

The super group $\mathbb{R}^{0|1}$, the parity changed tangent bundle ΠTM , and the de Rham complex $\Omega^* M$.

Notation. We denote by \mathcal{S} the category of super manifolds in the sense of Deligne and Morgan. Their paper [DM] is also our main reference for other fundamental notions and facts in super algebra and geometry. For example, we denote by Π the *parity reversal functor* that reverses the grading on (super) modules.¹ Clearly, Π also makes sense for sheaves of modules over the sheaf of functions on a super manifold.

Inner morphisms from $\mathbb{R}^{0|1}$ to M and the bundle ΠTM . Let M be a super manifold. The *super manifold $\underline{\mathcal{S}}(\mathbb{R}^{0|1}, M)$ of morphisms* from $\mathbb{R}^{0|1}$ to M is defined (up to canonical isomorphism) by the property that it is a representing object for the contravariant functor

$$S \mapsto \mathcal{S}(\mathbb{R}^{0|1} \times S, M).$$

More explicitly, we have natural bijections

$$\mathcal{S}(\mathbb{R}^{0|1} \times S, M) \cong \mathcal{S}(S, \underline{\mathcal{S}}(\mathbb{R}^{0|1}, M))$$

The following proposition implies that $\underline{\mathcal{S}}(\mathbb{R}^{0|1}, M)$ exists for all M .

Proposition 1. $\underline{\mathcal{S}}(\mathbb{R}^{0|1}, M) = \Pi TM$.

Proof. Denote by \mathcal{A} the category of commutative \mathbb{R} -super algebras. For a super manifold X we denote the global sections of its function sheaf by O_X . We split the proof of the desired bijection into the following natural correspondences:

$$\begin{aligned} \mathcal{S}(\mathbb{R}^{0|1} \times S, M) &\stackrel{(1)}{\longleftrightarrow} \mathcal{A}(O_M, O_{\mathbb{R}^{0|1} \times S}) \\ &\stackrel{(2)}{\longleftrightarrow} \mathcal{A}(O_M, \Lambda^*(\mathbb{R}) \otimes O_S) \\ &\stackrel{(3)}{\longleftrightarrow} \{ (f, g) \mid f \in \mathcal{A}(O_M, O_S) \text{ and } g \text{ is an odd derivation } O_M \rightarrow O_S \text{ w.r.t. } f \} \\ &\stackrel{(4)}{\longleftrightarrow} \{ (f, g) \mid f \in \mathcal{S}(S, M) \text{ and } g \text{ is an odd section of } f^* TM \} \\ &\stackrel{(5)}{\longleftrightarrow} \mathcal{S}(S, \Pi TM) \end{aligned}$$

We will use the following:

Proposition 2. *The functor $\Gamma : \mathcal{S} \rightarrow \mathcal{A}$, $M \mapsto O_M$, is bijective on Hom-sets.*

The *proof* of this proposition is based on the existence of partitions of unity on super manifolds and can be found in [BBH], see Corollary 1.5 in chapter III. \square

Now we are ready to prove the bijections (1) to (5):

- (1) follows directly from proposition 2.
- (2) just uses the definition of products of super manifolds and $O_{\mathbb{R}^{0|1}} = \Lambda^*(\mathbb{R})$.
- (3) To see this, we decompose $\varphi : O_M \rightarrow \Lambda^*(\mathbb{R}) \otimes O_S = O_S[\theta]$ as a sum

$$\varphi = f + \theta g, \text{ with } f, g : O_M \rightarrow O_S.$$

¹Let M be a (super) module over a commutative super algebra A . We would like to remind the reader that the left module structure of ΠM differs from the left module structure on M by a sign: We have $a(\Pi m) = (-1)^{|a|}\Pi(am)$ for $a \in A$ and $m \in M$. This makes the map $\Pi : M \rightarrow \Pi M$ which is the identity on the set underlying M into an *odd* A -module homomorphism.

Here θ is the usual odd coordinate on \mathbb{R}^{01} . Note that f preserves the grading, whereas g reverses it. For $a, b \in \mathcal{O}_M$ we have $\varphi(ab) = f(ab) + \theta g(ab)$, and since φ is an algebra homomorphism this is also equal to

$$\varphi(a)\varphi(b) = (f(a) + \theta g(a))(f(b) + \theta g(b)) = f(a)f(b) + \theta(g(a)f(b) + (-1)^{|a|}f(a)g(b)).$$

Comparing the coefficients we conclude that f is an algebra homomorphism and that g is an odd derivation w.r.t. f . Conversely, any such pair (f, g) defines an algebra map φ . It is clear that the bijection is natural w.r.t. super algebra maps $\mathcal{O}_S \rightarrow \mathcal{O}_{S'}$.

- (4) Proposition 2 takes care of the part involving f . The statement concerning g follows from the next lemma by looking at the odd part of the global sections.

Lemma 3. *Let $(|f|, f) : S \rightarrow M$ be a map of super manifolds. Then*

$$f^*TM \cong \text{Der}_f(|f|^{-1}\mathcal{O}_M, \mathcal{O}_S) \text{ as sheaves of } \mathcal{O}_S\text{-modules.}$$

Here $\text{Der}_f(|f|^{-1}\mathcal{O}_M, \mathcal{O}_S)$ denotes the sheaf of derivations from $|f|^{-1}\mathcal{O}_M$ to \mathcal{O}_S w.r.t. f .

The *proof* consists of putting together several standard isomorphisms of sheaves. The construction is indicated in [Ma], chapter 4, §1.10.

- (5) This is just the definition of the super manifold ΠTM in terms of its S -points. See [DM], page 72.

This finishes the proof of the proposition. We would like to point out that proposition 2 is not at all crucial for the proof. One can write down the equivalences in terms of maps of sheaves (instead of their restriction to global sections), the only thing that changes is that the notation becomes a bit more complicated. \square

Let us write down the above natural bijection more explicitly for super domains $M = U \subset \mathbb{R}^{p|q}$. In this case we can make the identification $\Pi TU \cong U \times \mathbb{R}^{q|p}$. Going through the above bijections one sees that if the morphism $\varphi : S \times \mathbb{R}^{01} \rightarrow U$ is given by coordinates

$$(x_1 + \theta \hat{x}_1, \dots, x_p + \theta \hat{x}_p, \eta_1 + \theta \hat{\eta}_1, \dots, \eta_q + \theta \hat{\eta}_q) \in (\mathcal{O}_{S \times \mathbb{R}^{01}}^{ev})^p \times (\mathcal{O}_{S \times \mathbb{R}^{01}}^{odd})^q,$$

then its image in $\tilde{\varphi} \in \mathcal{S}(S, U \times \mathbb{R}^{q|p})$ has coordinates

$$(x_1, \dots, x_p, \eta_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{\eta}_q, \hat{x}_1, \dots, \hat{x}_p) \in (\mathcal{O}_S^{ev})^p \times (\mathcal{O}_S^{odd})^q \times (\mathcal{O}_S^{ev})^q \times (\mathcal{O}_S^{odd})^p.$$

The translation action of \mathbb{R}^{01} . Using the additive group structure

$$\mu : \mathbb{R}^{01} \times \mathbb{R}^{01} \longrightarrow \mathbb{R}^{01}, (\eta, \theta) \mapsto \eta + \theta$$

we make the 'super point' \mathbb{R}^{01} into a super Lie group. For every $\eta : S \rightarrow \mathbb{R}^{01}$ there is a map

$$r_\eta : \mathbb{R}^{01} \times S \xrightarrow{\text{id} \times \eta} \mathbb{R}^{01} \times \mathbb{R}^{01} \xrightarrow{\mu} \mathbb{R}^{01},$$

called 'right translation by η '. From this, we obtain an action

$$\mathbb{R}^{01} \times \Pi TM \longrightarrow \Pi TM$$

by pre-composing by right translations: To an S -point

$$(\eta, \varphi) \in \mathcal{S}(S, \mathbb{R}^{01}) \times \mathcal{S}(\mathbb{R}^{01} \times S, M) \cong \mathcal{S}(S, \mathbb{R}^{01} \times \Pi TM)$$

it associates the composition

$$\varphi_\eta = \varphi(r_\eta \times \text{id}) \in \mathcal{S}(S, \Pi TM) \cong \mathcal{S}(\mathbb{R}^{01} \times S, M).$$

In local coordinates this just means that φ_η is given by replacing θ in the coordinate representation of φ by $\theta + \eta$. Hence locally, say on a super domain U , the action can be written in coordinates as

$$\mathbb{R}^{01} \times U \times \mathbb{R}^{q|p} \longrightarrow U \times \mathbb{R}^{q|p}, (\eta, x_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{x}_p) \mapsto (x_1 + \eta \hat{x}_1, \dots, \eta_q + \eta \hat{\eta}_q, \hat{\eta}_1, \dots, \hat{x}_p).$$

Differentiating the action with respect to η we obtain the action of the generator ∂_η of the super Lie algebra of $\mathbb{R}^{0|1}$ on the algebra of functions on ΠTM . It acts as the vector field with local representation

$$(x_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{x}_p) \mapsto (\hat{x}_1, \dots, \hat{\eta}_q, 0, \dots, 0)$$

which we may write as

$$D = \sum_{i=1}^p \hat{x}_i \frac{\partial}{\partial x_i} + \sum_{i=1}^q \hat{\eta}_i \frac{\partial}{\partial \eta_i}.$$

Writing the coordinates as $(y_1, \dots, y_{p+q}) := (x_1, \dots, \eta_q)$ and letting $\partial_i := \frac{\partial}{\partial y_i}$ this takes the simple form

$$D = \sum_{i=1}^{p+q} \hat{y}_i \partial_i$$

Note that since $\partial_\eta^2 = \frac{1}{2}[\partial_\eta, \partial_\eta] = 0$ applying D twice annihilates all functions on ΠTM .

The de Rham complex $\Gamma(\Omega^* M)$. There are two different possible conventions for the grading of the algebra $\Omega^* M$ of differential forms on M . We will work with the one that makes $\Gamma(\Omega^* M)$ into a commutative super algebra and leads to an *odd* de Rham differential d . This seems to be a natural choice, since we want to relate d to the action of the odd vector field D on the commutative super algebra $\mathcal{O}_{\Pi TM}$.

Let V be a module over the super commutative algebra A . Following the convention of Bernstein-Leites we define the exterior algebra on V to be

$$\Lambda_A^*(V) := \text{Sym}_A(\Pi V).$$

Here $\text{Sym}_A(W)$ is the quotient of the tensor algebra on W by the ideal generated by all super commutators $w_1 \otimes w_2 - (-1)^{|w_1||w_2|} w_2 \otimes w_1$. The commutative super algebra $\Lambda_A^*(V)$ has the universal property that giving a super algebra map from $\Lambda_A^*(V)$ to any commutative A -super algebra B is the same as giving an A -module map $\Pi V \rightarrow B$, see [Ma], chapter 3, §2.5.

Now let $\Omega^1 M = \text{Hom}_{\mathcal{O}_M}(TM, \mathcal{O}_M)$ be the cotangent sheaf of M and define

$$\Omega^* M = \Lambda_{\mathcal{O}_M}^*(\Omega^1 M).$$

Clearly, \mathcal{O}_M and $\Pi \Omega^1 M$ are subsheaves of $\Lambda_{\mathcal{O}_M}^*(\Omega^1 M)$ in a natural way. From the usual (even) 'differential' $d_{ev} : \mathcal{O}_M \rightarrow \Omega^1 M$ that is characterized by

$$d_{ev} f(X) = (-1)^{|f||X|} X(f) \text{ for all } X \in TM$$

we obtain an odd differential d by composing with $\Pi : \Omega^1 M \rightarrow \Pi \Omega^1 M$,

$$d := \Pi d_{ev} : \mathcal{O}_M \longrightarrow \Pi \Omega^1 M.$$

A de Rham differential on $\Omega^* M$ is an extension of d whose square is zero and which satisfies the Leibniz rule. We will see below that such an extension indeed exists.

Differential forms on M and functions on ΠTM . The next step is to interpret differential forms on M as functions on ΠTM . This will be done using

Lemma 4. *There is an embedding of sheaves of \mathcal{O}_M -super algebras*

$$\iota : \Omega^* M \hookrightarrow \mathcal{O}_{\Pi TM}$$

that maps onto the functions that are polynomial on every fiber.

Proof. Let x_1, \dots, η_q be local coordinates on M . We have canonically associated coordinates

$$(x_1, \dots, \eta_q, \hat{\eta}_1, \dots, \hat{\eta}_q, \hat{x}_1, \dots, \hat{x}_p)$$

on ΠTM . Recall that the \hat{x}_i 's are odd, whereas the $\hat{\eta}_j$'s are even. On the other hand, a local basis for the \mathcal{O}_M -module $\Omega^1 M$ is given by $dx_1, \dots, d\eta_q$. According to the convention we picked for the definition of the de Rham complex, the dx_i and $d\eta_j$ have odd and even parity, resp. Hence we can define a map of super \mathcal{O}_M -modules $\iota_0 : \Pi\Omega^1 M \rightarrow \mathcal{O}_{\Pi TM}$ by prescribing $dx_i \mapsto \hat{x}_i$ and $d\eta_j \mapsto \hat{\eta}_j$. It is not hard to check that this is independent of the coordinate system chosen.² According to the defining property of $\text{Sym}_{\mathcal{O}_M}(\Pi\Omega^1 M)$ the map ι_0 extends to a unique homomorphism of \mathcal{O}_M -algebras $\iota : \Omega^* M \rightarrow \mathcal{O}_{\Pi TM}$. It is clear that ι is injective. \square

The map ι is surjective if and only if M is purely even, i.e. a usual manifold.

Interpreting D as the de Rham differential on $\Omega^* M$. Since D is an odd vector field on ΠTM we have the Leibniz rule

$$D(fg) = (Df)g + (-1)^{|f|} f(Dg) \text{ for all functions } f, g \text{ on } \Pi TM.$$

Furthermore, we already know that $D^2 = 0$. Hence the restriction of D to $\Omega^* M$ deserves to be called a de Rham differential once we have shown

Lemma 5. *The restriction of D to $\mathcal{O}_M \subset \mathcal{O}_{\Pi TM}$ is the odd differential d . More precisely, we have*

$$D = \iota d : \mathcal{O}_M \rightarrow \mathcal{O}_{\Pi TM}.$$

Proof. It is clear from the local representation of D that the image of D is contained in $\iota(\Pi\Omega^1 M) \subset \mathcal{O}_{\Pi TM}$. The claim is equivalent to showing that the composition

$$\tilde{D} := \Pi^{-1} \iota^{-1} D : \mathcal{O}_M \rightarrow \Omega^1 M$$

is equal to d_{ev} , i.e. for all $f \in \mathcal{O}_M$ we have to check that

$$\tilde{D}f(X) = (-1)^{|f||X|} X(f) \text{ for all } X \in TM.$$

It suffices to prove this for (local) basis vector fields $\partial_j = \frac{\partial}{\partial y_j}$, $j = 1, \dots, p+q$, where the y_i are local coordinates on M . We first compute

$$\tilde{D}f = \Pi^{-1} \iota^{-1} D(f) = \Pi^{-1} \iota^{-1} \left(\sum_{i=1}^{p+q} \hat{y}_i \partial_i f \right) = \Pi^{-1} \iota^{-1} \left(\sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)} (\partial_i f) \hat{y}_i \right)$$

Since ι and Π are even and odd \mathcal{O}_M -module maps, resp., we get

$$\tilde{D}f = \Pi^{-1} \left(\sum_{i=1}^{p+q} (-1)^{(|y_i|+1)(|f|+|y_i|)} (\partial_i f) dy_i \right) = \sum_{i=1}^{p+q} (-1)^{|y_i|(|f|+1)} (\partial_i f) d_{ev} y_i$$

Applying this 1-form to ∂_j and using $(d_{ev} y_i) \partial_j = (-1)^{|y_i||y_j|} \delta_{ij}$ yields

$$\tilde{D}f(\partial_j) = (-1)^{|y_j|(|f|+1)+|y_j|} = (-1)^{|y_j||f|} \partial_j(f)$$

as desired. \square

²In fact, one can see this using the (global!) vector field D considered above: The map ι_0 is equal to the composition

$$\Pi\Omega^1 M \xrightarrow{\Pi^{-1}} \Omega^1 M \hookrightarrow \Omega^1(\Pi TM) \xrightarrow{\tilde{D}} \mathcal{O}_{\Pi TM}, \text{ where } \tilde{D}(\omega) = (-1)^{|\omega|} \omega(D).$$

If M is purely even the cohomological degree of $\alpha \in \Omega^*M$ is equal to the parity of its image in $\mathcal{O}_{\Pi TM}$ modulo 2. Hence the Leibniz rule above is exactly the (graded) Leibniz rule for differential forms, and so D is equal to the usual de Rham differential on Ω^*M .³

The action of the full automorphism group $\underline{\text{Aut}}(\mathbb{R}^{01})$. We used the translation action of \mathbb{R}^{01} on itself to define an action of \mathbb{R}^{01} on $\Pi TM = \underline{\mathcal{S}}(\mathbb{R}^{01}, M)$. In fact, the whole super Lie group $\underline{\text{Aut}}(\mathbb{R}^{01})$ of automorphisms of the super manifold \mathbb{R}^{01} acts on ΠTM . We briefly describe this action.

By definition, $\underline{\text{Aut}}(\mathbb{R}^{01})$ is the super Lie group representing the group-valued functor

$$S \mapsto \text{Aut}_S(\mathbb{R}^{01} \times S, \mathbb{R}^{01} \times S).$$

Here $\text{Aut}_S(\mathbb{R}^{01} \times S, \mathbb{R}^{01} \times S)$ is the group of automorphisms of $\mathbb{R}^{01} \times S$ that are compatible with the projection to S . We have the following result whose (easy) proof we will not give.

Lemma 6. *We have $\underline{\text{Aut}}(\mathbb{R}^{01}) \cong \mathbb{R}^{01} \rtimes \mathbb{R}^\times$, where \mathbb{R}^\times acts on \mathbb{R}^{01} by multiplication.*

Here the \mathbb{R}^{01} part corresponds to the translation action we described before, whereas $t \in \mathbb{R}^\times$ acts as $\theta \mapsto t\theta$. Using the S -point formalism and proposition 1 one sees that there is a natural action of $\underline{\text{Aut}}(\mathbb{R}^{01})$ on ΠTM . In local coordinates, the action of \mathbb{R}^\times on ΠTM is given by

$$(t, y_1, \dots, y_{p+q}, \hat{y}_1, \dots, \hat{y}_{p+q}) \mapsto (y_1, \dots, y_q, t\hat{y}_1, \dots, t\hat{y}_{p+q}).$$

Differentiating w.r.t. t yields that the vector field G that implements the action of the even left-invariant vector field $t \frac{\partial}{\partial t} + \theta \frac{\partial}{\partial \theta}$ on $\mathbb{R}^{01} \rtimes \mathbb{R}^\times$ can locally be expressed as

$$G = \sum_{i=1}^{p+q} \hat{y}_i \frac{\partial}{\partial \hat{y}_i}.$$

Now, every function f on ΠTM which is polynomial on every fiber is locally a finite sum of functions of the form

$$f = g \hat{y}_1^{i_1} \dots \hat{y}_{p+q}^{i_{p+q}}, \text{ where } g \in \mathcal{O}_M.$$

Using the product rule we find

$$G(f) = \left(\sum_{k=1}^{p+q} i_k \right) f.$$

Hence on Ω^*M the vector field G acts the grading operator, i.e. it maps α of cohomological degree s to $s\alpha$.

REFERENCES

- [BBH] Claudio Bartocci, Ugo Bruzzo, Daniel Hernández-Ruipérez, *The Geometry of Supermanifolds*
- [DM] Pierre Deligne, John W. Morgan, *Notes on Supersymmetry (following Joseph Bernstein)*
- [Ma] Yuri Manin, *Gauge Theory and Complex Geometry*

³Note that in the purely even case our definition of Ω^*M coincides with the usual one (when ignoring the grading).