THE LANDWEBER EXACT FUNCTOR THEOREM, STACKS, AND THE PRESHEAF OF ELLIPTIC HOMOLOGY THEORIES

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1. Introduction

The goal of this talk is to construct the presheaf of elliptic homology theories on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. This sets the stage for many of the later talks of the workshop where the objective will be to turn this presheaf into to a sheaf of E_∞-ring spectra (using obstruction theory).

Even though we use the language of stacks, much of this talk is closely related to the classical story of elliptic cohomology. Constructing a presheaf of homology theories on $\mathcal{M}_{ell}$ means to associate to every elliptic curve $\mathcal{C}$ (satisfying a certain flatness condition) a homology theory $Ell^{\mathcal{C}}$. The construction of $Ell^{\mathcal{C}}$ is based on Landweber's exact functor theorem, and we verify the assumptions of Landweber's theorem using the argument given by Landweber, Stong, Ravenel, and Franke, see [LRS], [Fr]. However, in the approach we describe (due to Hopkins and Miller) the use of Landweber's theorem is elegantly hidden in the statement that the morphism

$$\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

given by associating to an elliptic curve its formal group is flat. This result is the main ingredient in the construction of the presheaf of elliptic homology theories on $\mathcal{M}_{ell}$.

Let us briefly outline the content of the talk. In Section 2 we first review some generalities concerning even periodic cohomology theories and their formal groups. After that we turn to Quillen's theorem, the fundamental link between complex cobordism and formal group laws. We state a version of the theorem that indicates why the moduli stack of formal groups $\mathcal{M}_{FG}$ comes up in connection with complex cobordism. We conclude Section 2 by describing Landweber's exactness condition and Landweber's exact functor theorem which gives a nice criterion for when it is satisfied.

We begin Section 3 by introducing some basics about stacks that are needed for the construction of the presheaf of elliptic homology theories. We then express Landweber's exactness condition in the language of stacks: a formal group law over a ring R is Landweber exact if and only if the corresponding map $\operatorname{Spec} R \to \mathcal{M}_{FG}$ is flat. Using this and the flatness of the map $\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ we see that every flat morphism $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$ gives rise to a homology theory $Ell^{\mathcal{C}}$. We also explain how these fit together to give a presheaf of homology theories on $\mathcal{M}_{ell}$.

Finally, in Section 4 we prove, following Hopkins and Miller, that the map $\mathfrak{F}$ is flat. The proof is based on some facts about elliptic curves and their formal group laws and the Landweber exact functor theorem.

Before getting started, let us fix some terminology. Whenever we talk about stacks, we mean stacks defined on the Grothendieck site *Aff* of affine schemes with the flat topology. In this Grothendieck topology, a covering is a faithfully flat map $\operatorname{Spec} S \to \operatorname{Spec} R$. Since we will not be able to supply all the details about stacks that are needed, we will occasionally refer the reader to the papers [Goe] and [Nau] by Goerss and Naumann. Throughout these notes, we will refer to 1-morphisms between stacks simply as morphisms.

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2. Periodic cohomology theories, formal groups, and the Landweber exact functor theorem

We begin by outlining the relationship between even periodic cohomology theories and formal groups.

Definition 2.1. A multiplicative cohomology theory h^* is *even periodic* if $h^k(pt) = 0$ for odd integers k and if there exists a unit $u \in h^*(pt)$ of degree $|u| = 2$.

Remark 2.2. Let $h^0 := h^0(pt)$.

(1) Using the Atiyah-Hirzebruch spectral sequence we obtain (non-canonical) isomorphisms

$$\iota_k : h^0((\mathbb{CP}^\infty)^k) \xrightarrow{\cong} h^0[[x_1, ..., x_k]].$$

The choice of $\iota := \iota_1$ determines in a canonical way a choice of ι_k for $k > 1$.

(2) The map $\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifying the (exterior) tensor product of two copies of the universal line bundle induces a map

$$h^0[[x]] \underset{\iota^{-1}}{\cong} h^0(\mathbb{CP}^\infty) \underset{\mu_*}{\to} h^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \underset{\iota_2}{\cong} h^0[[x_1, x_2]].$$

The image $F_h(x_1, x_2)$ of x under this map defines a formal group law over the ring h^0.

(3) The formal group underlying the formal group law $F_h(x_1, x_2)$ is independent of the choice of ι and is called the formal group associated with h^*.

More information about this can be found in [Re]. The formal group associated with h^* can also be defined without choosing the coordinate $\iota : h^0(\mathbb{CP}^\infty)) \cong h^0[[x]]$, see [Lu].

Examples 2.3.

(1) The formal group associated with the periodic Eilenberg-MacLane spectrum

$$HP := \bigvee_{m \in \mathbb{Z}} \Sigma^{2m} H\mathbb{Z}$$

is the additive formal group.

(2) The formal group associated with complex K-theory is the multiplicative formal group.

(3) The periodic complex cobordism spectrum

$$MP := \bigvee_{m \in \mathbb{Z}} \Sigma^{2m} MU.$$

has a canonical complex orientation and the associated formal group law $F_{MP}(x_1, x_2)$ over $MP_0 = MP^0$ is, according to Quillen's theorem, the universal formal group law.

Remark 2.4. As in the case of MU_*, one can show that

$$MP_0 \rightrightarrows MP_0 MP$$

is a Hopf algebroid and that $MP_*(X)$ is a comodule (with $\mathbb{Z}$-grading) over it for all spaces X.

Recall that every Hopf algebroid $L \rightrightarrows W$ defines a functor from rings to groupoids by associating to a ring R the groupoid $\mathcal{P}_{(L,W)}(R)$ whose set of objects is $\mathrm{Hom}(L, R)$ and whose set of morphisms is $\mathrm{Hom}(W, R)$. The structure maps of $\mathcal{P}_{(L,W)}(R)$ (for example, the source and target maps $\mathrm{Hom}(W, R) \rightrightarrows \mathrm{Hom}(L, R)$), are induced by the structure maps of $L \rightrightarrows W$. We chose the notation $\mathcal{P}_{(L,W)}$ here, since the association

$$\mathrm{Spec}(R) \mapsto \mathcal{P}_{(L,W)}(R)$$

defines a prestack on *Aff*.

Now, consider the functor FGL from rings to groupoids that associates to a ring R the category of formal group laws over R and their isomorphisms; on morphisms, FGL is defined by pushforward. The following theorem is the fundamental link between formal groups laws and complex cobordism.

We will use the short-hand $(L, W) := (MP_0, MP_0 MP)$ for the Hopf algebroid associated with periodic complex cobordism.

Theorem 2.5 (Quillen, Landweber, Novikov)**.** *The Hopf algebroid (L, W) corepresents the functor FGL, i.e. we have a natural isomorphism of functors $FGL \cong \mathcal{P}_{(L,W)}$.*

For a proof in the non-periodic case we refer to [Ko]. More information about the periodic case can be found in [St].

Translating the theorem into the language of stacks, we obtain the following result.

Corollary 2.6. *The stack of formal groups $\mathcal{M}_{FG}$ is equivalent to the the stack associated to the Hopf algebroid (L, W).*

Proof. The theorem tells us that the prestack of formal group laws is isomorphic to $\mathcal{P}_{(L,W)}$. Note that there is a morphism from the stackification of the prestack of formal group laws to the stack of formal groups given by forgetting the choice of a (local) coordinate. Furthermore, since every formal group locally admits a coordinate, this morphism is an equivalence. This implies the claim. □

We have seen that every even periodic cohomology theory gives rise to a formal group. Now we want to try to associate to a formal group law F a cohomology theory. Recall that, according to Quillen's theorem, a formal group law over a ring R is the same as a ring homomorphism $F : L \to R$. One way to get a cohomology theory from F is to start with the cohomology theory MP^* corresponding to the universal formal group law and to tensor it with R, using the L-algebra structure on R defined by F. The functor $R \otimes_L MP^*(\_)$ is again a cohomology theory, provided that the exactness of the Mayer-Vietoris sequence is preserved.

We will describe Landweber's theorem in the homological setting, i.e. we address the question when the functor $R \otimes_L MP_*(\_)$ is a homology theory. This is certainly the case if R is flat over L. However, a much weaker condition will do. The point is that the L-modules $MP_*(X)$ that we tensor with R are of a very special type: as we remarked above, they are comodules over (L, W). This motivates the following definition.

Definition 2.7. A formal group law $F : L \to R$ is *Landweber exact* if the functor

$$M \mapsto R \otimes_L M$$

is exact on the category of (L, W)-comodules.

Using this language, we have: if F is a Landweber exact formal group law, then the functor

$$X \mapsto R \otimes_L MP_*(X)$$

defines a homology theory. Landweber's theorem gives a useful characterization of Landweber exact formal group laws. In order to formulate it, write the p-series $[p]_{F_u}(x) := x +_{F_u} ... +_{F_u} x$ (where we sum p copies of x) of the universal formal group law F_u over L as

$$[p]_{F_u}(x) =: \sum_{n \geq 1} a_n x^n$$

and let $v_i := a_{p^i}$ for all $i \geq 0$. Define ideals $I_n \subset R$ by $I_n := v_0 R + ... + v_{n-1} R$ for all $n \geq 0$.

Theorem 2.8 (Landweber, 1973)**.** *If for all primes p and for all integers $n \geq 0$ the map*

$$v_n : R/I_n \to R/I_n$$

is injective, then F is Landweber exact.

For the proof we refer to [La]. Note that Landweber only considers the functor $M \mapsto R \otimes_L M$ on the category of (L, W)-comodules that are finitely presented as L-modules. However, since every (L, W)-comodule is a direct limit of comodules of this type, the result holds in general, cf. Remark 9.6 in [Mi].

Examples 2.9.

(1) The multiplicative formal group is Landweber exact over $\mathbb{Z}$. The corresponding cohomology theory is complex K-theory.

(2) The additive formal group is not Landweber exact over $\mathbb{Z}$. However, it becomes Landweber exact if we allow rational coefficients instead.
(3) We will see that many formal group laws arising from elliptic curves are Landweber exact. They give rise to so-called elliptic cohomology theories. In the next section we will see that all these theories can be put together to form a presheaf of (co)homology theories on the moduli stack of elliptic curves.

3. Landweber exactness, stacks, and elliptic (co)homology

The goal of this section is to express Landweber's exactness condition in the language of stacks and to construct the presheaf of elliptic homology theories on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. We begin with some preliminary material about stacks.

Remark 3.1. Fibered products[1] exist in the 2-category of stacks. Given morphisms of stacks $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ and $\mathcal{G} : \mathcal{M}' \to \mathcal{N}$ we have a commutative pullback diagram

$$\begin{array}{ccc} \mathcal{M} \times_{\mathcal{N}} \mathcal{M}' & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \mathcal{F} \\ \mathcal{M}' & \xrightarrow{\mathcal{G}} & \mathcal{N}, \end{array}$$

where the fibered product $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'$ is defined as follows. For an affine scheme U, the objects of the groupoid $\mathcal{M} \times_{\mathcal{N}} \mathcal{M}'(U)$ are given by triples (m, m', ϕ), where $m \in \mathcal{M}(U)$, $m' \in \mathcal{M}'(U)$, and ϕ is an isomorphism $\phi : \mathcal{F}(m) \cong \mathcal{G}(m')$. Morphisms $(m_1, m_1', \phi_1) \to (m_2, m_2', \phi_2)$ are pairs (ψ, ψ'), where $\psi : m_1 \to m_2$ and $\psi' : m_1' \to m_2'$ such that $\mathcal{G}(\psi')\phi_1 = \phi_2\mathcal{F}(\psi)$.

Example 3.2. If the stacks involved come from Hopf algebroids, we can compute the fibered product in terms of a pushout of Hopf algebroids, i.e. we have a pullback square

$$\begin{array}{ccc} \mathcal{M}_{(S,T)} & \longrightarrow & \mathcal{M}_{(L_1,W_1)} \\ \downarrow & & \downarrow \\ \mathcal{M}_{(L_2,W_2)} & \longrightarrow & \mathcal{M}_{(L,W)}, \end{array}$$

where $S = L_1 \otimes_L W \otimes_L L_2$ and $T = W_1 \otimes_L W \otimes_L W_2$.

Remark 3.3. We will frequently use the following fact. If $\mathcal{M}_{(L,W)}$ is a stack associated with a Hopf algebroid and $a : \operatorname{Spec} A \to \mathcal{M}_{(L,W)}$ is any morphism of stacks, then a factors *locally* through the canonical map $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$, i.e. there exists a covering $k : \operatorname{Spec} S \to \operatorname{Spec} A$ such that $ak : \operatorname{Spec} S \to \mathcal{M}_{(L,W)}$ factors through c. This follows from the definition of the stackification functor.

Definition 3.4.

(1) A morphism $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is *representable* if for all morphisms $a : \operatorname{Spec} A \to \mathcal{N}$ from an affine scheme to $\mathcal{N}$ the fibered product $\mathcal{M} \times_{\mathcal{N}} \operatorname{Spec} A$ is equivalent to an affine scheme $\operatorname{Spec} P$.
(2) In this case, we call $\mathcal{F}$ *flat* (resp. a *covering*) if for all a the morphism of affine schemes $\mathcal{F}_a$ in the pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} P & \longrightarrow & \mathcal{M} \\ \downarrow \mathcal{F}_a & & \downarrow \mathcal{F} \\ \operatorname{Spec} A & \xrightarrow{a} & \mathcal{N} \end{array}$$

is flat (resp. a covering).

[1] Also knows as 'pullbacks', 'homotopy pullbacks', or '2-category pullbacks'.

Remark 3.5.

(1) In the case $\mathcal{N} = \mathcal{M}_{(L,W)}$ it suffices to check the representability of $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ on the morphism $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)} = \mathcal{N}$. To see this, assume that $\mathcal{M} \otimes_{\mathcal{N}} \operatorname{Spec} L \cong \operatorname{Spec} P$ is affine. If the morphism $a : \operatorname{Spec} A \to \mathcal{N}$ factors through $\operatorname{Spec} L$, then $\mathcal{M} \otimes_{\mathcal{N}} \operatorname{Spec} A \cong \operatorname{Spec} P \otimes_{\operatorname{Spec} L} \operatorname{Spec} A$ is affine. In the general case, we can find a covering $k : \operatorname{Spec} S \to \operatorname{Spec} A$ such ak factors through c. Hence the pullback of $\mathcal{M}$ to $\operatorname{Spec} S$ is affine and it follows from descent theory for affine schemes that the same is true for the pullback of $\mathcal{M}$ to $\operatorname{Spec} A$.

(2) Similarly, we can check whether a representable morphism $\mathcal{F} : \mathcal{M} \to \mathcal{M}_{(L,W)}$ is flat (or a covering) by checking on c. Since the properties 'flat' and 'faithfully flat' are local for the flat topology, it suffices to prove that $\mathcal{F}_c$ (faithfully) flat implies that $\mathcal{F}_a$ is flat for any $a : \operatorname{Spec} A \to \mathcal{M}_{(L,W)}$ in the case in where a factors through c. In this case, however, this follows since 'flat' and 'faithfully flat' are stable under base change.

Examples 3.6.

(1) The description of the fibered product of stacks coming from Hopf algebroids given in Example 3.2 implies that every map $\operatorname{Spec} R \to \mathcal{M}_{(L,W)}$ is representable. Hence it always makes sense to ask whether or not a map from an affine scheme into $\mathcal{M}_{FG}$ is flat.

(2) Let (L, W) be a Hopf algebroid and consider the canonical map $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$. From Example 3.2 we see that

$$\begin{array}{ccc} \operatorname{Spec} W & \xrightarrow{\eta_L} & \operatorname{Spec} L \\ \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle c} \\ \operatorname{Spec} L & \xrightarrow{c} & \mathcal{M}_{(L,W)} \end{array}$$

is a pullback diagram. Hence, if η_L (or, equivalently, η_R) is a faithfully flat map, then $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$ is a covering. For example, we see that $\operatorname{Spec} MP_0 \to \mathcal{M}_{FG}$ is a cover.

We will now define (pre)sheaves on stacks, more details can be found in [Goe]. Define the category $\mathit{Aff}/\mathcal{M}$ of affine schemes over the stack $\mathcal{M}$ as follows. The objects are simply morphisms $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}$. The morphisms, say from $\mathcal{F}_1$ to $\mathcal{F}_2$, are 2-commutative triangles, i.e. pairs (h, ϕ) consisting of a morphism $h : \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ and a 2-isomorphism ϕ between $\mathcal{F}_1$ and $\mathcal{F}_2 h$. We make $\mathit{Aff}/\mathcal{M}$ into a Grothendieck site by declaring a morphism (h, ϕ) to be a covering if h is a covering in Aff.

Definition 3.7. A *presheaf* on a stack $\mathcal{M}$ is a contravariant functor M on the category $\mathit{Aff}/\mathcal{M}$. A presheaf S on $\mathcal{M}$ is a *sheaf* if for every covering $V \to U$ in $\mathit{Aff}/\mathcal{M}$ we have an equalizer diagram

$$S(U) \to S(V) \rightrightarrows S(V \times_U V).$$

For example, the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of the stack $\mathcal{M}$ is the sheaf of rings defined by

$$\mathcal{O}_{\mathcal{M}}(r : \operatorname{Spec} R \to \mathcal{M}) := R.$$

Sheaves of modules over $\mathcal{O}_{\mathcal{M}}$ are defined in the obvious way. The notion of quasi-coherent sheaves of modules over $\mathcal{O}_{\mathcal{M}}$ and the correspondence in the following proposition is explained in [Goe], Section 1.3; see also [Nau], Section 3.4. In the following we assume that $\eta_L : L \to W$ is faithfully flat, i.e. that $c : \operatorname{Spec} L \to \mathcal{M}_{(L,W)}$ is a covering.

Proposition 3.8. *The category of comodules over (L, W) is equivalent to the category of quasi-coherent sheaves on $\mathcal{M}_{(L,W)}$.*

This implies that for every space X the comodule $MP_*(X)$ defines a quasi-coherent sheaf over the stack $\mathcal{M}_{FG}$.

For simplicity, we will call a sequence of quasi-coherent sheaves on $\mathcal{M}_{(L,W)}$ exact if the corresponding sequence of (L, W)-comodules is exact. By definition of the correspondence in Proposition 3.8 this is equivalent to asking that the pullback of the sequence to $\operatorname{Spec} L$ under $\operatorname{Spec} L \to \mathcal{M}_{(L,W)}$ is exact. Here the pullback functor is defined in the obvious way: $\mathcal{F}^* S(r) := S(\mathcal{F} r)$.

Proposition 3.9. *A representable morphism $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{(L,W)}$ is flat if and only if the pullback functor $\mathcal{F}^*$ from the category of quasi-coherent sheaves on $\mathcal{M}_{(L,W)}$ to the category of quasi-coherent sheaves on* $\operatorname{Spec} R$ *is exact.*

Proof. Since both conditions are local with respect to the flat topology, we may assume that $\mathcal{F}$ factors through $\operatorname{Spec} L$ by Remark 3.3. Now consider the pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} R \otimes_L W & \xrightarrow{c_{\mathcal{F}}} & \operatorname{Spec} R \\ \Big\downarrow \mathcal{F}_c & & \Big\downarrow \mathcal{F} \\ \operatorname{Spec} L & \xrightarrow{c} & \mathcal{M}_{(L,W)}. \end{array}$$

Since c is a covering the map $c_{\mathcal{F}}$ is faithfully flat. Hence $\mathcal{F}^*$ is exact if and only if $(\mathcal{F}c_{\mathcal{F}})^* = (c\mathcal{F}_c)^*$ is exact. By our definition of exact sequences on $\mathcal{M}_{(L,W)}$ this means precisely that $\mathcal{F}_c^*$ is exact on (L, W)-comodules. This is certainly true if $\mathcal{F}$ and hence $\mathcal{F}_c$ are flat. Conversely, if $\mathcal{F}^*$ is exact then the functor $N \mapsto \mathcal{F}^*N = R \otimes_L N$ is exact on (L, W)-comodules. Note that one can use the coalgebra structure of W to make every L-module of the form $W \otimes_L M$ (for some L-module M) into a (L, W)-comodule. Using this and the flatness of $\eta_R : L \to W$ it follows that the functor $M \mapsto R \otimes_L W \otimes_L M$ is exact on L-modules. Hence $\mathcal{F}_c$ is flat, i.e. $\mathcal{F}$ is flat. □

Recall that $\mathcal{M}_{FG} \cong \mathcal{M}_{(L,W)}$, where $(L, W) = (MP_0, MP_0MP)$. Hence if we are given a formal group law $F : L \to R$, we have a corresponding morphism of stacks

$$\mathcal{F} : \operatorname{Spec} R \to \operatorname{Spec} L \xrightarrow{c} \mathcal{M}_{(L,W)} \cong \mathcal{M}_{FG}.$$

Corollary 3.10. *A formal group law $F : MP_0 \to R$ over R is Landweber exact if and only if the corresponding morphism $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{FG}$ is flat.*

Corollary 3.11. *Let $\mathcal{F} : \operatorname{Spec} R \to \mathcal{M}_{FG}$ be any flat morphism. Then the functor*

$$X \mapsto \mathcal{F}^* MP_*(X)$$

defines a homology theory.

Now we turn to the definition of the presheaf of elliptic homology theories on $\mathcal{M}_{ell}$. We will use the following theorem that will be proved in Section 4.

Theorem 3.12 (Hopkins, Miller)**.** *The morphism $\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ that maps a familiy $\mathcal{C}$ of elliptic curves to the associated formal group $\mathcal{F}_{\mathcal{C}}$ is flat.*

Theorem 3.12 implies that for any flat morphism $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$ the composition

$$\mathcal{F}_{\mathcal{C}} : \operatorname{Spec} R \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

is again a flat. By Corollary 3.11 we hence obtain a homology theory for every such $\mathcal{C}$.

Definition 3.13. Given a flat map $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$, define by

$$Ell_*^{\mathcal{C}}(X) := \mathcal{F}_{\mathcal{C}}^* MP_*(X)$$

the *elliptic homology theory associated with the elliptic curve $\mathcal{C}$ over R.*

Remark 3.14. Note that if $\mathcal{F}$ factors through the cover $\operatorname{Spec} L \to \mathcal{M}_{FG}$ then, by construction, $Ell^{\mathcal{C}}$ has coefficients $Ell_*^{\mathcal{C}}(pt) = R \otimes_L MP_*(pt) \cong R[u^{\pm 1}]$ and the associated cohomology theory is even periodic. For a general morphism $\mathcal{F}$, the theory $Ell_*^{\mathcal{C}}(\_)$ might be merely weakly even periodic, cf. [Lu], Remark 1.6.

The *presheaf of elliptic homology theories* is defined on the Grothendieck site of flat affine schemes over $\mathcal{M}_{ell}$. This is the sub-site of $\mathit{Aff}/\mathcal{M}_{ell}$ whose objects are flat morphisms $\mathcal{C} : \operatorname{Spec} R \to \mathcal{M}_{ell}$. The value of this presheaf on an object is defined to be $Ell_*^{\mathcal{C}}$. In other words: for every space X we simply evaluate the quasi-coherent sheaf $\mathfrak{F}^* MP_*(X)$ on $\mathcal{C}$. On morphisms, we define the presheaf in the same way: for every space X and 2-commutative triangle (h, ϕ), say with $h : \mathcal{C}_1 \to \mathcal{C}_2$, we evaluate the sheaf $\mathfrak{F}^* MP_*(X)$ on (h, ϕ) to get an induced map

$$Ell_*^{\mathcal{C}_2}(X) \to Ell_*^{\mathcal{C}_1}(X).$$

This is nice and functorial since $\mathfrak{F}^* MP_*(X)$ is a sheaf on $\mathcal{M}_{ell}$. The work of defining the induced maps is hidden in Proposition 3.8, which allowed us to consider the comodule $MP_*(X)$ as a sheaf on $\mathcal{M}_{FG}$ (the explicit construction of the induced maps can be found in [Goe], Section 1.3).

4. The map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat

Recall that the moduli stack of elliptic curves $\mathcal{M}_{ell}$ is an open substack of the moduli stack of generalized elliptic curves $\mathcal{M}_{Weier}$. The latter is isomorphic to the stack associated with the Hopf algebroid (A, Γ), where $A := \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is the ring over which the universal Weierstrass curve lives and $\Gamma := A[u^{\pm 1}, r, s, t]$ parametrizes isomorphisms of Weierstrass curves. The open substack of elliptic curves $\mathcal{M}_{ell} \hookrightarrow \mathcal{M}_{Weier}$ is given by the locus where the discriminant Δ is non-zero. Consequently, $\mathcal{M}_{ell} \cong \mathcal{M}_{(\tilde{A}, \tilde{\Gamma})}$, where $\tilde{A} := A[\Delta^{-1}]$ and $\tilde{\Gamma} := \Gamma[\Delta^{-1}]$. For more information about (generalized) elliptic curves, see Appendix B in [AHS] and the references therein.

In order to make sense of the statement that the map $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat, we have to show that it is representable. It is easy to see that $\mathcal{M}_{ell} \hookrightarrow \mathcal{M}_{Weier}$ is representable and hence the representability of $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is implied by the following theorem.

Theorem 4.1. *The map $\mathcal{M}_{Weier} \to \mathcal{M}_{FG}$, $\mathcal{C} \mapsto \mathcal{F}_{\mathcal{C}}$, is representable.*

Proof. We will construct a pullback diagram of prestacks

$$\begin{array}{ccccc} \operatorname{Spec} A[b_4, b_5, ...] & \longrightarrow & \operatorname{Spec} A & \longrightarrow & \mathcal{P}_{(A,\Gamma)} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} L & \longrightarrow & \mathcal{P}_{FG^{(5)}} & \longrightarrow & \mathcal{P}_{(L,W)}. \end{array}$$

Stackifying this diagram yields the corresponding pullback diagram of the associated stacks.[2] It will be clear from the definition of the horizontal arrows in the bottom row that their composition is the usual cover $c : \operatorname{Spec} L \to \mathcal{M}_{FG}$. This implies the claim since the representability of $\mathcal{M}_{Weier} \to \mathcal{M}_{FG}$ can be checked on the cover c, see Remark 3.5.

The *prestack $\mathcal{P}_{FG^{(5)}}$ of formal group laws with a parameter modulo degree* 5 in the bottom row is corepresented by the Hopf algebroid

$$FG^{(5)} := (L^{(5)}, L^{(5)}[t_0^{\pm 1}, t_1, t_2, ...]),$$

where $L^{(5)} := L[u_1^{\pm 1}, u_2, u_3, u_4]$.

In order to complete the proof it suffices to show that the square on the right hand side is a pullback diagram. It then follows from the formula in Example 3.2 that the prestack in the top left of the diagram is isomorphic to an affine scheme (which turns out to be $\operatorname{Spec} A[b_4, b_5, ...]$).

We need the following lemma.

Lemma 4.2. *Given a Weierstrass curve $\mathcal{C}$ over R with a local parameter u modulo degree* 5 *near the identity, there exists a unique Weierstrass curve $\tilde{\mathcal{C}}$ over R and a unique isomorphism*

$$\psi : \mathcal{C} \to \tilde{\mathcal{C}}$$

that maps u to the canonical local parameter x/y near the identity (modulo degree 5*).*

[2]The reader comfortable with stacks might prefer to argue directly on the level of stacks, using essentially the same argument we use for the prestacks, but working 'locally' whenever appropriate.

Proof. We will show that there is a unique transformation ψ that maps u to $v := x/y$. $\tilde{\mathcal{C}}$ is then defined of be the image of $\mathcal{C}$ under ψ. Recall that isomorphisms of Weierstrass curves over R are given by transformations of 2-dimensional projective space over R that are of the form

$$M(u,s,r,t) = \begin{pmatrix} u^2 & 0 & r \\ su & u^3 & t \\ 0 & 0 & 1 \end{pmatrix},$$

see [AHS]. It is not difficult to compute that the following expansions hold near the identity:

$$\begin{array}{lcrcl} U_u = M(u,0,0,0) & \text{satisfies} & U_u(v) & = & u^{-1}v + O(v^2) \\ S_s = M(1,0,s,0) & \text{satisfies} & S_s(v) & = & v - sv^2 + O(v^3) \\ R_r = M(1,r,0,0) & \text{satisfies} & R_r(v) & = & v + rv^3 + O(v^4) \\ T_t = M(1,0,0,t) & \text{satisfies} & T_t(v) & = & v - tv^4 + O(v^5). \end{array}$$

In order to prove the last two, one uses that $y^{-1} = v^3 + O(v^4)$ near the identity, see [Si], IV §1. Now, given any local parameter $u = u_1 v + u_2 v^2 + ...$ modulo degree 5, we can apply the transformation U_{u_1} to make $u_1 = 1$. Similarly, we can apply transformations S_s, R_r, and T_t to achieve $u_2 = u_3 = u_4 = 0$. The composition of these transformations is the isomorphism ψ we wanted to construct. Furthermore, from the expansions for U_u, S_s, R_r, and T_t given above it is clear that the choices for u, s, r, and t are unique. Since any transformation $M(u,r,s,t)$ can be written as

$$M(u,r,s,t) = T_t R_r S_s U_u$$

for unique elements $t, r, s, u \in R$ this implies that ψ is unique. □

Now we can explain the pullback square on the right hand side of the diagram. Recall that for an affine scheme $U = \operatorname{Spec} R$ the objects of the groupoid $\mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}(U)$ are quadruples $(\mathcal{C}, F, u, \phi)$, where $\mathcal{C} : A \to R$ is a Weierstrass curve, $F : L \to R$ is a formal group law, $u : \mathbb{Z}[u_1^{\pm 1}, u_2, u_3, u_4] \to R$ is a local parameter modulo degree 5 for F, and $\phi : F_{\mathcal{C}} \to F$ is an isomorphism of formal group laws. We claim that the functor

$$i_U : \operatorname{Spec} A\,(U) \to \mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}(U),\ \mathcal{C} \mapsto (\mathcal{C}, \mathcal{F}_{\mathcal{C}}, x/y, id),$$

is an equivalence of categories. To see this, note that the inclusion of the full subcategory of $\mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}(U)$ whose objects are of the form $(\mathcal{C}, F_{\mathcal{C}}, u, id)$ is obviously essentially surjective. This together with the existence of ψ in Lemma 4.2 shows that i_U is essentially surjective. Furthermore, the uniqueness of ψ means that i_U is fully faithful, so i_U is an equivalence of categories. The functors i_U fit together to give an equivalence of prestacks $i : \operatorname{Spec} A \to \mathcal{P}_{(A,\Gamma)} \times_{\mathcal{P}_{(L,W)}} \mathcal{P}_{FG^{(5)}}$. Thus we have constructed the pullback square on the right hand side. □

Now that we have shown that $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is representable, we are ready to prove that it is flat. The crucial ingredient is the following lemma.

Lemma 4.3 (Franke, Landweber, Ravenel, Stong)**.** *The formal group law*

$$\tilde{F} : L \to \tilde{A}$$

of the universal smooth Weierstrass curve $\mathcal{C}_{univ}$ over $\tilde{A} := A[\Delta^{-1}]$ is Landweber exact.

Proof. We apply Theorem 2.8, the Landweber exact functor theorem. We have to check that for all primes p and integers $n \geq 0$, the map $v_n : \tilde{A}/I_n \to \tilde{A}/I_n$ is injective. This can be done as follows.

(1) Since $\tilde{A}$ is torsion-free, multiplication by $v_0 = p$ defines an injective endomorphism of $\tilde{A}$.
(2) Consider the endomorphism of $\tilde{A}/p\tilde{A} \cong \mathbb{F}_p[a_1, a_1, ..., a_6, \Delta^{-1}]$ given by v_1. Since $\tilde{A}/p\tilde{A}$ is an integral domain, v_1 is injective if and only if the image of v_1 in $\tilde{A}/p\tilde{A}$ is not zero. This, in turn, is equivalent to the condition that the mod p reduction of $\mathcal{C}_{\text{univ}}$ has a fiber that is not supersingular. This is certainly true; in fact, most elliptic curves are not supersingular.
(3) We finish the proof by showing that the image of v_2 in $\tilde{A}/(p\tilde{A} + v_1\tilde{A})$ is a unit. This shows that multiplication by v_2 is injective as required and that the conditions for all v_n, $n \geq 3$, are trivial, since the ring $\tilde{A}/(p\tilde{A} + v_1\tilde{A} + v_2\tilde{A})$ is zero. Assume the image of v_2 is not a unit in $\tilde{A}/(p\tilde{A} + v_1\tilde{A})$. Then there exists a maximal ideal $\mathfrak{m} \subset A$ containing (the images of) p,

v_1, and v_2. Hence the quotient $\tilde{A}/\mathfrak{m}$ is a field (of characteristic p) and the reduction of $\mathcal{C}_{\text{univ}}$ defines an elliptic curve over $\tilde{A}/\mathfrak{m}$ whose associated formal group law has height > 2. This contradicts the fact that the height of the formal group law of an elliptic curve over a field of characteristic $p > 0$ is either 1 or 2, see [Si], Chapter IV, Corollary 7.5. $\square$

Now we are ready to complete the *proof of Theorem 3.12*. Since the structure maps $\tilde{A} \rightrightarrows \tilde{\Gamma}$ of the Hopf algebroid $(\tilde{A}, \tilde{\Gamma})$ are faithfully flat, the canonical map $c : \operatorname{Spec} \tilde{A} \to \mathcal{M}_{ell}$ is a covering. Hence $\mathfrak{F} : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat if and only the composition

$$\tilde{\mathcal{F}} : \operatorname{Spec} \tilde{A} \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

is flat. By Corollary 3.10, $\tilde{\mathcal{F}}$ is flat if and only if the formal group law $\tilde{F} : L \to \tilde{A}$ is Landweber exact, which was shown in Lemma 4.3. $\square$

Remark 4.4. In the following talks we will see that the presheaf *Ell* on $\mathcal{M}_{ell}$ can be turned into a sheaf of E_∞-ring spectra (if we replace the flat topology by the étale topology). The spectrum *TMF* is defined as the global sections of this sheaf. It can be shown that the presheaf *Ell* and the corresponding sheaf of E_∞-ring spectra extend to the Deligne-Mumford compactification $\bar{\mathcal{M}}_{ell}$ of $\mathcal{M}_{ell}$. The (-1)-connected cover of the global sections of this sheaf is *tmf*.

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