 4, it follows that

$$\tilde{e}_1 \cdots \tilde{e}_{n-1} = (e_1 e_n) \cdots (e_{n-1} e_n) = e_1 \cdots e_n =: e$$

is also the volume element in C_n. Writing $M_0^{odd} = e_i \cdot M_0^{ev}$ for some i, our claim follow from $ee_i = -e_i e$. Finally, the module $M_0 \perp M_0^{op}$ has a C_{n+1}-action given by the element $e_{n+1} = f\alpha$, where f flips the two summands and α is the grading involution.

For part (2) one again needs to know that in the case that there are two graded irreducible C_n-modules M_0, M_1, they differ from each other by flipping the grading. This was proven above. Part (3) is obvious. □

Proof of Theorem thm:quasifib *71.* We begin by proving that the fibers of p are contractible. Fix $V \in \mathrm{Conf}_n^{\mathrm{fin}}$. The fiber $p^{-1}(V)$ consists of all $W \in \mathrm{Conf}'_n$ such that $V_\lambda = W_\lambda$ for $\lambda \in \mathbb{R}$ and V_∞ is the orthogonal sum of W_∞ and $W_{-\infty}$. Since $W_{-\infty} = \alpha(W_\infty)$, where α is the grading involution on $\mathcal{H}_n$, we may identify $p^{-1}(V)$ with the space of decompositions of the graded C_n-module V_∞ of the form $V_\infty = W_\infty \perp \alpha(W_\infty)$, where W_∞ is an ungraded C_n-submodule of V_∞ that for $n \equiv 3\ (4)$ satisfies the (AS) condition: both irreducible C_n-modules appear infinitely often in W_∞.

Without the (AS) condition, it is straightforward to show that this space of decompositions of V_∞ is homeomorphic to the following space of C_{n+1}-structures on V_∞:

$$\widetilde{\mathfrak{C}}_{n+1}(V_\infty) := \{\, e_{n+1} \in O(V_\infty) \mid e_{n+1}^2 = -\mathbb{1}, e_{n+1} e_i = -e_i e_{n+1},\ i = 1, \dots, n \,\}$$

Namely, given e_{n+1}, one can define $W_{\pm\infty}$ to be the (± 1)-eigenspaces of $e_{n+1}\alpha$ (and vice versa). Under this correspondence, the (AS) condition translates into the requirement that e_{n+1} defines a C_{n+1} module structure on V_∞ which contains both graded irreducibles infinitely often. We denote this subspace of $\widetilde{\mathfrak{C}}_{n+1}(V_\infty)$ simply by $\mathfrak{C}_{n+1}(V_\infty)$ and observe that all these module structures e_{n+1} on V_∞ are isomorphic.

We show in the following 4 steps that the fibre $p^{-1}(V) \approx \mathfrak{C}_{n+1}(V_\infty)$ is contractible under our assumptions.

Step 1: By our basic assumption, the ambient Hilbert space $\mathcal{H}_n$ contains all graded irreducible C_n-modules infinitely often. Since V was a finite configuration to start with, it follows that V_∞ has the same property and by part (1) of Lemma 73 it follows that $\mathfrak{C}_{n+1}(V_\infty)$ is not empty.
Step 2: Since any two points in $\mathfrak{C}_{n+1}(V_\infty)$ lead to C_{n+1}-module structures on V_∞ that are isomorphic, the orthogonal group $O_{C_n}(V_\infty)$ acts transitively (by conjugation) on $\mathfrak{C}_{n+1}(V_\infty)$. The stabilizer of a particular C_{n+1}-structure is $O_{C_{n+1}}(V_\infty)$ and hence

$$\mathfrak{C}_{n+1}(V_\infty) \approx O_{C_n}(V_\infty)/O_{C_{n+1}}(V_\infty)$$

We shall show that this space is contractible, as a quotient of two contractible groups.
Step 3: As a C_n-module, V_∞ contains both graded irreducible C_n-modules infinitely often, that's what we need by part (3) of Lemma 73 for the contractibility of the larger group $O_{C_n}(V_\infty)$.
Step 4: For the smaller group $O_{C_{n+1}}(V_\infty)$, the (AS) condition tells us again that the assumptions of part (3) of Lemma 73 are satisfied.

To finish the proof of Theorem 71, it remains to show that p is indeed a quasi-fibration. We will use the criterion in Theorem 76 but first we give the relevant definitions.

def:quasifib **Definition 74.** A map $p : E \twoheadrightarrow B$ is a *quasi-fibration* if for all $b \in B$, $i \in \mathbb{N}$ and $e \in p^{-1}(b)$, p induces an isomorphism

$$\pi_i(E, p^{-1}(b), e) \xrightarrow{\cong} \pi_i(B, b).$$

From the long exact sequence of homotopy groups for a pair it follows that p is a quasi-fibration exactly if there is a long exact homotopy sequence connecting fibre, total space and base space of p, just like for a fibration. However, p does not need to have any (path) lifting properties as the following example shows.

Example 75. The prototypical example of a quasi-fibration that is not a fibration is the projection of a 'step'

$$(-\infty, 0] \times \{0\} \cup \{0\} \times [0, 1] \cup [0, \infty) \times \{1\} \subset \mathbb{R}^2$$

onto the x-axis. Even though all fibers have the same homotopy type (they are contractible), the map doesn't have the lifting property of a fibration, since it is impossible to lift a path that passes through the origin.

The following sufficient condition for a map to be a quasi-fibration is an easy consequence of the results of [DT]:

DT criterion **Theorem 76.** *Let $p : Y \to X$ be a continuous map between Hausdorff spaces and $X_0 \subset X_1 \subset X_2 \subset ...$ an increasing sequence of closed subsets of*

X s.t. $X = \operatorname{colim}_{i\to\infty} X_i$. Assume further that for all $i \geq 0$ the map $p|_{Y_{i+1}\setminus Y_i}$ is a Serre fibration, where $Y_i := p^{-1}(X_i)$, and that there exists an open neighborhood N_i of X_i in X_{i+1} and homotopies $d_t = d_t^{(i)} : N_i \to N_i$ and $D_t^{(i)} = D_t : p^{-1}(N_i) \to p^{-1}(N_i)$ s.t.

1. *D covers d, i.e. $p \circ D_t = d_t \circ p$ for all t.*
2. *$D_0 = \mathrm{id}$, $D_t(Y_i) \subset Y_i$ for all t, and $D_1(p^{-1}(N_i)) \subset Y_i$*
3. *For every $x \in N_i$, the map $D_1 : p^{-1}(x) \to p^{-1}(d_1(x))$ is a weak homotopy equivalence.*

Then p is a quasi-fibration.

Proof. According to Satz 2.15 in [DT] p is a quasi-fibration provided that $p|_{Y_i}$ is a quasi-fibration for all $i \geq 0$. To see this, we proceed by induction on i. It is clear that $p|_{Y_0}$ is a quasi-fibration, since, by assumption, it is a Serre fibration. Now assume that we already know that $p|_{Y_i}$ is a quasi-fibration. Applying Hilfssatz 2.10 in [DT] with $B = N_i$, $B' = X_i$, $q = p|_{p^{-1}(N_i)}$, $D = D$, and $d = d$, implies that $p|_{p^{-1}(N_i)}$ is a quasi-fibration. Now, applying the Korollar of Satz 2.2 in [DT] with $U = N_{i+1}$ and $V = Y_{i+1} \setminus Y_i$ we see that $p|_{Y_{i+1}}$ is a quasi-fibration. (Note that p is a quasi-fibration over $U \cap V$, since it is even a Serre fibration.) □

Now, in order to apply Theorem 76 we filter $\mathrm{Conf}_n^{\mathrm{fin}}$ by the closed subspaces

$$X_i := \{\, V \in \mathrm{Conf}_n^{\mathrm{fin}} \mid \dim(V) := \dim_{C_n}(\oplus_{\lambda\in\mathbb{R}} V_\lambda) \leq 2i \,\}.$$

For each i consider the continuous function

$$L_i : X_{i+1} \to [0,\infty], V \mapsto L_i(V) := \sup\{\, r \mid \dim_{C_n}(\oplus_{\lambda\in(-r,r)} V_\lambda) \leq 2i \,\}.$$

Clearly, $L_i^{-1}(\infty) = X_i$. Define $N_i := L_i^{-1}((1,\infty])$. The homotopies $d^{(i)}$ and $D^{(i)}$ are now easy to find. Namely, consider the homotopies on $\mathrm{Conf}_n^{\mathrm{fin}}$ and Conf'_n induced by the family of maps h_t defined in the proof of Proposition 29 (in the case of Conf'_n, note that the formula defining h_t indeed also determines a homotopy on $\widetilde{\mathbb{R}}$ and hence on Conf'_n). It is easy to check that almost all assumptions of the Dold-Thom theorem are hold: (1) and (2) obviously satisfied. Condition (3) is trivial, since all fibers are contractible. The map $p|_{X_{i+1}\setminus X_i}$ is a fiber bundle, in particular, it is a Serre fibration. The crucial point in the proof of this is that dimension $\dim(V)$ of configurations in $X_{i+1} \setminus X_i$ is constant. This makes it possible to choose an open neighborhood N of V in $X_{i+1} \setminus X_i$ such that the orthogonal projection $P_{V_\infty} : W_\infty \to V_\infty$ is an isomorphism for all $W \in N$. It is easy to write down a local trivialization of p over such an N; this shows that $p|_{X_{i+1}\setminus X_i}$ is a fiber bundle.

The only condition in the Dold-Thom theorem that is violated is that $\mathrm{Conf}_n^{\mathrm{fin}}$ is not the colimit over the subspaces X_i (cf. Remark 30). However, this is not a serious issue: we can endow $\mathrm{Conf}_n^{\mathrm{fin}}$ and Conf'_n with the colimit topologies w.r.t. the filtrations X_i and $p^{-1}(X_i)$ and apply the Dold-Thom theorem to see that p is a quasi-fibration in this case. It follows directly from the definition of a quasi-fibration that the same also holds if we consider the original topologies, since the identity maps $\operatorname{colim}_{i\to\infty} X_i \to \mathrm{Conf}_n^{\mathrm{fin}}$ and $\operatorname{colim}_{i\to\infty} Y_i \to \mathrm{Conf}'_n$ are homotopy equivalences (see Remark colim Confk 30 for the $\mathrm{Conf}_n^{\mathrm{fin}}$ case, the same argument works in the case Conf'_n). This completes the proof of Theorem thm:quasifib 71. □

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