

UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Supersymmetry in the Stolz-Teichner Project on Elliptic Cohomology**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Mathematics

by

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2006

For my mother

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## ABSTRACT OF THE DISSERTATION

### **Supersymmetry in the Stolz-Teichner Project on Elliptic Cohomology**

by

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Doctor of Philosophy in Mathematics

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We investigate the role of supersymmetry in the Stolz-Teichner project on elliptic cohomology. We systematically develop the theory of super semigroups of (self-adjoint, compact) operators on Hilbert space and apply it to give a simplified proof of the relation between  $K$ -theory and supersymmetric Euclidian field theories of dimension  $(0+1|1)$  described in [ST]. Furthermore, we present results towards a more geometric description of these theories. We then turn to the role of supersymmetry in the conjectural relation between conformal field theory and elliptic cohomology, addressing the question raised in [ST] as to which notion of ‘super conformal surfaces’ is appropriate in the CFT approach to elliptic cohomology. We explain how the classical notion of SUSY curves gives rise to a notion of supersymmetric conformal field theories that satisfies the requirements demanded by Stolz and Teichner. The key observation we use is that every complex supermanifold has an underlying  $cs$  manifold. It should be pointed out that our definition of supersymmetric CFTs is not quite complete at this point. However, we investigate in detail the aspects relevant to the holomorphicity properties of the partition functions of such supersymmetric CFTs.

# Chapter 1

## Introduction

### 1.1 Elliptic cohomology and conformal field theory

Ever since the introduction of elliptic cohomology theories in the 1980s, the question of their geometric nature has been an intriguing problem to many mathematicians. While the homotopy-theoretic development of these theories has flourished, their geometric meaning has remained mathematically mysterious. The main hope is to establish a relationship between elliptic cohomology and an index theory for Dirac operators on loop space. The expectation that such a connection should exist stems from the interpretation of the Witten genus as the index of a heuristic  $S^1$ -equivariant Dirac operator on loop space. The finite-dimensional analogue of and model for this conjectural relation is the interplay between  $K$ -theory and Dirac operators described by the famous index theory of Atiyah and Singer.

#### Elliptic cohomology and the Witten genus

The origin of elliptic cohomology lies in Serge Ochanine's work on strongly multiplicative ('elliptic') genera and their relation to elliptic curves and modular forms, cf. [HBJ]. An important hint as to what the geometric significance of these invariants should be was given by Edward Witten who heuristically interpreted the universal elliptic genus  $\varphi(M)$  of a spin manifold  $M$  as the 'index of the  $S^1$ -equivariant signature operator on the free loop space  $\mathcal{L}M$ ', see [Wi3]. In the same paper, he also derived the index of a

formal  $S^1$ -equivariant Dirac operator on  $\mathcal{L}M$ . This lead to the *Witten genus*, a manifold invariant that, just like the universal elliptic genus, has close connections to modular forms. Witten's considerations resulted in the expectation that it should be possible to develop an index theory on the loop space  $\mathcal{L}M$  and that there should be a corresponding cohomology theory, *elliptic cohomology*, taking the place of  $K$ -theory in this context.

Since  $K$ -theory is the model for many considerations concerning elliptic cohomology, let us briefly recall the scenario in this case. Here the bridge between topology and analysis is the  $\hat{A}$ -genus which associates to each oriented closed manifold  $M$  a rational number  $\hat{A}(M)$ . Now, assume that  $M$  is a spin manifold and denote by  $\mathcal{D}$  the associated Dirac operator on  $M$ . Then the Atiyah-Singer index theorem says that

$$\hat{A}(M) = \text{ind}(\mathcal{D}),$$

where  $\text{ind}(\mathcal{D}) = \dim(\ker \mathcal{D}) - \dim(\text{coker } \mathcal{D})$  is the index of  $\mathcal{D}$ . In particular, we see that for a spin manifold  $\hat{A}(M)$  is an integer. The resulting homomorphism  $\hat{A} : \Omega_*^{\text{spin}} \rightarrow \mathbb{Z}$  on spin bordism can be refined to give a ring map  $\hat{A} : \Omega_*^{\text{spin}} \rightarrow KO^{-*}(pt)$  to the coefficient ring of real  $K$ -theory. Even better, this homomorphism extends to a family version, i.e. a map of spectra  $\hat{A} : M\text{Spin} \rightarrow ko$ .<sup>1</sup>

One would like to create a similar setup for the Witten genus  $W$ . The Witten genus  $W(M)$  of an oriented closed manifold  $M$  is a formal power series  $W(M) \in \mathbb{Q}[[q]]$ . If  $M$  is string, i.e.  $M$  is spin and the characteristic class  $\frac{p_1}{2}(M)$  is zero, then  $W(M)$  is the  $q$ -expansion of an integral modular form of weight  $\frac{1}{2} \dim M$ . This leads to a ring map  $\Omega_*^{\text{string}} \rightarrow \text{MF}_{*/2}^{\mathbb{Z}}$  from string bordism to the ring of integral modular forms. Now the natural question arises: is there a cohomology theory that plays the role of  $ko$  in the case of the Witten genus? The coefficient ring of a good candidate should be closely related to  $\text{MF}_{*/2}^{\mathbb{Z}}$ , say rationally isomorphic.<sup>2</sup> Furthermore, the theory should admit a map from  $M\text{String}$  that gives a family version of the Witten genus.

Such a theory indeed exists and was constructed by Hopkins and his collaborators (unpublished, but see [Ho1], [AHS], [Ho2] for some information). They call the theory *tmf* for topological modular forms. It is the universal elliptic cohomology theory

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<sup>1</sup>Using the map  $\hat{A} : M\text{Spin} \rightarrow ko$  one can associate to every family  $F$  of spin  $n$ -manifolds parametrized by a space  $S$  an element  $\hat{A}(F) \in KO^{-n}(S)$  using the Pontryagin-Thom construction.

<sup>2</sup>It is natural to ask for a rational isomorphism, because  $\text{MF}_{*/2}^{\mathbb{Z}}$  does not carry any torsion information.

in the sense that it maps to all theories arising from elliptic curves. As in the  $K$ -theory case, there is also a periodic version of the theory; it is denoted  $\mathrm{TMF}$  and has period  $576 = 24^2$ . Extensive calculations using these theories have led to a good understanding of many homotopy-theoretic aspects of elliptic cohomology.

Of course, it would be very interesting to understand the abundant information encoded in  $tmf$  cohomology in geometric terms. Additional motivation for studying the geometric side of elliptic cohomology stems from the conjecture that the Witten genus is an obstruction for metrics of positive Ricci curvature on string manifolds. This is described in the beautiful paper [St] by Stephan Stolz. Unfortunately, since the construction of  $tmf$  and the string orientation  $MString \rightarrow tmf$  are purely homotopy-theoretic, they do not explain of the geometry behind elliptic cohomology. In the same way as  $KO^*(X)$  can be defined in terms of vector bundles over  $X$ , one would like to find ‘elliptic objects over  $X$ ’ that represent elements in elliptic cohomology  $\mathrm{TMF}^*(X)$ . Also, in view of the string orientation of  $\mathrm{TMF}$ , we would expect that there is an object of this type canonically associated to every string manifold  $M$ , representing the Thom class of  $M$  in elliptic cohomology. The string condition also comes up in the physical interpretation of the situation by Witten, to which we now return.

## The work of Witten and Segal

When studying supersymmetry breaking in the early 1980s, Witten noticed connections between supersymmetries in supersymmetric quantum mechanical systems and Dirac operators on spin manifolds.<sup>3</sup> He also considered the case of supersymmetric quantum field theories and pointed out that the supersymmetry operators of the supersymmetric nonlinear  $\sigma$ -model in  $1+1$ -dimensions may be regarded as Dirac-like operators on the loop space of the target manifold of the theory, cf. [Wi1], §4. This interpretation became relevant to the theory of elliptic cohomology when Witten used it to explain several key properties of the universal elliptic genus, see [Se2], [Wi2], [Wi3]. An important aspect of the QFT point of view in Witten’s work (as opposed to the ‘Dirac operator on loop space’ perspective by itself) is that it illuminates the modularity properties of

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<sup>3</sup>Note that quantum mechanics may be considered as quantum field theory in  $0+1$ -dimensions. In this way quantum mechanics is united with the formalism of QFT.

the universal elliptic genus and the Witten genus, cf. [Wi4]. For later reference, note that in the QFT picture the Witten genus  $W(M)$  of a string manifold  $M$  appears as the *partition function* of the supersymmetric nonlinear  $\sigma$ -model associated with  $M$ .

Based on the QFT viewpoint, Segal suggested a definition of *elliptic objects* in terms of holomorphic conformal field theories, see [Se2], §6. The mathematical notion of conformal field theory he employs is developed in his fundamental paper [Se1], where he defines a CFT as a (projective) functor from the bordism category of conformal surfaces to topological vector spaces. In order to obtain a notion of CFTs over a space  $X$ , he endows the objects and morphisms of the bordism category with maps to  $X$ . Unfortunately, it is not clear why this approach should yield a cohomology theory, the main problem being the excision axiom, see Section 1.1 in [ST]. Furthermore, it seems that holomorphic CFTs are too rigid to give elliptic cohomology, cf. [Se3]. Due to these problems, the development of a geometric description of elliptic cohomology was relatively stagnant throughout the 1990s. About five years ago, the QFT-based approach to elliptic cohomology was revived by the work of Stephan Stolz and Peter Teichner.

## 1.2 The Stolz-Teichner project and supersymmetry

In [ST] they introduce a modification of Segal's notion of elliptic objects called *Clifford elliptic objects*. There are three central new ideas in their paper. The most important one is probably the extension of CFTs to *3-tier theories*, addressing the excision problem of Segal's elliptic objects. We will not discuss this here, instead we refer the reader to [ST], especially chapters 4 and 5, for more information. Another new aspect is that their notion of conformal field theories consistently includes spin structures on bordisms and  $\mathbb{Z}_2$ -gradings on the associated Hilbert spaces. On the one hand, this enables Stolz and Teichner to introduce theories of *degree  $n$* , the second new concept in their approach. On the other hand, the spin structures and  $\mathbb{Z}_2$ -gradings are the first step towards *supersymmetry*, the third new feature presented in [ST].<sup>4</sup> It is this aspect of the Stolz-Teichner project that we are going to address in this dissertation.

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<sup>4</sup>In fact, spin structures,  $\mathbb{Z}_2$ -gradings, as well as supersymmetry are already mentioned in Segal's work from the late 1980s, cf. [Se1], page 25, and [Se2], sections 4 and 6. However, not only is the treatment of these aspects much more explicit in [ST], but also their concrete mathematical purpose is explained.

Before saying more about it, we would like to point out the following alternative formulation of the question for a geometric description of elliptic cohomology. Instead of constructing elliptic objects over  $X$  one might ask for geometric models for the spaces making up the spectrum  $\mathrm{TMF}$ , in a similar way as the Grassmannian and Fredholm operators give geometric models for  $BO$ . This is somewhat less satisfying than having genuine objects over each space  $X$ , but potentially an easier task. From this point of view, the philosophy behind the Stolz-Teichner project may be stated as follows: there are topological spaces of field theories

$$\mathcal{CFT}_n := \left\{ \begin{array}{l} \text{Supersymmetric 3-tier conformal} \\ \text{field theories of degree } n \end{array} \right\}$$

so that  $\{\mathcal{CFT}_{-n}\}_{n \in \mathbb{Z}}$  constitutes an  $\Omega$ -spectrum homotopy equivalent to  $\mathrm{TMF}$ . Of course, before one can possibly prove a result like this, it is necessary to specify what exactly is meant by a supersymmetric 3-tier CFT of degree  $n$ . A complete definition has, to this date, not been given. The main obstacles remaining seem to lie in the 3-tier aspect of the definition. Our goal here is to make a suggestion as to what kind of ‘supersymmetry’ is appropriate in the definition of these theories. In order to understand what this task entails, let us explain the purpose of supersymmetry in [ST].

### The role of supersymmetry in the Stolz-Teichner project

If the spaces  $\mathcal{CFT}_{-n}$  are supposed to yield  $\mathrm{TMF}$ , we should make sure that the coefficient rings of the two spectra coincide. The latter is rationally isomorphic to the ring of weak integral modular forms<sup>5</sup>

$$\mathrm{wMF}_*^{\mathbb{Z}} = \mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}] / (c_4^3 - c_6^2 - (12)^3 \Delta).$$

Thus, an important criterion guiding the construction of the spaces  $\mathcal{CFT}_n$  is that for each  $n$  there should be a rational isomorphism

$$Z : \pi_0(\mathcal{CFT}_n) \longrightarrow \mathrm{wMF}_{n/2}^{\mathbb{Z}}.$$

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<sup>5</sup>Weak modular forms are sometimes also called modular functions. A weak modular form is *integral* if all coefficients occurring in its  $q$ -expansion are integers.

According to the physical intuition, this map should be given by associating to a CFT  $E$  its partition function  $Z_E$ . Consequently, all elements in  $\mathcal{CFT}_n$  should have the property that their partition function is a weak integral modular form of weight  $n/2$ .

As pointed out above, Segal's original notion of elliptic objects was formulated using holomorphic CFTs. This ensures that the partition function of such a theory is holomorphic, which is certainly necessary if we want it to be a modular form. In fact, according to [Se1], §4, a CFT is holomorphic if and only if its partition function is holomorphic. Hence, in order to allow theories that are less rigid than holomorphic ones, but still have holomorphic partition functions, the CFT axioms must be modified. The suggestion of Stolz and Teichner is to allow a  $\mathbb{Z}_2$ -grading on the Hilbert spaces involved and to replace the trace in Segal's CFT axioms by the super trace. In this framework, the partition function of a CFT can be a weak modular form even though the theory is not holomorphic. The hope is that this makes it possible to find a class of CFTs with holomorphic partition functions in which deformations are possible.

This philosophy is supported by Theorem 3.3.14 in [ST] which says that the partition function of a CFT  $E$  in the sense of Stolz-Teichner is a weak integral modular form provided  $E$  satisfies the following condition. Let  $D_0 := \{ q \in \mathbb{C} \mid 0 < |q| < 1 \}$  be the punctured unit disc. Consider the semigroup homomorphism

$$\phi_E : D_0 \longrightarrow \text{HS}^{sa}(\mathcal{H})$$

defined by evaluating  $E$  on annuli whose boundary parametrizations are given by rigid rotations of  $S^1$ , see Section 4.1 for details. The assumption Stolz and Teichner need to prove Theorem 3.3.14 is that the derivative of  $\phi_E$  in  $\bar{q}$ -direction is the square of an *odd* operator (at 1). Note that this is trivially satisfied if  $E$  is holomorphic. The whole  $\phi_E$  business will be explained in detail in Section 4.1, and all we need to notice at this point is that the required condition looks a lot like a typical supersymmetry relation. Consequently, Stolz and Teichner suggest to consider CFTs that are 'supersymmetric' in an appropriate sense. Summarizing, we might say that Stolz and Teichner introduce a certain notion of supersymmetry in order to characterize a class of not necessarily holomorphic CFTs whose partition functions are weak integral modular forms.

We have already mentioned that  $K$ -theory is an important model for many

considerations concerning elliptic cohomology. This is also true for the supersymmetry aspect and a part of [ST] is devoted to the relationship between supersymmetric Euclidian field theories of dimension  $(0 + 1|1)$  and  $K$ -theory. It turns out that supersymmetry is indispensable also in this context. As in the CFT case, the partition function of a supersymmetric theory has special properties that one cannot expect from arbitrary EFTs: it is constant and, in fact, integer-valued. More importantly, supersymmetry ensures that the spaces  $\mathcal{EFT}_n$  of supersymmetric EFTs of degree  $n$  constructed by Stolz and Teichner have an interesting homotopy type: the analogous spaces of non-supersymmetric theories are contractible, whereas the spaces  $\mathcal{EFT}_{-n}$  constitute an  $\Omega$ -spectrum representing  $K$ -theory.

Now that we have hinted at the role of supersymmetry in the Stolz-Teichner project on elliptic cohomology, we are ready to say what our work, which to a large extent directly pertains to and is based on the work of Stolz and Teichner, is about.

### **How does this thesis fit into the picture?**

In order to put the current work into context, let us recall in more detail the parts of [ST] related to the definition of field theories and supersymmetry. In Section 2.1 Stolz and Teichner formulate the usual Atiyah-Segal axioms for QFTs and introduce the main examples they are interested in: Euclidian field theories of dimension  $0 + 1$  and conformal field theories of dimension  $1 + 1$ . These are based on the bordism categories  $\mathcal{EB}^1$  and  $\mathcal{CB}^2$  of 1-dimensional Euclidian bordisms and 2-dimensional conformal bordisms, respectively. Sections 2.2 and 2.3 introduce Clifford algebras associated with objects and Fock spaces associated with bordisms in the categories  $\mathcal{EB}^1$  and  $\mathcal{CB}^2$ . More precisely, this is done under the assumption that all manifolds involved are equipped with spin structures. The Clifford algebras and Fock spaces are then used to define (Clifford linear) EFTs and CFTs of degree  $n$ . Roughly speaking, such a theory associates (in a Clifford linear fashion) an operator to each pair  $(\Sigma, \eta)$  consisting of a spin bordism  $\Sigma$  and an element  $\eta$  in the  $-n^{\text{th}}$  tensor power of its Fock space. In particular, the Fock spaces take care of the ‘conformal anomaly’ in the CFT case.

In Chapter 3 Stolz and Teichner demonstrate the usefulness of the notion of theories of degree  $n$  and the necessity of supersymmetry. The first main result, see Section

3.2, concerns the relation between  $K$ -theory and supersymmetric EFTs of degree  $n$ : after defining the space  $\mathcal{EFT}_n$  of such theories, Stolz and Teichner show that it is homotopy equivalent to the  $-n^{\text{th}}$  space in the  $K$ -theory spectrum. Depending on whether the target category consists of real or complex Hilbert spaces, one obtains real or complex  $K$ -theory.

The treatment of supersymmetric (‘susy’) EFTs in [ST] is rather brief and in our chapters 2 and 3 we give an extended discussion of the supersymmetry aspect in the EFT context. The approach to susy EFTs in [ST] is mostly operator-theoretic: even though a discussion of Euclidian super intervals is included, susy EFTs of degree  $n$  are essentially characterized as super semigroup homomorphisms  $\mathbb{R}_{>0}^{1|1} \longrightarrow \text{HS}_{C_n}^{sa}(\mathcal{H})$ , i.e. as super semigroups of self-adjoint,  $C_n$ -linear Hilbert-Schmidt operators on a  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H}$ . In Chapter 2 and Section 3.1 we study such super semigroups of operators systematically, adding detail to the work of Stolz and Teichner on the one hand and giving a more elementary proof of their result concerning the spaces  $\mathcal{EFT}_n$  on the other hand. The remainder of Chapter 3 is devoted to developing a better geometric understanding of susy EFTs: we give results towards an Atiyah-Segal style characterization of supersymmetric Euclidian field theories.

The second main result in [ST] related to supersymmetry is the aforementioned Theorem 3.3.14 concerning CFTs  $E$  with holomorphic partition functions. Here supersymmetry is mainly present as the (operator-algebraic) condition imposed on the homomorphism  $\phi_E$ . However, as already pointed out, Stolz and Teichner expect that this condition should be satisfied for certain ‘supersymmetric’ conformal field theories. Furthermore, they suggest that this class of supersymmetric theories should arise in a natural way from an appropriate notion of a ‘super conformal structure’ on the bordisms of the domain category. This is formulated in Hypothesis 3.3.13 in [ST]. The aim of our chapters 4 and 5 is to explain how SUSY structures in the sense of Manin, see [Ma2], lead to a notion of supersymmetric CFTs so that the conditions of Hypothesis 3.3.13 are satisfied. We should point out that Stolz and Teichner (unpublished) also suggested a definition of susy CFTs which is specifically designed to have the properties they are looking for. The main point of our approach is to exhibit a connection between supersymmetry in the Stolz-Teichner project and the well-established notion of SUSY curves

(a.k.a. super Riemann surfaces). All this will be explained in greater detail below.

An extended summary, chapter by chapter, will be given in 1.4 and 1.5. As the reader might infer from what we have said up to now, a central problem is what the definition of a supersymmetric quantum field theory should be. Before explaining our idea for the definition in the next section, we want to conclude this one with the following remark about our terminology concerning field theories.

By a EFT or CFT we usually mean a theory in the sense of Stolz and Teichner. Whenever there is a ‘degree  $n$ ’ in the game, there is no ambiguity and we directly refer to their definition (for example, this applies to sections 3.1 and 4.1). In all sections that deal with geometric aspects of supersymmetry, however, we do not include the degree  $n$  in our discussion and the reader should mainly have theories of degree 0 in mind (this applies to sections 3.2 - 3.4 and all of Chapter 5). Note that even for degree 0 theories the domain and target categories feature spin structures and  $\mathbb{Z}_2$ -gradings. One should keep this in mind when comparing a ‘usual CFT of degree 0’ in our sense to Segal’s CFTs in [Se1]. Finally, the bordism categories in [ST] include automorphisms of the objects in addition to genuine bordisms. These ‘infinitely thin bordisms’ are omitted in our approach; in fact, allowing *odd* automorphisms results in unbounded operators (cf. the general formula for susy EFTs in 2.1 evaluated at  $t = 0$ ) and this makes for some unpleasant technical difficulties that we have avoided.

### 1.3 Supersymmetric QFTs and enriched categories

In this section we explain our general strategy for the definition of supersymmetric QFTs and outline how we will proceed in the examples we are interested in. The basic idea is to produce a supersymmetric version of the axiomatic approach to quantum field theory by Atiyah and Segal.

In mathematical language, a *quantum field theory* is usually described as a functor  $E$  from a bordism category  $\mathcal{B}$  to a category of (topological) vector spaces  $\mathcal{V}$  satisfying certain additional axioms, cf. [At], [Se1], [ST]. It is natural to require the dependence of the operators associated with bordisms to be nicely behaved (e.g. to be continuous or holomorphic). In order to make sense of such a condition one needs to

endow the morphism sets of  $\mathcal{B}$  and  $\mathcal{V}$  with additional structure (e.g. a topology or the structure of a complex manifold). Expressed in the language of category theory this amounts to making  $\mathcal{B}$  and  $\mathcal{V}$  into categories enriched over an appropriate category  $\mathcal{K}$  (e.g. topological spaces or complex manifolds) and requiring  $E$  to be a  $\mathcal{K}$ -functor.

Recall that a *category  $\underline{\mathcal{D}}$  enriched over  $\mathcal{K}$*  (or  *$\mathcal{K}$ -category*) consists of the following data: (1) A class of objects  $|\underline{\mathcal{D}}|$ , (2) for each pair of objects  $Y_1, Y_2 \in |\underline{\mathcal{D}}|$  a *morphism object*  $\underline{\mathcal{D}}(Y_1, Y_2)$  in  $\mathcal{K}$ , and (3) *composition morphisms* in  $\mathcal{K}$

$$\underline{\mathcal{D}}(Y_2, Y_3) \times \underline{\mathcal{D}}(Y_1, Y_2) \longrightarrow \underline{\mathcal{D}}(Y_1, Y_3)$$

satisfying the usual associativity axiom.<sup>6</sup> A  *$\mathcal{K}$ -functor*  $F$  between two  $\mathcal{K}$ -categories  $\mathcal{D}_1$  and  $\mathcal{D}_2$  associates to every object  $Y \in |\mathcal{D}_1|$  an object  $F(Y) \in |\mathcal{D}_2|$ , and for every pair  $Y_1, Y_2 \in |\mathcal{D}_1|$  it gives a morphism in  $\mathcal{K}$

$$F : \underline{\mathcal{D}}(Y_1, Y_2) \longrightarrow \underline{\mathcal{D}}(F(Y_1), F(Y_2)).$$

These morphisms are assumed to be compatible with the composition morphisms in the obvious way. We have omitted the discussion of identity elements in the definition, since they do not play an important role for our considerations. For a complete description of enriched categories we refer to [Ke].

For example, if  $\mathcal{K}$  is the category of topological spaces this just means that  $E$  is a continuous functor - a condition that every reasonable theory should satisfy. In the case of CFTs, taking  $\mathcal{K}$  to be the category of complex manifolds yields holomorphic conformal field theories. Our plan is to obtain a notion of *supersymmetric* quantum field theories by choosing  $\mathcal{K}$  to be a category  $\mathcal{S}$  of supermanifolds.

## Turning $\mathcal{B}$ and $\mathcal{V}$ into $\mathcal{S}$ -categories

Making the target category  $\mathcal{V}$  into an  $\mathcal{S}$ -category  $\underline{\mathcal{V}}$  is usually not difficult. For example, we will be interested in the case where  $\mathcal{V}$  is the category of  $\mathbb{Z}_2$ -graded Hilbert spaces and bounded operators between them. Then the space of bounded operators

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<sup>6</sup>In order to make sense of this, we need to assume that finite products exist in  $\mathcal{K}$ . We should point out that the definition a  $\mathcal{K}$ -category works for any monoidal structure  $\otimes$  on the category  $\mathcal{K}$ . However, in all cases we are going to consider the monoidal structure  $\otimes$  is given by the categorial product  $\times$  and we have thus restricted ourselves to this case.

from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  is a  $\mathbb{Z}_2$ -graded Banach space and can be interpreted as a supermanifold in a natural way. Roughly speaking, the  $\mathbb{Z}_2$ -grading gives the splitting into even and odd coordinates. Clearly, supermanifolds of this type will be infinite-dimensional. We will deal with this problem in a quite simple fashion by considering them as ‘generalized supermanifolds’, see Section 2.1 for details.

The much more interesting case is the bordism category  $\mathcal{B}$ . One way to construct the *morphism supermanifolds*  $\underline{\mathcal{B}}(Y_1, Y_2)$  is to consider *families of bordisms* between  $Y_1$  and  $Y_2$  parametrized by supermanifolds  $S$ . In order for this approach to yield genuine morphism supermanifolds we should start out with a bordism category  $\mathcal{B}$  whose morphisms are supermanifolds. Once a notion of ‘families’ is specified, we can define  $\underline{\mathcal{B}}(Y_1, Y_2)$  by the universal property that there are bijections

$$\mathcal{S}(S, \underline{\mathcal{B}}(Y_1, Y_2)) \cong \left\{ \begin{array}{l} \text{Families of bordisms between } Y_1 \text{ and } Y_2 \\ \text{parametrized by } S \text{ modulo isomorphism} \end{array} \right\},$$

naturally in  $S$ . In other words,  $\underline{\mathcal{B}}(Y_1, Y_2)$  is a representing object for the functor that associates to a supermanifold  $S$  the set on the right hand side.<sup>7</sup> By the Yoneda lemma, lifting composition of bordisms to a map of supermanifolds

$$\underline{\mathcal{B}}(Y_2, Y_3) \times \underline{\mathcal{B}}(Y_1, Y_2) \longrightarrow \underline{\mathcal{B}}(Y_1, Y_3)$$

amounts to defining gluing of families.

As is well-known, the existence of  $\underline{\mathcal{B}}(Y_1, Y_2)$  is problematic when automorphisms are present, and in these cases one might end up with a ‘super stack’ rather than a supermanifold. In the cases we are interested in, automorphisms do not appear as long as all components of the bordisms under consideration have non-empty boundary. An example of what can happen in the case of closed components is described in Section 3.3, where we will see that the connected endomorphisms of the empty set in the category of Euclidian super bordisms are parametrized by a ‘super orbifold’. A strategy less elegant than allowing stacks is to simply ignore closed components and to enrich  $\mathcal{B}(Y_1, Y_2)$  ‘away from closed components’.

Another thing that can happen is that  $\underline{\mathcal{B}}(Y_1, Y_2)$  is a supermanifold, but infinite-dimensional. For instance, in the context of CFTs the boundary parametrizations give

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<sup>7</sup>This association is indeed a (weak) functor: on morphisms it is defined by pullback of families.

rise to infinite-dimensional spaces, cf. [Se1], §2 and §4. Clearly, this is a case we would like to allow. Our considerations concerning supersymmetric CFTs in Chapter 5 certainly do not go deep enough to force us to deal with technical questions like this. However, it seems that for a complete treatment of supersymmetric CFTs in our spirit the development of a theory of infinite-dimensional supermanifolds will be necessary.

Altogether, we see that the category  $\mathcal{S}$  will have to be more general than just (finite-dimensional) supermanifolds.

### Supersymmetric quantum field theories

Now, assume that we managed to produce the  $\mathcal{S}$ -categories  $\underline{\mathcal{B}}$  and  $\underline{\mathcal{V}}$ . Then we may define a *supersymmetric quantum field theory* as an  $\mathcal{S}$ -functor  $\underline{E} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{V}}$  satisfying the typical additional field theory axioms, e.g. the compatibility between disjoint unions in  $\underline{\mathcal{B}}$  and tensor products in  $\underline{\mathcal{V}}$ . For details, see Section 2.1 in [ST].

#### Remarks.

1. Note that every  $\mathcal{S}$ -category/functor has an underlying ordinary category/functor, because every supermanifold has an underlying set. Consequently, every supersymmetric QFT  $\underline{E}$  has an underlying non-supersymmetric QFT  $E : \mathcal{B} \rightarrow \mathcal{V}$ .
2. Of course, the ‘additional axioms’ now have to be formulated in the context of  $\mathcal{S}$ -categories. One thing to notice is that one should not necessarily expect the anti-involution to be a contravariant  $\mathcal{S}$ -functor. For example, in the case of holomorphic CFTs the anti-involution is given by passing to the complex conjugate of a Riemann surface; this is certainly not an holomorphic operation on moduli space.
3. It is possible to ignore the representability question of the ‘family functor’ altogether and to still obtain a definition of supersymmetric QFTs. This is the approach currently pursued by Stolz and Teichner. We will say a bit more about this in Section 5.3.
4. Working with representing objects will allow us to think of supersymmetric CFTs as operator-valued functions on the moduli space of SUSY curves (or on super

Teichmüller space) that are compatible with the gluing operation. This connects our approach with a point of view sometimes taken in theoretical physics.

### Examples: Supersymmetric EFTs and CFTs

Although we hope that the formalism described above will also be useful in other contexts, we will only consider the two examples of supersymmetric QFTs that are related to  $K$ -theory and elliptic cohomology: susy EFTs of dimension  $(0 + 1|1)$  and susy CFTs of dimension  $(1 + 1|1)$ . In both cases we start with non-supersymmetric theories which we want to turn into supersymmetric ones. More precisely, we would like supersymmetric extensions of our theories in the sense that the underlying non-susy QFTs are EFTs and CFTs of the type we started out with. The non-supersymmetric theories we look at are based on the bordism categories whose morphisms are spin bordisms with a Riemannian (in physics lingo: Euclidian) or conformal structure, resp. (see [ST] and also [Se1], §7). In order to have a chance to turn these into interesting  $\mathcal{S}$ -categories, we have to replace them by equivalent bordism categories of supermanifolds; in both our examples we will add one odd (‘Fermionic’) dimension.

(1) *Supersymmetric EFTs*. In the EFT case we replace Euclidian spin bordisms by  $(1|1)$ -dimensional supermanifolds equipped with a *super Euclidian structure*, see Section 3.2. Although the resulting category  $\mathcal{E}$  is equivalent to the category of Euclidian spin bordisms, even as a topological category, we see that something interesting has happened once we look at the associated category  $\underline{\mathcal{E}}$  enriched over (real) supermanifolds. For example, while the semigroup of Riemannian spin intervals (i.e. of connected bordisms from a point to itself) is  $\mathbb{R}_{>0} \times \mathbb{Z}_2$ , the super semigroup of Euclidian super intervals is  $\mathbb{R}_{>0}^{1|1} \rtimes \mathbb{Z}_2$ , i.e. it has one Fermionic dimension that we did not notice before. All this is explained in Section 3.3.

Unfortunately, I was not able to extend the ‘additional structures’ (see Section 2.1. in [ST]) needed to formulate the ‘additional axioms’ to the  $\mathcal{S}$ -category  $\underline{\mathcal{E}}$ : I do not know how to define the (anti)involutions in the super case. The reason behind this seems to be that all automorphisms of  $\mathbb{R}^{1|1}$  respecting the standard super Euclidian structure are orientation-preserving. Consequently, I was not able to give a formally complete definition of susy EFTs following the recipe explained above. However, I hope that

the considerations in Chapter 3 give some indication as to what the geometry behind supersymmetric Euclidian field theories looks like.

(2) *Supersymmetric CFTs.* In the CFT case we start out with the bordism category whose morphisms are Riemann surfaces with boundary equipped with a spin structure. It is well-known (see [Ma2] and [LR]) that there is a very interesting notion of families of super Riemann surfaces (a.k.a. SUSY curves) such that families over a point are the same as conformal spin surfaces. Our suggestion is that the bordism category  $\mathcal{C}$  based on SUSY curves leads to a notion of supersymmetric CFTs with the properties Stolz and Teichner are looking for.

This calls for some explanation, because Remark 3.3.18 in [ST] contains an argument as to why SUSY curves are *not* the right thing in the context of elliptic cohomology. The point made is that SUSY curves give too much supersymmetry, making the partition function of such a theory not only holomorphic (as desired) but constant and thus boring. In this argument a supersymmetric CFT is, roughly speaking, considered as an operator-valued function on the *real analytic supermanifold* underlying the moduli space of SUSY curves. What we have in mind is something slightly different: we want to look at the *cs manifold* underlying the moduli space of SUSY curves.<sup>8</sup>

Recall that our reason to introduce supersymmetry was that we did not want to restrict ourselves to holomorphic CFTs. Thus we would like to consider operator-valued functions on the moduli space of SUSY curves with parametrized boundaries  $\mathcal{M}$  that are not holomorphic, but merely smooth. In order to do so, we have to consider  $\mathcal{M}$ , which is (something like) a complex supermanifold, as a smooth supermanifold. Now, instead of passing to the underlying real supermanifold we propose to look at the associated *cs manifold*  $\mathcal{M}_{cs}$ , which should be thought of as the underlying smooth supermanifold with complex-valued functions. In other words, we want to make  $\mathcal{C}$  into a category  $\underline{\mathcal{C}}$  enriched over *cs manifolds* rather than over real (or complex) supermanifolds. We

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<sup>8</sup>*cs manifolds* are smooth supermanifolds with complex function rings. The notion does not have an (interesting) analogue in the non-super world, since there it does not matter whether one looks at smooth functions with real values or with complex values. In the super case, however, it does matter and one obtains a notion different from usual (real) supermanifolds. It seems that the category of *cs manifolds* is almost as natural of a ‘super’ extension of smooth manifolds as the category of (real) supermanifolds. In the literature, however, they do not seem to appear very often and the only reference we know of is the article of Deligne and Morgan, [DM]. More details about *cs manifolds* and their relation to complex supermanifolds can be found in sections 4.2 and 4.4.

should say that not much of this plan has been realized yet. For example, an important step would be the construction of the appropriate moduli space  $\mathcal{M}$ . A more detailed description of our ideas and further remarks can be found in Chapter 5.

## Super mathematics

The above outline for the definition of supersymmetric QFTs makes clear that super geometry plays an important role in our approach. We use the notion of supermanifolds based on the language of algebraic geometry, i.e. for us a supermanifold is a topological space with a sheaf of  $\mathbb{Z}_2$ -graded ('super') commutative algebras that locally looks like smooth functions tensored with an exterior algebra. Our main reference for super mathematics is the elegant *Notes on Supersymmetry* [DM] by Pierre Deligne and John Morgan, based on lectures by Joseph Bernstein. We will use the theory developed there without further reference. In particular, we will perform the formal computations typical in super geometry without providing additional explanations.

Other good sources for super mathematics in general and especially for complex supermanifolds and SUSY curves are Yuri Manin's books [Ma1], [Ma2]. More detailed accounts of basic super analysis can be found in [Lei] and [Var]. Further references will be given when we need them.

## 1.4 Outline of chapters 2 and 3

Chapters 2 and 3 are devoted to the relationship between supersymmetric Euclidian field theories of dimension  $(0 + 1|1)$  and  $K$ -theory. As explained in the previous section, I was not able to formulate a purely supersymmetric version of the Atiyah-Segal axioms that describes susy EFTs. Because of this, our considerations start out from an operator-theoretic point of view: at first, we will think of susy EFTs as super semigroups of self-adjoint Hilbert-Schmidt operators on a  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H}$ . This seems quite natural in view of the bijective correspondence between Euclidian field theories of dimension  $0 + 1$  and semigroups of self-adjoint Hilbert-Schmidt operators, see Proposition 3.1.1 in [ST]. This point of view motivates our study of *super semigroups of operators*, short SGOs, in Chapter 2.

## Chapter 2

It seems to us that SGOs are interesting objects to study, not only because of their relation to susy EFTs, but also since they are *the* odd analogue of semigroups of operators in the world of super mathematics. In any case, in Section 2.1 we introduce SGOs and give some examples arising from Dirac-type operators. In 2.2 we prove the existence of infinitesimal generators for super semigroups of compact, self-adjoint operators and explain how such generators may be interpreted as configurations on the real line. This is the basic step towards the classification of the SGOs that we are interested in.<sup>9</sup> In 2.3 we move on to study the space of SGOs on a fixed Hilbert space  $\mathcal{H}$ . We also look at the corresponding space of infinitesimal generators and at the space of  $C^*$ -homomorphisms introduced in [HG] and then show that the three spaces are homeomorphic. In 2.4 we define spaces  $\mathcal{SGO}_n$ ,  $n \in \mathbb{Z}$ , of super semigroups of self-adjoint, compact operators on a separable Hilbert space  $\mathcal{H}$  that commute with an action of the Clifford algebra  $C_n$  on  $\mathcal{H}$ . We then prove the main result of the chapter: for  $n \geq 1$  there are *homeomorphisms*

$$\mathcal{SGO}_n \approx \bar{\Omega}_{n-1}.$$

Here the latter is a completion of the Milnor space  $\Omega_{n-1}$  and is, in fact, homotopy equivalent to it. The spaces  $\Omega_n$  are introduced in Milnor's book on Morse theory [Mi], and it is shown there that  $\Omega_n$  is homotopy equivalent to the loop space of  $\Omega_{n-1}$ ; this is nothing but Bott's periodicity theorem. Thus we can conclude that the spaces  $\{\mathcal{SGO}_{-n}\}_{n \in \mathbb{Z}}$  form an  $\Omega$ -spectrum representing real  $K$ -theory  $KO$ . In particular, the space of all compact, self-adjoint SGOs on a separable Hilbert space is homotopy equivalent to  $BO \times \mathbb{Z}$ .

## Chapter 3

In Section 3.1 we introduce susy EFTs as super semigroups of self-adjoint Hilbert-Schmidt operators and use the results of Chapter 2 to deduce that the spaces  $\{\mathcal{EFT}_{-n}\}_{n \in \mathbb{Z}}$  of susy EFTs of degree  $n$  also represent the real  $K$ -theory spectrum. Hence we recover Theorem 1.0.1 in [ST]. Note that our proof of Theorem 1.0.1 is more direct

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<sup>9</sup>In particular, our discussion provides a precise proof of Lemma 3.2.14 in [ST]: the 'sketch of proof' given there does not include an argument for the surjectivity of the homeomorphism  $C^*(C_0(\mathbb{R}), \mathcal{K}_{C_n}(H)) \rightarrow \text{Hom}(\mathbb{R}_+^{1|1}, \mathcal{K}_{C_n}^{sa}(H))$ .

than the proof given in [ST]: it does not use the work of Higson and Guentner [HG] cited there. Furthermore, it shows that the *homeomorphism type* of  $\mathcal{EFT}_n$ ,  $n \geq 1$ , is closely related to the ‘small’ standard models for the classifying spaces of  $K$ -theory, cf. [Mi].

In the remainder of Chapter 3 we present some results towards an interpretation of susy EFTs in more geometric terms, following the strategy outlined in Section 1.3. In 3.2 we define families of Euclidian super curves and bordisms. In our formulation we have tried to stress the analogy between (real) Euclidian super curves and SUSY curves, compare definitions 5 and 7. In Section 3.2 we prove the main result of the chapter: we compute that the super semigroup of Euclidian super intervals is  $\mathbb{R}_{>0}^{1|1} \rtimes \mathbb{Z}_2$ . We also describe the super orbifold of closed connected Euclidian super curves. These computations are the basic steps in the construction of the  $\mathcal{S}$ -category  $\underline{\mathcal{E}}$ . We mainly investigate Euclidian super curves through their relation to the theory of odd ordinary differential equations on supermanifolds. We have summarized the basic facts about this in the ‘appendix’ 3.5. Finally, in Section 3.4 we conclude the chapter with some remarks on the definition of susy EFTs.

## 1.5 Outline of chapters 4 and 5

In Chapters 4 and 5 we turn to the role of supersymmetry in the conjectural relation between conformal field theory and elliptic cohomology. More precisely, our goal is to propose an answer to the question raised in [ST] as to which notion of supersymmetry on conformal spin surfaces gives the conditions of Hypothesis 3.3.13 in [ST]. The philosophy of our approach may be summarized as saying that the classical notion of  $N = 1$  supersymmetry is a good candidate - if one interprets the complex analytic moduli space of SUSY curves as a smooth supermanifold in the right way, namely, as a *cs* manifold.

### Chapter 4

In Section 4.1 we recall some basic material concerning partition functions of CFTs, their relation to semigroups of annuli, and the supersymmetry condition imposed on the homomorphism  $\phi_E$  that we already mentioned in Section 1.2. In 4.2 we intro-

duce  $cs$  manifolds, which play a central role in our approach. In Section 4.3 we make the first step towards finding the appropriate notion of supersymmetry: we describe a  $cs$  semigroup  $\bar{\mathfrak{A}}_{cs}$  that contains the semigroup  $D_0$  and has the property that a semigroup homomorphism

$$\phi : D_0 \longrightarrow \text{HS}(\mathcal{H}), \quad \phi(z, \bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0},$$

extends to  $\bar{\mathfrak{A}}_{cs}$  if and only if  $\bar{L}_0$  is the square of an odd operator  $\bar{G}_0$  that commutes with  $L_0$ . This is precisely the condition we wanted  $\phi_E$  to satisfy. We also point out the relation between the  $cs$  semigroup  $\bar{\mathfrak{A}}_{cs}$  and the ‘super conformal structures’ suggested by Stolz and Teichner after [ST] was written. Our considerations show, in particular, that their notion of ‘super conformal structure’ indeed satisfies the second condition in Hypothesis 3.3.13.

In 4.4 we point out the relation between complex supermanifolds and  $cs$  manifolds that is fundamental to our idea of connecting the notion of supersymmetry in the Stolz-Teichner project to complex analytic super geometry, more specifically, to SUSY curves. Namely, we prove that every complex supermanifold  $M$  has an underlying (‘smooth’)  $cs$  manifold  $M_{cs}$  obtained by forgetting the notion of holomorphicity on  $M$ . In particular, see 4.5, this enables us to identify the  $cs$  semigroup  $\bar{\mathfrak{A}}_{cs}$  as the  $cs$  semigroup underlying a complex super semigroup  $\bar{\mathfrak{A}}$ . It is this observation that allows us to interpret  $\bar{\mathfrak{A}}_{cs}$  as a  $cs$  semigroup SUSY curves in Chapter 5.

## Chapter 5

We should begin with the warning that much of the material contained in Chapter 5 is not as mathematically developed as it should be and apologize to the reader for our lack of mathematical rigor. This applies especially to Section 5.3, where we present ideas rather than mathematical theory. However, we wanted to include this material here, because we think that it gives strong evidence that SUSY curves and supersymmetry in the Stolz-Teichner project are closely related.

While Chapter 4 only dealt with the aspects of supersymmetric CFTs directly related to their partition functions, we now want to look at supersymmetric CFTs as a whole. Our suggestion for a definition follows the plan described in Section 1.3. Thus, the first step is to define families of SUSY bordisms; this is done in 5.1. As an example,

we consider a super semigroup of particularly simple SUSY annuli, thereby completing the discussion of Chapter 4. In Section 5.2 we compare SUSY curves to the conformal  $cs$  surfaces suggested by Stolz and Teichner, giving some indication that there is probably a close connection between the two concepts. Finally, in 5.3 we outline our definition of supersymmetric conformal field theories. We do so without giving any substantial amount of details and concentrate on making our ideas plausible. Roughly speaking, we want to define a susy CFT as an operator-valued function on  $\bar{\mathcal{M}}_{cs}$ , the  $cs$  manifold underlying the complex conjugate of the moduli space of SUSY curves with parametrized boundaries, that is compatible with the gluing operation. The point of taking the complex conjugate is that we want supersymmetry in  $\bar{z}$ -direction rather than in  $z$ -direction. Finally, we remark on the advantages and disadvantages of our approach, especially when compared to the definition proposed by Stolz and Teichner (unpublished).

## Chapter 2

# Super Semigroups of Operators and $K$ -Theory

We study super semigroups of operators following the outline in Section 1.4. Although we only deal with the case of real SGOs, we should point out that the arguments also work in the complex case. For example, the space of SGOs (or susy EFTs) on a complex Hilbert space is homotopy equivalent to  $BU \times \mathbb{Z}$ .

### 2.1 Super semigroups of operators

In this section we will define super semigroups of operators using as little super mathematics as possible. We will only need basic definitions and results from the theory of supermanifolds, as can be found in Chapter 2 of [DM].

#### The twisted super group $\mathbb{R}^{1|1}$

Define the ‘twisted’ super Lie group structure on  $\mathbb{R}^{1|1}$  by

$$\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1}, (t, \theta), (s, \eta) \mapsto (t + s + \theta\eta, \theta + \eta).$$

This super Lie group plays a special role in super geometry, the reason being the particular structure of its super Lie algebra:  $\text{Lie}(\mathbb{R}^{1|1}) \cong \mathbb{R}[D]$  is *the* super Lie algebra generated freely by one *odd* generator  $D$ . Thus,  $\mathbb{R}^{1|1}$  may be considered the *odd analogue of the*

*Lie group*  $\mathbb{R}$ . For example, integrating an odd vector field on a supermanifold  $M$  leads to a flow  $\mathbb{R}^{1|1} \times M \rightarrow M$ , and formulating the flow property involves the ‘twisted’ group structure, cf. Section 3.5.

From the definition of  $\mu$  it is clear that the open sub supermanifold  $\mathbb{R}_{>0}^{1|1}$  defined by the inclusion  $\mathbb{R}_{>0} \subset \mathbb{R}$  inherits the structure of a super semigroup.<sup>1</sup> Now we can already guess what a SGO should be: just as an ordinary semigroup of operators is a homomorphism from  $\mathbb{R}_{>0}$  to an algebra of operators, a super semigroup of operators will be a homomorphism from the super semigroup  $\mathbb{R}_{>0}^{1|1}$  to a ( $\mathbb{Z}_2$ -graded) operator algebra. In order to make sense of such a homomorphism, we will consider the latter to be a generalized super semigroup using the ‘functor of points’ formalism (see [DM], §§2.8-2.9). Note that we, implicitly, already used the ‘functor of points’ language when writing down the group law  $\mu$ . The formula above tells us what the product of two elements in the group  $\text{Hom}(S, \mathbb{R}^{1|1})$  is. Since the rule holds functorially for all supermanifolds  $S$ , this defines the map  $\mu$  by the Yoneda lemma.

Finally, we would like to remark that the structure of  $\text{Lie}(\mathbb{R}^{1|1})$  and the existence of an *odd* infinitesimal generator  $\mathcal{D}$  for a SGO  $\Phi$ , which we will prove in Section 2.2, are closely related:  $\mathcal{D}$  is nothing but the image of  $D$  under the derivative of  $\Phi$ . However, making this precise requires some work (note that  $\Phi$  maps to an infinite-dimensional space!). We will avoid such problems altogether: the super Lie algebras do not appear in our argument.

## Generalized supermanifolds and super Lie groups

We will use the following, somewhat primitive, extension of the notion of supermanifolds:

**Definition 1.** A *generalized supermanifold*  $M$  is a contravariant functor from supermanifolds to sets.<sup>2</sup> Similarly, if  $M$  takes values in the category of (semi)groups, we call it a *generalized super (semi)group*. Morphisms in all these categories are natural transformations.

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<sup>1</sup>A *super (Lie) semigroup* is a supermanifold  $M$  together with an associative multiplication  $M \times M \rightarrow M$ . In terms of the functor of points language: the morphism sets  $\text{Hom}(S, M)$  carry semigroup structures, functorially in  $S$ .

<sup>2</sup>We use this simple notion here in order to avoid dealing with infinite-dimensional supermanifolds.

## Examples

1. The Yoneda lemma implies that supermanifolds are embedded as a full subcategory in generalized supermanifolds by associating to a supermanifold  $M$  the functor

$$S \mapsto M(S) := \text{Hom}(S, M).$$

The analogous statement holds for super (semi)groups. For example, we will consider  $\mathbb{R}_{>0}^{1|1}$  as a generalized super semigroup by identifying it with the contravariant functor

$$S \mapsto \text{Hom}(S, \mathbb{R}_{>0}^{1|1})$$

from supermanifolds to semigroups.

2. Every  $\mathbb{Z}_2$ -graded real Banach space  $B = B_0 \oplus B_1$  may be considered as a generalized supermanifold as follows. We define the value of the functor  $B$  on a super domain  $U = (|U|, C^\infty(-)[\theta_1, \dots, \theta_q]) \subset \mathbb{R}^{p|q}$  to be

$$B(U) := (C^\infty(|U|, B)[\theta_1, \dots, \theta_q])^{ev}.$$

The superscript *ev* indicates that we pick out the even elements, so that an element  $f \in B(U)$  is of the form  $f = \sum_I f_I \theta^I$ , where  $I \subset \{1, \dots, q\}$ ,  $\theta^I := \prod_{j \in I} \theta_j$ , and each  $f_I$  is a smooth map  $|U| \rightarrow B_{|I|}$ .

For a map  $\varphi : U' \rightarrow U$  between super domains, the map  $B(\varphi)$  is defined using a formal Taylor expansion, just as in the case of usual supermanifolds. This functor on super domains may be extended to the whole category of supermanifolds by gluing. It is not hard to see that for  $B = \mathbb{R}^s \oplus \mathbb{R}^t$  the set  $B(U)$  coincides with the usual morphism set  $\text{Hom}(U, \mathbb{R}^{s|t})$ .

3. If  $B$  is a  $\mathbb{Z}_2$ -graded Banach algebra,  $B(U)$  is an algebra and thus  $B$  is a generalized super semigroup. Again,  $B$  may be extended to all supermanifolds by gluing.

## Remarks.

1. Giving a morphism from an ordinary supermanifold  $T$  to a generalized supermanifold  $B$  amounts to prescribing the image of the universal element  $\text{id} \in \text{Hom}(T, T)$  in  $B(T)$ . Hence  $B(T)$  is exactly the set of morphisms from  $T$  to  $B$ .

2. Now assume that, in addition,  $T$  and  $B$  carry super (semi)group structures. A map  $\Phi : T \rightarrow B$  is a homomorphism if

$$\begin{array}{ccc} \mathrm{Hom}(S, T) \times \mathrm{Hom}(S, T) & \longrightarrow & \mathrm{Hom}(S, T) \\ \downarrow \Phi \times \Phi & & \downarrow \Phi \\ B(S) \times B(S) & \longrightarrow & B(S). \end{array}$$

commutes for all supermanifolds  $S$ . Again, it suffices to check the commutativity for the universal element

$$pr_1 \times pr_2 \in \mathrm{Hom}(T \times T, T) \times \mathrm{Hom}(T \times T, T).$$

**Definition 2.** Let  $\mathcal{H}$  be a  $\mathbb{Z}_2$ -graded Hilbert space, and denote by  $B(\mathcal{H})$  the Banach algebra of bounded operators on  $\mathcal{H}$  equipped with the  $\mathbb{Z}_2$ -grading inherited from  $\mathcal{H}$ .

1. A *super semigroup of operators* on  $\mathcal{H}$  is a morphism of generalized super semigroups

$$\Phi : \mathbb{R}_{>0}^{1|1} \longrightarrow B(\mathcal{H}).$$

As explained above, such a map is of the form  $\Phi = A + \theta B$ , where

$$A : \mathbb{R}_{>0} \rightarrow B^{ev}(\mathcal{H}) \text{ and } B : \mathbb{R}_{>0} \rightarrow B^{odd}(\mathcal{H})$$

are smooth maps. The homomorphism property amounts to certain relations between  $A$  and  $B$  (cf. the proof of Proposition 1).

2. If  $K \subset B(\mathcal{H})$  is a subset, we say  $\Phi$  is a *super semigroup of operators with values in  $K$*  if the images of  $A$  and  $B$  are contained in  $K$ .
3. If  $\mathcal{H}$  is a module over the Clifford algebra  $C_n$ , we say  $\Phi$  is  *$C_n$ -linear* if it takes values in  $C_n$ -linear operators.

**Examples.** SGOs arise in a natural way from Dirac operators. We give two examples of that type and then extract their characteristic properties to describe a more general class of examples. The verification of the SGO properties for these more general examples also includes the case of Dirac operators.

1. Let  $\mathcal{D}$  be the Dirac operator on a closed spin manifold  $X$ . There is a corresponding SGO on the Hilbert space of  $L^2$ -sections of the spinor bundle  $S$  over  $X$ . It is given by the super semigroup of operators

$$\mathbb{R}_{>0}^{1|1} \longrightarrow L^2(S), (t, \theta) \mapsto e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2} (= e^{-t\mathcal{D}^2 + \theta\mathcal{D}})$$

and takes values in the compact, self-adjoint operators  $K^{sa}(L^2(S)) \subset L^2(S)$ .

2. If  $\dim X = n$ , one can consider the  $C_n$ -linear spinor bundle and the associated  $C_n$ -linear Dirac operator (see [LM], chapter 2, §7). Using the same formula as in the previous example one obtains a  $C_n$ -linear SGO.
3. Now, let  $\mathcal{H}$  be any  $\mathbb{Z}_2$ -graded Hilbert space. For any closed subspace  $V_\infty \subset \mathcal{H}$  invariant under the grading involution and any odd, self-adjoint operator  $\mathcal{D}$  on  $V_\infty^\perp$  with compact resolvent<sup>3</sup>, there is a unique super semigroup of self-adjoint, compact operators  $\Phi = A + \theta B$  defined (using functional calculus) by

$$A(t) = e^{-t\mathcal{D}^2} \text{ and } B(t) = \mathcal{D}e^{-t\mathcal{D}^2} \text{ on } V_\infty^\perp$$

and  $A(t) = B(t) = 0$  on  $V_\infty$ . The first thing to check is that the maps  $A$  and  $B$  are indeed smooth; this follows easily using the fact that the map  $\mathbb{R}_{>0} \rightarrow C_0(\mathbb{R})$ ,  $t \mapsto e^{-tx^2}$ , is smooth and the standard functional calculus estimate  $\|f(\mathcal{D})\| \leq \|f\|$ . Since  $\mathcal{D}$  is self-adjoint, the same holds for  $A$  and  $B$ . Finally, we have to show that  $\Phi$  is a homomorphism. Let  $t, \theta, s, \eta$  be the usual coordinates on  $\mathbb{R}_{>0}^{1|1} \times \mathbb{R}_{>0}^{1|1}$ . It suffices to check the homomorphism property on the universal element  $pr_1 \times pr_2 = (t, \theta) \times (s, \eta)$ . The computation, which, of course, heavily uses that odd coordinates

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<sup>3</sup>Recall that the definition of the adjoint of an operator  $\mathcal{D}$  requires the domain of  $\mathcal{D}$  to be dense. Thus, if we write ‘self-adjoint’ we should always assume that the operator is densely defined. We say that the self-adjoint operator  $\mathcal{D}$  has *compact resolvent* if its spectrum consists of eigenvalues of finite multiplicity that do not have an accumulation point in  $\mathbb{R}$ . This is equivalent to the condition that for one ( $\Rightarrow$  all)  $\mu$  in the resolvent set of  $\mathcal{D}$  the resolvent  $(\mu - \mathcal{D})^{-1}$  is compact. Note that such an operator  $\mathcal{D}$  is necessarily unbounded if its domain is infinite-dimensional. Abusing terminology, we will also refer to  $\mathcal{D}$  as a ‘self-adjoint operator on  $\mathcal{H}$ ’, but what we mean is always a subspace  $V_\infty$  together with an operator  $\mathcal{D}$  on  $V_\infty^\perp$ .

$\theta$  and  $\eta$  square to zero, goes as follows (cf. [ST], page 38):

$$\begin{aligned}
\Phi(t, \theta)\Phi(s, \eta) &= (e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2})(e^{-s\mathcal{D}^2} + \eta\mathcal{D}e^{-s\mathcal{D}^2}) \\
&= e^{-t\mathcal{D}^2}e^{-s\mathcal{D}^2} + e^{-t\mathcal{D}^2}\eta\mathcal{D}e^{-s\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}e^{-s\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}\eta\mathcal{D}e^{-s\mathcal{D}^2} \\
&= e^{-(t+s)\mathcal{D}^2} + (\theta + \eta)\mathcal{D}e^{-(t+s)\mathcal{D}^2} + \theta\mathcal{D}\eta\mathcal{D}e^{-(t+s)\mathcal{D}^2} \\
&= (1 - \theta\eta\mathcal{D}^2)e^{-(t+s)\mathcal{D}^2} + (\theta + \eta)\mathcal{D}e^{-(t+s)\mathcal{D}^2} \\
&= e^{-(t+s+\theta\eta)\mathcal{D}^2} + (\theta + \eta)\mathcal{D}e^{-(t+s+\theta\eta)\mathcal{D}^2} \\
&= \Phi(t + s + \theta\eta, \theta + \eta)
\end{aligned}$$

The second to last equality uses the typical Taylor expansion in super geometry.

We call  $\mathcal{D}$  the *infinitesimal generator* of  $\Phi$ . We will see presently that every super semigroup of self-adjoint, compact operators has a unique infinitesimal generator and is hence one of our examples.

Note that if  $V_\infty$  is a  $C_n$ -submodule and if  $\mathcal{D}$  is  $C_n$ -linear, then  $A$  and  $B$  will also be  $C_n$ -linear, i.e.  $\Phi$  takes values in  $C_n$ -linear operators.

## 2.2 Infinitesimal generators and configurations

Next, we will construct infinitesimal generators for super semigroups of operators. We restrict ourselves to the compact, self-adjoint case, which makes the proof an easy application of the spectral theorem for compact, self-adjoint operators. However, invoking the usual theory of semigroups of operators it should not be too difficult to prove the result for more general SGOs.

**Proposition 1.** *Every super semigroup  $\Phi$  of compact, self-adjoint operators on a  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H}$  has a unique infinitesimal generator  $\mathcal{D}$  as in the third example above and is hence of the form*

$$\Phi(t, \theta) = e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}.$$

If  $\Phi$  is  $C_n$ -linear, so is  $\mathcal{D}$ .

We need the following technical lemma:

**Lemma 2.** *Let  $A, B : \mathbb{R}_{>0} \rightarrow K^{sa}(H)$  be smooth families of self-adjoint, compact operators on the Hilbert space  $H$ , and assume that the following relations hold for all  $s, t > 0$ :*

$$A(s+t) = A(s)A(t) \quad (2.1)$$

$$B(s+t) = A(s)B(t) = B(s)A(t) \quad (2.2)$$

$$A'(s+t) = -B(s)B(t). \quad (2.3)$$

*Then  $H$  decomposes uniquely into orthogonal subspaces  $H_\lambda$ ,  $\lambda \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ , such that on  $H_\lambda$*

$$A(t) = e^{-t\lambda^2} \quad \text{and} \quad B(t) = \lambda e^{-t\lambda^2}$$

*(where we set  $e^{-\infty} = 0$ ,  $\infty \cdot e^{-\infty} = 0$ ). For  $\lambda \in \mathbb{R}$ , the dimension of  $H_\lambda$  is finite. Furthermore, the subset of  $\mathbb{R}$  consisting of  $\lambda$ 's with  $H_\lambda \neq 0$  is discrete.*

*Proof.* The identities (2.1)–(2.3) show that all operators  $A(s), B(t)$  commute. We apply the spectral theorem for self-adjoint, compact operators to obtain a decomposition of  $H$  into simultaneous eigenspaces  $H_\lambda$  of the  $A(s)$  and  $B(t)$ ; the label  $\lambda$  takes values in  $\bar{\mathbb{R}}$  and will be explained presently. We define functions  $A_\lambda, B_\lambda : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  by

$$A(t)x = A_\lambda(t)x \quad \text{and} \quad B(t)x = B_\lambda(t)x \quad \text{for all } x \in H_\lambda.$$

Clearly,  $A_\lambda$  and  $B_\lambda$  are smooth and satisfy the same relations as  $A$  and  $B$ .

From (2.1) we see that  $A_\lambda$  is non-negative, and (2.3) shows  $A'_\lambda \leq 0$ , hence  $A_\lambda$  is decreasing. On the other hand, (2.1) implies  $A_\lambda(\frac{1}{n}) = \sqrt[n]{A_\lambda(1)}$ , so that

$$A_\lambda(0) := \lim_{t \rightarrow 0} A_\lambda(t) \quad \text{exists and equals 0 or 1.}$$

In the first case we conclude  $A_\lambda \equiv 0$  and thus also  $B_\lambda \equiv 0$ ; the label of the corresponding subspace is  $\lambda = \infty$ . In the second case, we have  $A_\lambda(1) \neq 0$  and using (2.1) again we compute

$$A'_\lambda(s) = \frac{A_\lambda(1)}{A_\lambda(1)} \lim_{t \rightarrow 0} \frac{A_\lambda(s+t) - A_\lambda(s)}{t} = \frac{A_\lambda(s)}{A_\lambda(1)} \lim_{t \rightarrow 0} \frac{A_\lambda(1+t) - A_\lambda(1)}{t} = -\lambda^2 A_\lambda(s),$$

where  $\lambda^2 := -A'_\lambda(1)/A_\lambda(1)$  defines the label  $\lambda$  up to choice of a sign. By uniqueness of solutions of ODEs, we must have

$$A_\lambda(t) = e^{-t\lambda^2}.$$

Finally, (2.3) gives

$$B_\lambda(t) = \sqrt{\lambda^2 e^{-2t\lambda^2}} = \lambda e^{-t\lambda^2},$$

picking the appropriate sign for  $\lambda$ . □

*Proof of Proposition 1.* Let  $\Phi = A + \theta B$  be a super semigroup of compact, self-adjoint operators. As before, we consider  $U = \mathbb{R}_{>0}^{1|1} \times \mathbb{R}_{>0}^{1|1}$  with coordinates  $t, \theta, s, \eta$ . For the universal element  $pr_1 \times pr_2 = (t, \theta) \times (s, \eta)$  the homomorphism property of  $\Phi$  gives that

$$\begin{aligned} \Phi(t + s + \theta\eta, \theta + \eta) &= A(t + s + \theta\eta) + (\theta + \eta)B(t + s + \theta\eta) \\ &= A(t + s) + A'(t + s)\theta\eta + (\theta + \eta)(B(t + s) + B'(t + s)\theta\eta) \\ &= A(t + s) + \theta B(t + s) + \eta B(t + s) + \theta\eta A'(t + s) \end{aligned}$$

equals

$$\begin{aligned} \Phi(t, \theta)\Phi(s, \eta) &= (A(t) + \theta B(t))(A(s) + \eta B(s)) \\ &= A(t)A(s) + \theta B(t)A(s) + \eta A(t)B(s) - \theta\eta B(t)B(s). \end{aligned}$$

Comparing the coefficients<sup>4</sup> yields exactly the relations in Lemma 2. Using the corresponding decomposition of  $\mathcal{H}$  into subspaces  $\mathcal{H}_\lambda$  we define the operator  $\mathcal{D}$  by letting  $\mathcal{D} = \lambda$  on  $\mathcal{H}_\lambda$ . From the construction it is clear that  $\mathcal{D}$  is the desired infinitesimal generator. Since  $A$  is even and  $B$  is odd, it follows that  $\mathcal{D}$  is an odd operator. If  $\Phi$  is  $C_n$ -linear, so is  $\mathcal{D}$ . □

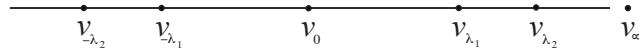
## Infinitesimal generators as configurations

Proposition 1 gives a bijective correspondence

$$\left\{ \begin{array}{l} \text{Super semigroups } \Phi \text{ of self-adjoint} \\ \text{compact operators on } \mathcal{H} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{Odd self-adjoint operators } \mathcal{D} \\ \text{on } \mathcal{H} \text{ with compact resolvent} \end{array} \right\}.$$

To be precise, each  $\mathcal{D}$  need only be defined on a closed (graded) subspace of  $\mathcal{H}$ . Each such operator  $\mathcal{D}$  can be visualized nicely as a *configuration*  $V = \{V_\lambda\}_{\lambda \in \mathbb{R}}$  on the real line indexed by orthogonal subspaces of  $\mathcal{H}$ . By this we mean a picture

<sup>4</sup>Just to make the formal aspect of this computation clearer, we would like to point out that the considered identity is an equation in the algebra  $K^{sa}(\mathcal{H})(\mathbb{R}_{>0}^{1|1} \times \mathbb{R}_{>0}^{1|1}) = C^\infty(\mathbb{R}_{>0} \times \mathbb{R}_{>0}, K^{sa}(\mathcal{H}))[\theta, \eta]^{ev}$ .



in which the labels  $V_\lambda$ ,  $\lambda \in \mathbb{R}$ , are finite-dimensional subspaces of  $\mathcal{H}$  that are mutually orthogonal. We require the subset of  $\lambda$ 's with non-zero  $V_\lambda$  to be discrete in  $\mathbb{R}$ ; however, infinity may be an accumulation point. Furthermore, we require the symmetry condition

$$V_{-\lambda} = \alpha(V_\lambda).$$

That is, the label at  $-\lambda$  is the image of the label at  $\lambda$  under the grading involution  $\alpha$  on  $\mathcal{H}$ . Sometimes we will think of the orthogonal complement all  $V_\lambda$  as a label  $V_\infty$  at infinity. Then  $\mathcal{H}$  is the Hilbert sum of all  $V_\lambda$ ,  $\lambda \in \bar{\mathbb{R}}$ .

The relation between infinitesimal generators  $\mathcal{D}$  and configurations is given by the spectral theorem for self-adjoint operators. Each label  $V_\lambda$  is just the  $\lambda$ -eigenspace of  $\mathcal{D}$ , and  $V_\infty$  is the orthogonal complement of the domain of  $\mathcal{D}$  (one might imagine that  $\mathcal{D} = \infty$  on  $V_\infty$ ). Since  $\mathcal{D}$  has compact resolvent, the subspaces  $V_\lambda$  are indeed finite-dimensional and the occurring  $\lambda$ 's are a discrete subset of  $\mathbb{R}$ . Finally,  $V_{-\lambda} = \alpha(V_\lambda)$  means precisely that  $\mathcal{D}$  is an odd operator.

## 2.3 Spaces of SGOs, configurations, and $C^*$ -homomorphisms

### Spaces of super semigroups of operators

Let  $\mathcal{H}$  be a  $\mathbb{Z}_2$ -graded Hilbert space and  $C \subset B(\mathcal{H})$  a subspace of the algebra of bounded operators on  $\mathcal{H}$ . We denote by  $\mathcal{SGO}(C)$  the set of super semigroups of operators with values in  $C$ . We endow  $\mathcal{SGO}(C)$  with the topology of uniform convergence on compact subsets, i.e.

$$\Phi_n = A_n + \theta B_n \longrightarrow \Phi = A + \theta B$$

if and only if for all compact  $K \subset \mathbb{R}_{>0}$  we have

$$A_n(t) \longrightarrow A(t) \text{ and } B_n(t) \longrightarrow B(t) \text{ uniformly on } K$$

with respect to the operator norm on  $B(\mathcal{H})$ .

## Configuration spaces

Denote by  $\text{Conf}(\mathcal{H})$  the set of all configurations on  $\mathbb{R}$  indexed by orthogonal subspaces of  $\mathcal{H}$ . We will now define a topology on  $\text{Conf}(\mathcal{H})$ ; the resulting topological space should be thought of as our model for the ‘space of infinitesimal generators’  $\mathcal{D}$  (using the identification introduced in the previous section).

For every bounded open subset  $U \subset \mathbb{R}$  define a semi-metric  $m_U$  on  $\text{Conf}(\mathcal{H})$  by

$$m_U(V, W) := \|V_U - W_U\|,$$

where  $V_U$  denotes the orthogonal projection onto the subspace  $\oplus_{\lambda \in U} V_\lambda$  of  $\mathcal{H}$  and  $\|\cdot\|$  is the operator norm. We define the topology on  $\text{Conf}(\mathcal{H})$  by declaring that a neighborhood subbasis of a configuration  $V \in \text{Conf}(\mathcal{H})$  is given by the sets

$$V_{U,\varepsilon} := \{ W \in \text{Conf}(\mathcal{H}) \mid m_U(V, W) < \varepsilon \},$$

where  $U \subset \mathbb{R}$  as above and  $\varepsilon > 0$ . In order to exclude undesired pathological behavior, we only allow  $U$ ’s such that  $\partial U \cap \{ \lambda \mid V_\lambda \neq 0 \} = \emptyset$ .

To get a better feeling for this topology, let us describe a neighborhood basis of a configuration  $V \in \text{Conf}(\mathcal{H})$ . Let  $K$  be a (large) positive real number such that  $V_K = 0$ . Denote by  $\lambda_1, \dots, \lambda_{l_K}$  the numbers in the open ball  $B_K(0)$  such that  $V_{\lambda_i} \neq 0$ . Let  $\delta > 0$  and  $\varepsilon > 0$  be (small) real numbers; we may choose  $\delta$  so small that  $B_\delta(\lambda_i) \cap B_\delta(\lambda_j) = \emptyset$  for  $i \neq j$ . Denote by  $V_{K,\delta,\varepsilon}$  the set of all configurations  $W$  such that  $m_{B_\delta(\lambda_i)}(V, W) < \varepsilon$  for all  $i$  and such that  $W_\lambda = 0$  for all  $\lambda \in B_K(0)$  that do not lie in one of the balls  $B_\delta(\lambda_i)$ . Thus, an element  $W \in V_{K,\delta,\varepsilon}$  almost looks like  $V$  on  $\bar{B}_K(0)$ : the only thing that can happen is that a label  $V_\lambda$  ‘splits’ into labels  $W_{\lambda_j}$  with  $|\lambda - \lambda_j|$  small ( $< \delta$ ) and  $\oplus_j W_{\lambda_j}$  close to  $V_\lambda$  ( $< \varepsilon$ ). In particular, we see that the topology on  $\text{Conf}(\mathcal{H})$  controls configurations well on compact subsets of  $\mathbb{R}$  but not near infinity. The sets  $V_{K,\delta,\varepsilon}$  form a neighborhood basis for  $V$ . The proof of this follows easily using the triangle inequality and the boundedness of the  $U$ ’s in the definition of the neighborhood subbases.

More information about configuration spaces can be found in [Mar]. As already mentioned, we will often think of elements in  $\text{Conf}(\mathcal{H})$  as unbounded operators  $\mathcal{D}$ . For example, we might speak of the functional calculus of  $\mathcal{D} \in \text{Conf}(\mathcal{H})$ : for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  we denote by  $f(\mathcal{D})$  the operator defined by  $f(\mathcal{D})|_{\mathcal{D}_\lambda} = \lambda$ .

### $C^*$ -homomorphisms

Another interpretation of the unbounded operators  $\mathcal{D}$  can be given in terms of  $C^*$ -homomorphisms, cf. [HG]. Let  $C_0(\mathbb{R})$  be the  $\mathbb{Z}_2$ -graded (real)  $C^*$ -algebra of real-valued continuous functions on the real line vanishing at infinity. Here the grading is given by the usual notion of ‘even’ and ‘odd’ functions, and the  $*$  is the identity. Now, if  $K$  is any other  $\mathbb{Z}_2$ -graded  $C^*$ -algebra, we can consider the set of grading-preserving  $C^*$ -homomorphisms  $C_{gr}^*(C_0(\mathbb{R}), K)$ , which we equip with the topology of pointwise convergence, i.e.

$$\varphi_n \longrightarrow \varphi \text{ if and only if } \varphi_n(f) \longrightarrow \varphi(f) \text{ for all } f \in C_0(\mathbb{R}).$$

It follows from the spectral theorem that every element in  $\varphi \in C_{gr}^*(C_0(\mathbb{R}), K(\mathcal{H}))$  is given by functional calculus on some infinitesimal generator  $\mathcal{D}$  (cf. [ST], Remark 3.2.13). This means that for all  $f \in C_0(\mathbb{R})$  we have

$$\varphi(f) = f(\mathcal{D}).$$

### Relations between the models

We have a triangle

$$\begin{array}{ccc} SGO(K^{sa}(\mathcal{H})) & \xleftarrow{R} & C_{gr}^*(C_0(\mathbb{R}), K(\mathcal{H})) \\ & \searrow I & \nearrow F \\ & \text{Conf}(\mathcal{H}) & \end{array}$$

Here  $I$  maps a super semigroup of operators to its infinitesimal generator,  $F$  is given by functional calculus,

$$F(\mathcal{D})(f) := f(\mathcal{D}),$$

and  $R$  is given by evaluating  $\varphi : C_0(\mathbb{R}) \rightarrow K(\mathcal{H})$  on the functions  $e^{-tx^2}$  and  $xe^{-tx^2}$ :

$$R(\varphi) := \varphi(e^{-tx^2}) + \theta\varphi(xe^{-tx^2}).$$

**Lemma 3.** *The maps  $I$ ,  $F$ , and  $R$  are homeomorphisms.*

*Proof.* From the previous discussion it is clear that the composition of the three arrows is the identity no matter where in the triangle we start. We complete the proof by showing that  $F$  and  $R$  are homeomorphisms.

1. To show that  $F$  is continuous, assume  $V_n \rightarrow V$  and fix  $f \in C_0(\mathbb{R})$ . We have to prove  $f(V_n) \rightarrow f(V)$ . Given  $\varepsilon > 0$ , choose  $K > 0$  such that  $|f(x)| < \varepsilon$  if  $|x| > K$ . Since the continuous map  $f$  is automatically uniformly continuous on compact sets, we can find a  $\delta > 0$  such that for all  $x \in B_K(0)$  we have  $|f(x) - f(y)| < \varepsilon$  provided  $|x - y| < \delta$ . The assumption  $V_n \rightarrow V$  tells us that  $V_n \in V_{K,\delta,\varepsilon}$  for large  $n$ . The claim now follows from the following estimate that holds for all  $W \in V_{K,\delta,\varepsilon}$ :

$$\begin{aligned}
\|f(V) - f(W)\| &= \left\| \sum_{\lambda \in \mathbb{R}} f(\lambda)V_\lambda - \sum_{\mu \in \mathbb{R}} f(\mu)W_\mu \right\| \\
&\leq \left\| \sum_{\lambda \in B_K(0), V_\lambda \neq 0} \left( f(\lambda)V_\lambda - \sum_{\mu \in B_\delta(\lambda)} f(\mu)W_\mu \right) \right\| + 2\varepsilon \\
&\leq \#\{\lambda \in B_K(0) \mid V_\lambda \neq 0\} \cdot \left( \max_{\lambda \in B_K(0)} f(\lambda) \cdot \varepsilon + \varepsilon \right) + 2\varepsilon \\
&\leq C \cdot \varepsilon,
\end{aligned}$$

where the constant  $C$  only depends on  $f$  and  $V$ . Abusing notation, we used the symbol  $V_\lambda$  here to denote the orthogonal projection onto  $V_\lambda$ . The first inequality follows by re-arranging the terms and using the triangle inequality together with  $|f(x)| < \varepsilon$  for  $|x| > K$ . The second inequality follows, since

$$\begin{aligned}
\|f(\lambda)V_\lambda - \sum_{\mu \in B_\delta(\lambda)} f(\mu)W_\mu\| &\leq \|f(\lambda)(V_\lambda - W_{B_\delta(\lambda)})\| + \left\| \sum_{\mu \in B_\delta(\lambda)} (f(\lambda) - f(\mu))W_\mu \right\| \\
&\leq \max_{\lambda \in B_K(0)} f(\lambda) \cdot \varepsilon + \varepsilon.
\end{aligned}$$

2. Now assume  $f(V_n) \rightarrow f(V)$  for all  $f$ . We need to show that for all admissible  $U$  we have  $m_U(V_n, V) \rightarrow 0$  as  $n \rightarrow \infty$ . Note that for a number  $\gamma \in \mathbb{R}$  that is an accumulation point of the set  $\bigcup_n \{\lambda \mid (V_n)_\lambda \neq 0\}$  we necessarily have  $V_\gamma \neq 0$ , because otherwise we would also have  $\|f(V_n) - f(V)\| \geq \frac{1}{2}$  for infinitely many  $n$  if we choose  $f$  to be a bump function with  $f(\gamma) = 1$  that is concentrated near  $\gamma$ . This together with  $\partial U \cap \{\lambda \mid V_\lambda \neq 0\} = \emptyset$  implies that there is a neighborhood

$\text{nb}(\partial U)$  of  $\partial U$  such that  $(V_n)_\lambda \neq 0$  for  $\lambda \in \text{nb}(\partial U)$  occurs only for finitely many  $n$ . Now, choose a function  $f \in C_0(\mathbb{R})$  such that  $f|_{\mathbb{R} \setminus U} = 0$  and  $f|_{U \setminus \text{nb} U} = 1$ . By construction,  $f(V_n) = \chi_U(V_n)$  for large  $n$ , where  $\chi_U$  denotes the indicator function for  $U$ . The same identity holds for  $V$  and hence we can conclude

$$\lim_{n \rightarrow \infty} m_U(V_n, V) = \lim_{n \rightarrow \infty} \|\chi_U(V_n) - \chi_U(V)\| = \lim_{n \rightarrow \infty} \|f(V_n) - f(V)\| = 0.$$

3. The continuity of  $R^{-1}$  follows from the following assertion. We claim that

$$f(\mathcal{D}_n) \longrightarrow f(\mathcal{D}) \text{ for all } f \in C_0(\mathbb{R}) \iff e^{-\mathcal{D}_n^2} \longrightarrow e^{-\mathcal{D}^2} \text{ and } \mathcal{D}_n e^{-\mathcal{D}_n^2} \longrightarrow \mathcal{D} e^{-\mathcal{D}^2}.$$

‘ $\implies$ ’ is obvious. To see ‘ $\impliedby$ ’, note that the assumption implies that  $f(\mathcal{D}_n) \rightarrow f(\mathcal{D})$  for all  $f$  that can be written as a polynomial in the functions  $e^{-x^2}$  and  $x e^{-x^2}$ . Furthermore, since  $e^{-x^2}$  and  $x e^{-x^2}$  generate  $C_0(\mathbb{R})$  as a  $C^*$ -algebra, cf. [HG], Remark 1.4, the set of such  $f$  is dense in  $C_0(\mathbb{R})$ . Using that  $\|f(\mathcal{D})\| \leq \|f\|$  for all  $\mathcal{D}$  and the triangle inequality we can deduce that  $f(\mathcal{D}_n) \rightarrow f(\mathcal{D})$  holds for all  $f \in C_0(\mathbb{R})$ .

4. The continuity of  $R$  amounts to showing that if  $f(\mathcal{D}_n) \rightarrow f(\mathcal{D})$  for all  $f$ , then  $e^{-t\mathcal{D}_n^2} \rightarrow e^{-t\mathcal{D}^2}$  and  $\mathcal{D}_n e^{-t\mathcal{D}_n^2} \rightarrow \mathcal{D} e^{-t\mathcal{D}^2}$  *uniformly* for all  $t$  in a compact subset  $K \subset \mathbb{R}_{>0}$ . As before, we can use  $\|f(\mathcal{D})\| \leq \|f\|$  and the triangle inequality to see that for a given  $\varepsilon > 0$  we can find  $N$  such that we do not only have  $\|f(\mathcal{D}_n) - f(\mathcal{D})\| \leq \varepsilon$  for all  $n \geq N$ , but that this estimate also holds for all  $g$  in a small neighborhood of  $f$ . This together with the compactness of  $K$  and the continuity of the maps  $t \mapsto e^{-tx^2}$  and  $t \mapsto x e^{-tx^2}$  implies the claim.

This completes the proof. □

**Remark.** The arguments in the last parts of the proof can be used to show that we could also have equipped  $\mathcal{SGO}(K^{sa}(\mathcal{H}))$  with the topology that controls *all* derivatives of a super semigroup map  $\Phi$  and still would have obtained the same topological space. We find this interesting, because this is the topology that one usually considers on spaces of smooth maps.

## 2.4 SGOs and $K$ -theory

Now we will now determine the homotopy type of the space of  $\mathcal{SGO}(K^{sa}(\mathcal{H}))$  of compact, self-adjoint SGOs on a separable  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H}$ . More generally, we will consider spaces of Clifford linear SGOs.

Let us fix some terminology. In the following, we denote by  $\mathcal{H}_n$  a  $\mathbb{Z}_2$ -graded separable Hilbert space that is also a graded  $C_n$ -module such that every generator  $e_i$  of  $C_n$  acts as a bounded, skew-adjoint operator. We assume that each irreducible graded module of  $C_n$  occurs infinitely many times in  $\mathcal{H}$  (this is only a condition if  $n \equiv 0 \pmod{4}$ ; in this case there are two such modules, whereas in all other cases there is a unique irreducible graded  $C_n$ -module). In order to get a meaningful definition of  $\mathcal{H}_n$  for all integers  $n$  we let  $C_n := C_{-n}^{\text{op}}$  for  $n < 0$ . Here  $(\cdot)^{\text{op}}$  denotes the opposite algebra in the graded sense, cf. [ST], Section 2.2.

### Definition 3.

1. For all integers  $n$  denote by  $\mathcal{SGO}_n$  the space of  $C_n$ -linear, self-adjoint, compact SGOs on  $\mathcal{H}_n$ ,

$$\mathcal{SGO}_n := \mathcal{SGO}(K_{C_n}^{sa}(\mathcal{H}_n)).$$

2. Let  $\text{Conf}_n$  be the corresponding space of  $C_n$ -linear infinitesimal generators.

### Remarks.

1. The definitions of  $\mathcal{SGO}_n$  and  $\text{Conf}_n$  are essentially independent of  $\mathcal{H}_n$ , since all possible choices for  $\mathcal{H}_n$  are isomorphic.
2. It follows from the classification of Clifford algebras (see [LM], Chapter 1) that for all  $n \in \mathbb{Z}$  the graded algebras  $C_n$  and  $C_{n+8}$  are Morita equivalent. This implies  $\mathcal{SGO}_n \approx \mathcal{SGO}_{n+8}$ , i.e. the spaces  $\mathcal{SGO}_n$  are 8-periodic, cf. [AS], Theorem 5.1.

From the triangle of the last section (with  $\mathcal{H} = \mathcal{H}_n$ ) we obtain, for each integer  $n$ , a triangle of homeomorphisms

$$\begin{array}{ccc} \mathcal{SGO}_n & \xleftarrow[\approx]{R} & C^*(C_0(\mathbb{R}), K_{C_n}(\mathcal{H}_n)) \\ & \searrow \approx I & \nearrow \approx F \\ & & \text{Conf}_n \end{array}$$

by looking at the corresponding subspaces of  $C_n$ -linear objects.

We will now prove that for  $n \geq 1$  the space  $\text{Conf}_n$  (and hence  $\mathcal{SGO}_n$ ) is homeomorphic to a completion of the Milnor space  $\Omega_{n-1}$ . To do so, it will be convenient to choose the following model for  $\mathcal{H}_n$ . For each  $n \geq 1$ , let  $H_n$  be a separable Hilbert space that is a  $C_{n-1}$ -module such that each irreducible representation of  $C_{n-1}$  appears with infinite multiplicity. Tensoring with  $C_n$  we obtain a graded  $C_n$ -module

$$\mathcal{H}_n := H_n \otimes_{C_{n-1}} C_n.$$

Here  $C_n$  acts on  $\mathcal{H}_n$  by multiplication from the right. The tensor product is formed using the embedding  $C_{n-1} \hookrightarrow C_n$  defined by the identification

$$C_{n-1} \xrightarrow{\cong} C_n^{ev}, \quad e_i \mapsto e_n e_i, \quad \text{for } i = 1, \dots, n-1.$$

It will be useful to consider  $\Omega_{n-1}$  as a subspace of the group of orthogonal operators  $O(H_n)$ . To do so, we interpret  $H_n$  as a module over  $C_{n-2}$  via

$$C_{n-2} \xrightarrow{\cong} C_{n-1}^{ev}, \quad \tilde{e}_i \mapsto e_{n-1} e_i,$$

where we denote by  $\tilde{e}_i$ ,  $i = 1, \dots, n-2$ , the standard generators of  $C_{n-2}$ .<sup>5</sup> It is easy to see that for  $n \geq 2$  the space  $\Omega_{n-1}$  may be described as the space of orthogonal operators  $J$  on  $H_n$  satisfying

- $J^2 = -1$ , or, equivalently,  $J$  is skew-adjoint.
- $J$  anti-commutes with  $\tilde{e}_1, \dots, \tilde{e}_{n-2}$ .
- $J = e_{n-1}$  on a subspace of finite codimension

(cf. [AS], pages 8-9, for all similar description). In the case  $n = 1$  we have, by definition,

$$\Omega_0 = \{ A \in O(H_1) \mid A \equiv \mathbf{1} \text{ modulo finite rank operators} \}.$$

Denote  $\bar{\Omega}_{n-1}$  be the closure of  $\Omega_{n-1}$  in  $O(H_n)$  with respect to the operator norm. This space differs from  $\Omega_{n-1}$  in that the last condition is replaced by

---

<sup>5</sup>The point of the tildes is that in the proof of Proposition 4 it will be useful to have different notations for elements in  $C_{n-2}$  and  $C_{n-1}$ .

- $J \equiv e_{n-1}$  modulo compact operators.

The completion  $\bar{\Omega}_n$  is homotopy equivalent to  $\Omega_n$ . This follows, for example, from the argument is given in the proof of Proposition 5 below.

**Proposition 4.** *For all  $n \geq 1$  we have homeomorphisms*

$$\text{Conf}_n \approx \bar{\Omega}_{n-1}.$$

*Proof.* Let us begin with the easiest case,  $n = 1$ . Since  $C_1 = \mathbb{C}$ ,  $\mathcal{H}_1 = H_1 \otimes \mathbb{C}$  is just the complexification of the real Hilbert space  $H_1$  and the grading involution on  $\mathcal{H}_1$  is given by complex conjugation. We use this to consider  $\mathcal{H}_1$  as a complex Hilbert space. Furthermore, the  $C_1$ -linearity of the operators  $\mathcal{D}$  in  $\text{Conf}_1$  means precisely that they are complex-linear operators on  $\mathcal{H}_1$ .

We claim that there is a homeomorphism

$$C : \text{Conf}_1 \xrightarrow{\approx} \{ A \in O(H_1) \mid A \equiv \mathbf{1} \text{ modulo } K(H_1) \} = \bar{\Omega}_0.$$

that is essentially given by the Cayley transform. More precisely, for  $\mathcal{D} \in \text{Conf}_1$  we consider the operator  $c(\mathcal{D})$  obtained by functional calculus using the homeomorphism

$$c : \bar{\mathbb{R}} \xrightarrow{\approx} S^1, x \mapsto \frac{x+i}{x-i}.$$

Since  $c$  takes values in  $S^1$ ,  $c(\mathcal{D})$  is unitary. Next, observe that the homeomorphism  $c$  maps  $\bar{\mathbb{R}}$  to  $S^1$  in such a way that  $c(\infty) = 1$ ,  $c(0) = -1$ , and  $c(-\lambda) = \overline{c(\lambda)}$  for all  $\lambda \in \mathbb{R}^\times$ . Thus, that  $\mathcal{D}$  is an odd operator means precisely that the conjugate of the  $\mu$ -eigenspace of  $c(\mathcal{D})$  is the  $\bar{\mu}$ -eigenspace of  $c(\mathcal{D})$ , i.e.  $c(\mathcal{D})$  commutes with complex conjugation. Hence  $c(\mathcal{D})$  is the complexification of an orthogonal operator  $C(\mathcal{D})$  on  $H_1$ . It is clear from the definition of  $\text{Conf}_1$  that  $C(\mathcal{D})$  is equal to  $\mathbf{1}$  modulo compact operators. From our construction we immediately see that  $C$  is invertible: we take the complexification of an operator  $A$  and take the inverse of the Cayley transform. Looking at the topologies, the continuity of both  $C$  and its inverse is not very surprising. For the sake of completeness, we give the proof in the next three paragraphs; the reader is invited to skip these.

Since the map  $O(H_1) \rightarrow U(\mathcal{H}_1)$ ,  $A \mapsto A \otimes \text{id}_{\mathbb{C}}$ , is an isometry, the continuity of  $C$  and  $C^{-1}$  amounts to showing that

$$\mathcal{D}_n \rightarrow \mathcal{D} \text{ as configurations} \iff c(\mathcal{D}_n) \rightarrow c(\mathcal{D}) \text{ w.r.t. the operator norm.}$$

Because  $c(\mathcal{D}) \rightarrow c(\mathcal{D}_n)$  if and only if  $c(\mathcal{D}) - \mathbf{1} \rightarrow c(\mathcal{D}_n) - \mathbf{1}$ , the implication ‘ $\Rightarrow$ ’ follows from the argument in part 1 of the proof of Lemma 3 (with  $V = \mathcal{D}$ ,  $W = \mathcal{D}_n$  for  $n$  large, and  $f = c - 1$ ).

Conversely, assume  $c(\mathcal{D}_n) \rightarrow c(\mathcal{D})$ . It follows from the triangle inequality that for every neighborhood  $U$  of the spectrum  $\sigma(c(\mathcal{D}))$ , there is an  $N$  such that  $\sigma(c(\mathcal{D}_n)) \subset U$  for all  $n \geq N$ . This implies that for every bounded subset of  $\mathbb{R}$ , say  $B_K(0)$ , there is an  $N$  such that for  $\mu \in B_K(0)$  and  $n \geq N$  the label  $(\mathcal{D}_n)_\mu$  can be non-trivial only if  $\mu$  lies within proximity of a  $\lambda$  with  $\mathcal{D}_\lambda \neq 0$ , say within a  $\delta$ -neighborhood. Now, in order to conclude that  $\mathcal{D}_n \rightarrow \mathcal{D}$  in the configuration topology, we must show that for  $\lambda \in B_K(0)$  with  $\mathcal{D}_\lambda \neq 0$  and  $n$  large

$$\| \mathcal{D}_\lambda - \sum_{\mu \in B_\delta(\lambda)} (\mathcal{D}_n)_\mu \|$$

becomes arbitrarily small. To do so, we choose  $K > 0$  so big that  $|c(x) - 1|$  is small for  $x$  outside of  $B_K(0)$ . Then an estimate similar to the one in part 1 of the proof of Lemma 3 can be used to show that if  $\|c(\mathcal{D}) - c(\mathcal{D}_n)\|$  is small, then the same is true for

$$\| \sum_{\lambda \in B_K(0), \mathcal{D}_\lambda \neq 0} c(\lambda) \cdot \left( \mathcal{D}_\lambda - \sum_{\mu \in B_\delta(\lambda)} (\mathcal{D}_n)_\mu \right) \|.$$

This reduces the problem to the following question. Consider an operator of the form

$$A = \sum_{\text{finite}} \gamma(P_\gamma - Q_\gamma),$$

where the  $P_\gamma$  are projections onto mutually orthogonal subspaces (same for the  $Q_\gamma$ ). Assume that  $\|A\|$  is small. We would like to show that in this case  $\|P_\gamma - Q_\gamma\|$  is small for all  $\gamma \neq 0$ . We may assume that the direct sum of the  $P_\gamma$  is the whole space  $\mathcal{H}$  by adding a term with  $\gamma = 0$ . Let  $h \in P_\rho(\mathcal{H})$ ,  $\|h\| = 1$ . We have

$$\|Ah\| = \left\| \sum_{\gamma} \gamma(P_\gamma - Q_\gamma)h \right\| = \left\| \sum_{\gamma} (\rho - \gamma)Q_\gamma h \right\|.$$

Since  $\|A\|$  is small, this means that  $\|Q_\gamma h\|$  is small for  $\gamma \neq \rho$ . This implies that  $\|Q_\rho h - h\|$  is small for  $h \in P_\rho(\mathcal{H})$ . Reversing the roles of  $P_\rho$  and  $Q_\rho$  gives that  $\|P_\rho h - h\|$  is small for  $h \in Q_\rho(\mathcal{H})$ ,  $\|h\| = 1$ . Taking these two results together it is easy to compute that  $\|P_\gamma - Q_\gamma\|$  is small for all  $h \in \mathcal{H}$ ,  $\|h\| = 1$ , which was what we wanted to see. This finishes the proof of the continuity of  $C$  and its inverse.

Now, let  $n \geq 2$ . Again, we can consider  $\mathcal{H}_n$  as the complexification of  $H_n$  via

$$\mathcal{H}_n = H_n \otimes_{C_{n-1}} C_n \cong H_n \otimes_{\mathbb{R}} \mathbb{C}.$$

In other words, we make  $\mathcal{H}_n = H_n \otimes_{C_{n-1}} C_n$  into a complex Hilbert space by defining multiplication by  $i$  to be given by the action of  $e_n \in C_n$ . As for  $n = 1$  we can now write down the map  $C$ , but we must also understand what the  $C_n$ -linearity of  $\mathcal{D}$  means for  $C(\mathcal{D})$ . Just as for  $n = 1$ , the relation  $\mathcal{D}e_n = e_n\mathcal{D}$  gives the  $\mathbb{C}$ -linearity of  $c(\mathcal{D})$ ; it does not lead to any condition on  $C(\mathcal{D})$ . We claim that the relations  $\mathcal{D}e_i = e_i\mathcal{D}$  for the remaining  $n - 1$  generators  $e_i$  of  $C_n$  imply that the generators  $e_i$  of  $C_{n-1}$  satisfy

$$e_i C(\mathcal{D}) = C(\mathcal{D})^{-1} e_i.$$

To see this, note that we have the relations<sup>6</sup>

$$e_i(\mathcal{D} \pm i) = (\mathcal{D} \mp i)e_i \text{ and } e_i(\mathcal{D} \pm i)^{-1} = (\mathcal{D} \mp i)^{-1}e_i$$

which together yield

$$e_i c(\mathcal{D}) = e_i(\mathcal{D} - i)(\mathcal{D} + i)^{-1} = (\mathcal{D} + i)(\mathcal{D} - i)^{-1}e_i = c(\mathcal{D})^{-1}e_i.$$

Note that all these are operators on  $\mathcal{H}_n$ . We assert that the same relation holds for the operator  $C(\mathcal{D})$  on  $H_n$ . First, note that since  $c(\mathcal{D})$  is  $\mathbb{C}$ -linear, i.e. it commutes with  $e_n$ , we have  $e_n e_i c(\mathcal{D}) = c(\mathcal{D})^{-1} e_n e_i$ . Next, one checks that under the isomorphism  $H_n \otimes_{C_{n-1}} C_n \cong H_n \otimes_{\mathbb{R}} \mathbb{C}$  the action of  $e_n e_i \in C_n$ ,  $i = 1, \dots, n - 1$ , corresponds to the automorphism  $e_i \otimes \text{id}$  of  $H_n \otimes \mathbb{C}$ . This together with the relation we computed for  $c(\mathcal{D}) = C(\mathcal{D}) \otimes \text{id}$  implies  $e_i C(\mathcal{D}) = C(\mathcal{D})^{-1} e_i$  for  $e_i \in C_{n-1}$ .

Hence we have constructed a map

$$C : \text{Conf}_n \longrightarrow \{ A \in O(H_n) \mid A \equiv \mathbf{1} \pmod{K(H_n)} \text{ and } e_i A = A^{-1} e_i \text{ for } i = 1, \dots, n - 1 \}$$

<sup>6</sup>The notation  $(\mathcal{D} \pm i)^{-1}$  best interpreted on each pair of eigenspaces  $V_\lambda \oplus V_{-\lambda}$  of  $\mathcal{D}$  separately; there it definitely makes sense and that is all we care about here.

and as in the case  $n = 1$  this is a homeomorphism.

The space on the right-hand side is not quite  $\bar{\Omega}_{n-1}$  yet. However, we claim that it can be identified with  $\bar{\Omega}_{n-1}$  by associating to an operator  $A$  the complex structure

$$J := e_{n-1}A \in \bar{\Omega}_{n-1}.$$

It is clear that  $J \equiv e_{n-1} \pmod{K(H_n)}$ . Furthermore,  $J$  is indeed a complex structure:

$$J^2 = e_{n-1}Ae_{n-1}A = e_{n-1}AA^{-1}e_{n-1} = -\mathbf{1}.$$

It remains to check that  $J$  anti-commutes with the generators  $\tilde{e}_1, \dots, \tilde{e}_{n-2}$  of  $C_{n-2}$ .

$$\tilde{e}_i J = e_{n-1}e_i e_{n-1}A = e_{n-1}e_i A^{-1}e_{n-1} = e_{n-1}Ae_i e_{n-1} = -e_{n-1}Ae_{n-1}e_i = -J\tilde{e}_i$$

This completes the proof. □

## Chapter 3

# Supersymmetric Euclidian Field Theories

In the first section of this chapter we give an ad-hoc definition of  $(0+1|1)$ -dimensional supersymmetric Euclidian field theories of degree  $n$  in terms of super semigroups of self-adjoint Hilbert-Schmidt operators. We then use the main result of Chapter 2 to deduce the homotopy type of the spaces  $\mathcal{EFT}_n$  of such theories. In the remainder of the chapter we attempt to give a geometric definition of susy EFTs (of degree 0) following the strategy laid out in Section 1.3.

### 3.1 The space of supersymmetric Euclidian field theories

**Definition 4.** Let  $\mathcal{H}$  be a  $\mathbb{Z}_2$ -graded Hilbert space.

1. A *supersymmetric Euclidian field theory*  $E$  of dimension  $(0 + 1|1)$  based on  $\mathcal{H}$  is a super semigroup of operators on  $\mathcal{H}$  that takes values in self-adjoint Hilbert-Schmidt operators  $\text{HS}^{\text{sa}}(\mathcal{H})$ .
2. If  $\mathcal{H}$  is a graded  $C_n$ -module and  $E$  is  $C_n$ -linear, we say that  $E$  has *degree*  $n$ .
3. For all  $n \in \mathbb{Z}$ , we denote by  $\mathcal{EFT}_n \subset \mathcal{SGO}_n$  the subspace of EFTs of degree  $n$ .

**Examples.** The examples of SGOs arising from infinitesimal generators  $\mathcal{D}$  we described in Section 2.1 define susy EFTs if the eigenvalues of  $\mathcal{D}$  converge to infinity sufficiently

fast. This is, for example, the case for Dirac operators on closed spin manifolds, cf. [LM], Chapter 3, §5.

**Proposition 5.** *We have homotopy equivalences*

$$\mathcal{EFT}_n \simeq \Omega_{n-1} \text{ for all } n \geq 1.$$

*In particular,  $\mathcal{EFT}_n$  represents the functor  $KO^{-n}$ .*

*Proof.* Define the space of finite rank configurations by

$$\text{Conf}_n^{fin} = \{ \{V_\lambda\}_{\lambda \in \mathbb{R}} \mid \dim(\oplus_{\lambda \in \mathbb{R}} V_\lambda) < \infty \} \subset \text{Conf}_n.$$

Similarly, denote by  $\mathcal{EFT}_n^{fin} \subset \mathcal{EFT}_n$  the subspace of super semigroups of finite rank operators. Using the homeomorphisms from sections 2.3 and 2.4 we obtain a diagram

$$\begin{array}{ccc} \mathcal{EFT}_n^{fin} & \hookrightarrow & \mathcal{EFT}_n & \hookrightarrow & \mathcal{SGO}_n \\ \uparrow \approx & & & & \uparrow \approx \\ \text{Conf}_n^{fin} & \hookrightarrow & & \longrightarrow & \text{Conf}_n \\ \downarrow \approx & & & & \downarrow \approx \\ \Omega_{n-1} & \hookrightarrow & & \longrightarrow & \bar{\Omega}_{n-1}. \end{array}$$

We will see presently that the horizontal arrow in the middle is a homotopy equivalence. Thus the same is true for the top and bottom rows. In fact, the homotopy involved preserves the subspace  $\mathcal{EFT}_n \subset \mathcal{SGO}_n$  which implies  $\mathcal{EFT}_n^{fin} \xrightarrow{\simeq} \mathcal{EFT}_n$  and hence the result. Now, consider the map  $h : \bar{\mathbb{R}} \times [0, 1] \longrightarrow \bar{\mathbb{R}}$  defined by

$$h(x, t) := \begin{cases} \frac{x}{1-t|x|} & \text{if } x \in (-\frac{1}{t}, \frac{1}{t}) \\ \infty & \text{else.} \end{cases}$$

Then the family of maps

$$H_t : \text{Conf}_n \longrightarrow \text{Conf}_n, \{V_\lambda\} \mapsto \{V_{h(\lambda, t)}\}$$

gives a homotopy from the identity on  $\text{Conf}_n$  to  $H_1$  whose image is  $\text{Conf}_n^{fin}$ . Thus, we see that the inclusion  $\iota : \text{Conf}_n^{fin} \hookrightarrow \text{Conf}_n$  is a homotopy equivalence with homotopy

inverse  $H_1$ . From the construction it is clear that the  $H_t$  define a homotopy on  $SGO_n$  that preserves the subspace  $\mathcal{EFT}_n$ . Hence  $H_1|_{\mathcal{EFT}_n}$  is a homotopy inverse to the inclusion  $\mathcal{EFT}_n^{fin} \hookrightarrow \mathcal{EFT}_n$ . Note that this argument actually works for all spaces that lie between  $EFT_n^{fin}$  and  $SGO_n$ .  $\square$

## 3.2 Families of Euclidian super bordisms

As explained in Section 1.3, the first step towards an Atiyah-Segal style definition of susy EFTs is to turn the category of 0 + 1-dimensional Euclidian spin bordisms  $\mathcal{EB}_0^1$ , see Definition 2.3.20 in [ST], into an  $\mathcal{S}$ -category  $\mathcal{E}$ . The objects of the underlying category  $\mathcal{E}$  are finite disjoint unions of super points  $\mathbb{R}^{0|1}$  equipped with an orientation.<sup>1</sup> A morphism in  $\mathcal{E}(Y_1, Y_2)$  is a bordism between  $Y_1$  and  $Y_2$  equipped with an odd vector field  $V$  (defined up to sign) such that  $[V, V]$  is nowhere vanishing. The basic example of such a vector field is the ‘standard Euclidian structure’  $V = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t}$  on  $\mathbb{R}^{1|1}$ ; note that it satisfies  $V^2 = \frac{\partial}{\partial t}$ . This description of the morphisms is not very precise, but this will be fixed presently: we will now, more generally, define families of Euclidian super bordisms parametrized by a supermanifold  $S$ . Taking  $S = pt$  then yields the complete definition of elements in  $\mathcal{E}(Y_1, Y_2)$ .

### Definition 5.

1. A *family of supermanifolds* of dimension  $(p|q)$  over a supermanifold  $S$  is a submersive morphism of supermanifolds  $\pi : F \rightarrow S$  with fiber dimension  $(p|q)$ .<sup>2</sup>
2. A *family of (real) Euclidian super curves* is a family of  $(1|1)$ -dimensional supermanifolds  $\pi : F \rightarrow S$  equipped with a vertical vector field  $V$ , defined up to sign, that generates a  $(0|1)$ -dimensional distribution<sup>3</sup>  $\mathcal{D} \subset T_{F/S}$  such that the Lie bracket

---

<sup>1</sup>By an orientation of a supermanifold we mean an orientation of the underlying reduced manifold. Note that our category  $\mathcal{E}$  has less objects than the category  $\mathcal{EB}_0^1$  considered by Stolz and Teichner who allow arbitrary closed  $(0|1)$ -dimensional supermanifolds. The restriction to disjoint unions of super points simplifies some technical questions.

<sup>2</sup>By the implicit function theorem, cf. [Lei], this is equivalent to requiring  $\pi$  to be isomorphic to a projection  $U \times V \rightarrow V$  locally in  $F$ . Of course, this definition of families is meaningful also in other categories of supermanifolds, e.g. for complex analytic ones.

<sup>3</sup>As defined in [DM], §3.5.

induces an isomorphism

$$\mathcal{D} \otimes \mathcal{D} \longrightarrow T_{F/S}/\mathcal{D}.$$

Here  $T_{F/S}$  denotes the sheaf of vertical tangent fields on  $F$ .

3. Now, let  $Y = \mathbb{R}^{0|1} \amalg \dots \amalg \mathbb{R}^{0|1}$ . A *family of super Euclidian zero bordisms*  $(\pi, \hat{F}, e)$  of  $Y$  is a family of Euclidian super curves  $\pi : F \rightarrow S$  together with the following data:  $\hat{F} \subset F_{\text{red}}$  is a subfamily of compact (1-dimensional) manifolds with boundary of the reduced family  $\pi_{\text{red}} : F_{\text{red}} \rightarrow S_{\text{red}}$  underlying  $\pi$ , and  $e = (|e|, e^*)$  is an embedding  $e : Y \times S \hookrightarrow F$  (over  $S$ ) such that  $|e|$  is a homeomorphism onto  $\partial\hat{F}$ . In order to ensure a well-defined gluing operation, we need the following compatibility between  $e$  and the Euclidian structure  $V$  on  $F$ . For simplicity, let us consider only the embedding of one boundary component  $e : \mathbb{R}^{0|1} \times S \hookrightarrow F$ . The additional assumption we impose is that  $e$  extends to a collar

$$\bar{e} : (-\epsilon, \epsilon)^{1|1} \times S \hookrightarrow F$$

that maps the standard Euclidian structure  $\frac{\partial}{\partial\theta} + \theta \frac{\partial}{\partial t}$  on  $(-\epsilon, \epsilon)^{1|1}$  to  $V$ .

4. It turns out that the (germ of the) geometric collar  $\bar{e}$  is unique when it exists, see Lemma 6. As a consequence,  $Y$  inherits an orientation from  $F$ : a component of  $Y$  is, say, positively oriented if  $|\bar{e}|$  maps  $(-\epsilon, 0] \times S$  to  $\hat{F}$ ; in the other case we declare it to be negatively oriented. Thus, we can define a *family of (oriented) Euclidian super bordisms* from an object  $Y_1$  to an object  $Y_2$  of  $\mathcal{E}$  to be a family of zero bordisms  $F$  of  $\bar{Y}_1 \amalg Y_2$  so that the orientations match. Here  $\bar{Y}_1$  denotes  $Y_1$  equipped with the opposite orientation.
5. A *morphism* between two families of bordisms  $F_1$  and  $F_2$  is a map of families  $(\tilde{\pi}_1, \hat{F}_1, e_1) \rightarrow (\pi_2, \hat{F}_2, e_2)$ , where  $\tilde{\pi}_1 : \tilde{F}_1 \rightarrow S$  is an open subfamily of  $\pi_1$  such that  $\hat{F}_1 \subset |\tilde{F}_1|$ . In particular, this means that a family  $(\pi, \hat{F}, e)$  is isomorphic to each family obtained from it by restricting  $\pi$  to an open subfamily  $\tilde{F}$  such that  $\hat{F} \subset |\tilde{F}|$ . In other words: all that matters is the germ of  $F$  around  $\hat{F}$ .

**Remarks.**

1. Using the (essentially) unique collars, we obtain a well-defined gluing operation of families of bordisms over a fixed supermanifold  $S$ . The point is that it is easy to glue sheaves along open subsets and the collars allow us to do precisely that.
2. If  $\pi : F \rightarrow S$  is a family of Euclidian super curves, then locally we can find a relative coordinate system  $(t, \theta)$  such that

$$V = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}.$$

We will refer to such a coordinate system as a ‘geometric chart’. One way to show the existence of such a geometric chart is to solve the ODE defined by  $V$  and then to check that the condition that  $V$  spans a distribution of dimension  $(0|1)$  means that the flow restricts to a local isomorphism with  $(-\epsilon, \epsilon)^{1|1} \times S$  equipped with the standard Euclidian structure  $\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$ . This is proved in Section 3.5, where we have collected some basic facts about odd ODEs on supermanifolds.

3. A family of Euclidian super curves  $F$  over a reduced base  $S$  is the same as a family of 1-dimensional Euclidian spin manifolds. To see this, note that we have  $\mathcal{O}_{F_{\text{red}}} = \mathcal{O}_F^{ev}$  since  $F$  has only one odd dimension. Hence there is a canonical map  $\mathcal{O}_{F_{\text{red}}} \hookrightarrow \mathcal{O}_F$  that allows us to consider modules over  $\mathcal{O}_F$  as  $\mathcal{O}_{F_{\text{red}}}$ -modules. Let  $E$  be the line bundle over  $F_{\text{red}}$  generated by  $V$ . By Lemma 9 below (see Section 3.5) we know that  $(V^2)_{\text{red}}$  is nowhere vanishing so that we obtain an isomorphism  $E^{\otimes 2} \xrightarrow{\cong} TF_{\text{red}}$ . The line bundle  $E$  together with this isomorphism defines a spin structure on  $F_{\text{red}}$  and  $(V^2)_{\text{red}}$  gives the Euclidian structure.

This implies that  $\mathcal{E}$  is equivalent to the category of 1-dimensional Euclidian spin bordisms, even as a category enriched over the category of manifolds. However, we will see that  $\mathcal{E}$  can be extended to a very interesting  $\mathcal{S}$ -category  $\underline{\mathcal{E}}$ .

4. The bordism category  $\mathcal{E}$  does not take Clifford algebras into account: a susy EFT defined on  $\underline{\mathcal{E}}$  is a theory of degree 0. However, work in progress by Stephan Stolz and Peter Teichner seems to show that bordism categories  $\mathcal{E}_n$  that give rise to susy EFTs of degree  $n$  can be obtained from  $\mathcal{E}$  by a certain twisting construction.

We conclude this section by proving the uniqueness of geometric collars claimed above. Again, the main tool is the theory of odd ODEs, see Section 3.5.

**Lemma 6.** *Let  $\partial_0 \hat{F} \subset \partial \hat{F}$  be a boundary component of  $\pi : F \rightarrow S$ . There is a bijective correspondence between the following items:*

- *Boundary embeddings  $e : \mathbb{R}^{0|1} \times S \hookrightarrow F$  for  $\partial_0 \hat{F}$  that extend to a geometric collar.*
- *Germ of geometric collars  $\bar{e} : (-\epsilon, \epsilon)^{1|1} \times S \hookrightarrow F$ .*
- *A choice of a vector field  $\tilde{V}$  representing the Euclidian structure  $V$  near  $\partial_0 \hat{F}$  together with a section  $\tilde{e} : S \rightarrow F$  of  $\pi$  so that  $|\tilde{e}| : |S| \xrightarrow{\cong} \partial_0 \hat{F}$ .*

**Remark.** The restriction to a single boundary component is clearly not necessary; we merely wanted to keep the notation simple.

*Proof of Lemma 6.* The germ of geometric collars  $\bar{e}$  extending  $e$  is unique. This follows, for example, from the computation of (local) automorphisms of  $\mathbb{R}^{1|1} \times S$  equipped with the standard Euclidian structure. We will determine these when proving Lemma 7 and refer to the remark after the proof. This establishes the first bijection.

The second bijection is given by associating to  $\bar{e}$  the image  $\tilde{V}$  of the standard vector field  $\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$  under  $\bar{e}$  and the restriction  $\tilde{e} := e|_{\mathbb{R}^{0|0} \times S} : S \rightarrow F$ . The inverse is given by solving the odd ODE defined by  $\tilde{V}$  for the family of initial conditions  $\tilde{e} : S \rightarrow F$ . The basics concerning odd ODEs on supermanifolds are laid out in the appendix 3.5.  $\square$

### 3.3 The supermanifold of Euclidian super intervals

Ideally, we would like to prove that  $\mathcal{E}$  admits an extension to a category  $\underline{\mathcal{E}}$  enriched over the category of supermanifolds  $\mathcal{S}$  and then define a susy EFT as an  $\mathcal{S}$ -functor from  $\underline{\mathcal{E}}$  to Hilbert spaces.<sup>4</sup> Unfortunately, as explained in 1.3, I do not know how to make a completely supersymmetric definition work, the problem being the definition of the (anti-)involution on  $\underline{\mathcal{E}}$ . Because of this, I do not see the point of giving a formally complete proof that considering families of Euclidian super bordisms leads to an  $\mathcal{S}$ -category  $\underline{\mathcal{E}}$ . However, we will construct the supermanifold of Euclidian super intervals

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<sup>4</sup>To be precise, the category of  $\mathbb{Z}_2$ -graded Hilbert spaces is a category enriched over generalized supermanifolds, see Section 2.1, so that we have to choose enrichment over generalized supermanifolds rather than ordinary supermanifolds.

explicitly and also briefly describe the super orbifold of Euclidian super circles. These are the two fundamental cases in the construction of  $\underline{\mathcal{E}}$ , since all families of Euclidian super bordisms are disjoint unions of families of intervals and circles; the rest is mostly bookkeeping. We begin with some preliminary computations.

### The automorphism group of super Euclidian $\mathbb{R}^{1|1}$

We want to determine the super Lie group  $\underline{\text{Aut}}(\mathbb{R}^{1|1}, V)$  of automorphisms of  $\mathbb{R}^{1|1}$  equipped with the standard super Euclidian structure  $V = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$ . It is defined by the universal property that families of automorphisms of  $(\mathbb{R}^{1|1}, V)$  parametrized by  $S$  are classified by maps from  $S$  to  $\underline{\text{Aut}}(\mathbb{R}^{1|1}, V)$ , i.e. we have group isomorphisms

$$\mathcal{S}(S, \underline{\text{Aut}}(\mathbb{R}^{1|1}, V)) \cong \text{Aut}_S(\mathbb{R}^{1|1} \times S),$$

naturally in  $S$ .

**Lemma 7.** *We have*

$$\underline{\text{Aut}}(\mathbb{R}^{1|1}, V) = \mathbb{R}^{1|1} \rtimes \mathbb{Z}_2,$$

where  $\mathbb{R}^{1|1}$  is equipped with the twisted group structure defined in Section 2.1 and the action of an element  $a \in \mathbb{Z}_2 = \{\pm 1\}$  on  $\mathbb{R}^{1|1}$  is given by  $\tau_a : \mathbb{R}^{1|1} \rightarrow \mathbb{R}^{1|1}$ ,  $(t, \theta) \mapsto (t, a\theta)$ .

*Proof.* Consider an automorphism  $\varphi$  of  $\mathbb{R}^{1|1} \times S$  over  $S$ ,

$$\varphi : \mathbb{R}^{1|1} \times S \longrightarrow \mathbb{R}^{1|1} \times S, \quad \varphi(t, \theta, s) = (F(t, \theta, s), G(t, \theta, s), s).$$

Consider the basis  $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\}$  for the vector fields on  $\mathbb{R}^{1|1}$ . Then the condition that the vector field  $V$  is, up to sign, preserved by  $\varphi$  yields the equation

$$\begin{pmatrix} \frac{\partial F}{\partial t} & \frac{\partial F}{\partial \theta} \\ -\frac{\partial G}{\partial t} & \frac{\partial G}{\partial \theta} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = a \begin{pmatrix} G \\ 1 \end{pmatrix},$$

with  $a \in \{\pm 1\}$ . The second component of this matrix identity gives

$$\frac{\partial G}{\partial t} = 0 \text{ and } \frac{\partial G}{\partial \theta} = a, \text{ and thus } G(t, \theta, s) = g(s) + a\theta.$$

This and the equation coming from the first component combine to

$$\frac{\partial F}{\partial t} = 1 \text{ and } \frac{\partial F}{\partial \theta} = ag, \text{ so that } F(t, \theta, s) = t + f(s) + a\theta g(s).$$

On the other hand, each choice of  $f \in \mathcal{O}_S^{ev}$ ,  $g \in \mathcal{O}_S^{odd}$  and  $a \in \mathbb{Z}_2$  gives rise to a family of automorphisms  $\varphi$ . Hence we obtain a bijection of sets

$$\text{Aut}_S(\mathbb{R}^{1|1} \times S) \xrightarrow{\cong} \mathcal{S}(S, \mathbb{R}^{1|1} \times \mathbb{Z}_2), \quad \varphi \mapsto (f, g, a).$$

It is easy to check that composition of morphisms induces the asserted group structure on the right hand side.  $\square$

In particular, the computation shows that every family of (local) automorphisms of  $(\mathbb{R}^{1|1}, V)$  that leaves  $\mathbb{R}^{0|1} \times S \subset \mathbb{R}^{1|1} \times S$  fixed is the identity, since in this case we must have  $a = 1$ ,  $g = 0$ , and consequently also  $f = 0$ . This statement was used in the proof of Lemma 6.

Lemma 7 says that the automorphisms of  $(\mathbb{R}^{1|1}, V)$  are generated by the right translations with respect to the twisted super group structure on  $\mathbb{R}^{1|1}$  and the ‘spin flip’  $(t, \theta) \mapsto (t, -\theta)$ . The ordinary Lie group underlying  $\underline{\text{Aut}}(\mathbb{R}^{1|1}, V)$  is  $\mathbb{R} \times \mathbb{Z}_2$ , the automorphism group of the Euclidian real line with spin structure. The additional Fermionic dimension that we see in the super case comes from the *odd translations*  $(t, \theta) \mapsto (t + \theta g, \theta + g)$ . Note that these occur only in families over a non-reduced supermanifolds  $S$ : if  $\mathcal{O}_S^{odd} = 0$ , then necessarily  $g = 0$ .

### The super semigroup of Euclidian super intervals

Let  $\mathcal{I} = \mathbb{R}_{>0}^{1|1} \rtimes \mathbb{Z}_2$  be the open sub super semigroup of  $\mathbb{R}^{1|1} \rtimes \mathbb{Z}_2$  defined by the inclusion  $\mathbb{R}_{>0} \times \mathbb{Z}_2 \subset \mathbb{R} \times \mathbb{Z}_2$ . This notation is motivated by the following result.

**Proposition 8.** *The functor from supermanifolds to sets*

$$S \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of connected Euclidian super} \\ \text{bordisms from } \mathbb{R}^{0|1} \text{ to } \mathbb{R}^{0|1} \text{ parametrized by } S \end{array} \right\}$$

*actually takes values in the category of semigroups, thanks to gluing of families of bordisms over  $S$ . This functor is represented by the super semigroup  $\mathcal{I}$ .*

Before we begin the proof, let us describe a class of examples of families of Euclidian super bordisms between two super points  $\mathbb{R}^{0|1}$ . For each pair of morphisms

$$\alpha, \omega : S \longrightarrow \mathbb{R}^{1|1} \rtimes \mathbb{Z}_2 = \underline{\text{Aut}}(\mathbb{R}^{1|1}, V)$$

such that  $pr_1\alpha(s) < pr_1\omega(s)$  for all  $s \in |S|$ , there is an associated family  $I_{\alpha,\omega} = (\pi, \hat{I}_{\alpha,\omega}, r_\alpha \amalg r_\omega)$  of Euclidian super bordisms from  $\mathbb{R}^{0|1}$  to  $\mathbb{R}^{0|1}$  defined as follows. We consider the trivial family  $\pi : I_{\text{tr}} = \mathbb{R}^{1|1} \times S \rightarrow S$  with the standard super Euclidian structure  $V = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$  together with the boundary embeddings  $r_\alpha, r_\omega$  determined by the geometric collars

$$\bar{r}_\alpha, \bar{r}_\omega : (-\epsilon, \epsilon) \times S \longrightarrow \mathbb{R}^{1|1} \times S$$

coming from the families of automorphisms  $\alpha$  and  $\omega$ . Consequently,

$$\hat{I}_{\alpha,\omega} = \{ (t, s) \mid pr_1\alpha(s) < t < pr_1\omega(s) \} \subset \mathbb{R} \times |S|.$$

*Proof of Proposition 8.* We claim that every family of compact Euclidian super intervals over a supermanifold  $S$  is isomorphic to exactly one of the form

$$I_\omega := I_{0,\omega}, \text{ where } \omega : S \longrightarrow \mathcal{I} \subset \mathbb{R}^{1|1} \rtimes \mathbb{Z}_2 = \underline{\text{Aut}}(\mathbb{R}^{1|1}, V)$$

and 0 denotes the identity element in the group  $\mathcal{S}(S, \underline{\text{Aut}}(\mathbb{R}^{1|1}, V))$  (i.e. the constant map to 0). Furthermore, it is easy to check that the pullback of  $I_\omega$  under a map  $g : S' \rightarrow S$  is given by  $I_{\omega g}$ . This implies that the moduli supermanifold of super intervals is  $\mathcal{I}$ .

Let us check that gluing of intervals corresponds to the semidirect product structure as asserted. Let  $I_{\omega_1}, I_{\omega_2} \in \mathcal{S}(S, \underline{\mathcal{E}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}))$  be given. Recall from 1.3 that we write the composition morphisms in the form  $\underline{\mathcal{E}}(Y_2, Y_3) \times \underline{\mathcal{E}}(Y_1, Y_2) \rightarrow \underline{\mathcal{E}}(Y_1, Y_3)$ . Hence the composite family  $I_{\omega_2} I_{\omega_1}$  is obtained by gluing the incoming boundary of  $I_{\omega_2}$  to the outgoing boundary of  $I_{\omega_1}$ . The automorphism  $\omega_1$  of  $\mathbb{R}^{1|1} \times S$  induces an isomorphism  $I_{\omega_2} \cong I_{\omega_1, \omega_2 \omega_1}$ . We conclude that we can obtain  $I_{\omega_2} I_{\omega_1}$  from  $I_{\omega_1}$  and  $I_{\omega_1, \omega_2 \omega_1}$  by gluing them via the identity map. Hence we see that, as desired,  $I_{\omega_2} I_{\omega_1} = I_{\omega_1} \cup I_{\omega_1, \omega_2 \omega_1} = I_{\omega_2 \omega_1}$ .

Let us now prove the *claim*. Let  $(\pi, \hat{F}, f_{\text{in}} \amalg f_{\text{out}})$  be a family of compact Euclidian super intervals, where  $\pi : F \rightarrow S$ ,  $\hat{F} \subset |F|$ , and  $f_{\text{in}}, f_{\text{out}} : \mathbb{R}^{0|1} \times S \hookrightarrow F$ . Denote by  $\tilde{V}_{\text{in}}$  the vector field that represents the Euclidian structure near the incoming

boundary and that is compatible with the embedding  $f_{\text{in}}$  (see Lemma 6). We assert that the Euclidian structure can even globally be represented by a vector field  $\tilde{V}$  (extending  $\tilde{V}_{\text{in}}$ ) and that integrating  $\tilde{V}$  with respect to the initial condition  $f_{\text{in}}|_{\mathbb{R}^{0|0} \times S}$  yields an isomorphism with  $I_\omega$ , for some  $\omega$ .

It suffices to prove this locally in  $S$ : assume that we have shown that for each  $s \in |S|$  there is a neighborhood  $|U_s| \subset |S|$  such that  $\tilde{V}_{\text{in}}$  extends to a vector field  $\tilde{V}_s$  representing the Euclidian structure on a neighborhood of  $\hat{F} \cap |\pi|^{-1}(|U_s|) \subset |\pi|^{-1}(|U_s|)$  and such that the family of integral curves  $\varphi_s$  for  $\tilde{V}$  compatible with the initial conditions determined by  $f_{\text{in}}$  gives an isomorphism of Euclidian super intervals

$$\varphi_s : \mathbb{R}^{1|1} \times U_s \supset I_{\omega_s} \longrightarrow \pi^{-1}(U_s).$$

Then the  $\tilde{V}_s$  fit together to give a global vector field  $\tilde{V}$  (this follows since they agree near the incoming boundary) and the uniqueness of solutions of ODEs implies that the  $I_{\omega_s}$  and  $\varphi_s$  can be glued up to give the desired global isomorphism.

Now, let  $s \in |S|$ . Since  $\hat{F} \cap |\pi|^{-1}(s)$  is compact, we can find a (connected) neighborhood  $|U_s|$  of  $s$  and a finite collection of geometric charts

$$\varphi_s^i : (a_i, b_i)^{1|1} \times U_s \xrightarrow{\cong} W_i \subset \pi^{-1}(U_s), \quad i = 0, \dots, k$$

covering a neighborhood of  $\hat{F} \cap |\pi|^{-1}(s)$ . We may assume that the collar of the incoming boundary  $\bar{f}_{\text{in}} =: \varphi_s^0$  is among the  $\varphi_s^i$  and that for  $i \neq j$  the chart domains  $|W_i|$  and  $|W_j|$  intersect non-trivially only if  $j = i + 1$ . Furthermore, we can choose the  $\varphi_s^i$  such that  $|W_i| \cap |W_{i+1}|$  is connected for all  $i$ . To see this, note that we can clearly achieve these properties for the sets  $|W_i| \cap |\pi|^{-1}(s)$  since  $\hat{F} \cap |\pi|^{-1}(s)$  is homeomorphic to a compact interval. However, once this is done, we get the same relations for the  $|W_i|$ , at least after making  $|U_s|$  smaller, if necessary.

It is now easy to see that the  $\varphi_s^i$  can be fused together to give  $\varphi_s$ . For example, the chart change between  $\varphi_s^0$  and  $\varphi_s^1$  is given by a right translation and/or a spin flip in  $\mathbb{R}^{1|1}$  (this follows from the computation in the proof of Lemma 7). If no spin flip is involved, the image of  $\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$  under  $\varphi_s^1$  agrees with  $\tilde{V}_{\text{in}}$  on  $W_0 \cap W_1$  and we directly get the extension of  $\tilde{V}_{\text{in}}$  to  $W_0 \cup W_1$ . Furthermore, we may compose  $\varphi_s^1$  with the right translation that gives the chart change in order to combine  $\varphi_s^0$  and  $\varphi_s^1$  into one chart with

image  $W_0 \cup W_1$ . If the chart change involves a spin flip we can reduce to the previous case by pre-composing the  $\varphi_s^1$  with the spin flip map  $(t, \theta) \mapsto (t, -\theta)$ . Applying this fusing process finitely many times yields  $\tilde{V}_s$  and  $\varphi_s$ . Since we started out with  $\varphi_s^0 = \bar{f}_{\text{in}}$  it is clear that the  $\varphi_s$  constructed in this way solves the ODE defined by  $\tilde{V}_s$  and that it satisfies the desired initial condition.

This completes the construction of  $\varphi$ . Uniqueness follows from the uniqueness of solutions of ODEs.  $\square$

### The orbifold of Euclidian super circles

To illustrate the problem with automorphisms in the case of closed components we show that *the functor that associates to a supermanifold  $S$  the set of isomorphism classes of families of closed connected Euclidian super curves over  $S$  is represented by the super orbifold*

$$\mathbb{R}_{>0}^{1|1} \rtimes \mathbb{Z}_2 / \mathbb{R}^{0|1} \rtimes \mathbb{Z}_2,$$

where  $\mathbb{R}^{0|1} \rtimes \mathbb{Z}_2$  acts on  $\mathbb{R}_{>0}^{1|1} \rtimes \mathbb{Z}_2$  by conjugation. Since we want the the underlying manifold to coincide with the moduli manifold of Euclidian spin circles  $\mathbb{R}_{>0} \times \mathbb{Z}_2$  (the circumference gives  $t \in \mathbb{R}_{>0}$ , the  $\mathbb{Z}_2$  accounts for the spin structure), we only allow families whose underlying circle bundle is trivial. Passing to the universal cover we see that the classification problem of such families of Euclidian super circles over  $S$  corresponds precisely to understanding families of conjugacy classes of subgroups of  $\mathbb{R}^{1|1} \rtimes \mathbb{Z}_2$  isomorphic to  $\mathbb{Z}$ . An easy calculation shows that these, in turn, are precisely classified by the above super orbifold.

### 3.4 Outlook: the definition of susy EFTs

So, where does our discussion of Euclidian super intervals leave us? As pointed out before, I do not know how to define the (anti)involutions on the  $\mathcal{S}$ -category  $\underline{\mathcal{X}}$  in the right way. This amounts to extending the (anti)involutions on Euclidian spin 1-manifolds to families of super Euclidian bordisms. The formula that comes to mind is to locally

replace the vector field

$$V = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \text{ by } \bar{V} = -\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}.$$

As desired, this flips the orientation: we have  $\bar{V}^2 = -\frac{\partial}{\partial t} = -V^2$ . However, this operation is not invariantly defined, since  $\bar{V}$  is *not* invariant under right translations on  $\mathbb{R}^{1|1}$  (=coordinate changes between geometric charts).<sup>5</sup>

Of course, we could directly define the (anti)involutions on the moduli super spaces we computed in the last section. For example, declaring the anti-involution to be the identity on the supermanifold of Euclidian super intervals  $\mathbb{R}_{>0}^{1|1} \rtimes \mathbb{Z}_2$  would force, as desired, the operators  $A$  and  $B$  defining the super semigroup homomorphism  $\Phi_E = A + \theta B$  to be self-adjoint. However, this definition is ungeometric and thus unsatisfactory. One way to get a geometric definition of the (anti)involutions might be to consider (1|1)-dimensional Euclidian *cs* manifolds: it seems that most of the considerations of the last two sections should go through for (1|1)-dimensional *cs* manifolds equipped with an odd vector field satisfying a certain reality condition. Then a candidate for the (anti)involutions would be to replace a family of bordisms by its complex conjugate. Unfortunately, I have nothing precise to say about this idea.

Another way to get around the (anti)involution problem is to consider supersymmetry only on the morphism set of connected bordisms from  $\mathbb{R}^{0|1}$  to itself. This means we would declare a susy EFT  $\underline{E}$  to be a usual EFT  $E$  together with an extension of the semigroup homomorphism

$$\phi_E : \mathbb{R}_{>0} \longrightarrow \text{HS}^{sa, ev}(\mathcal{H})$$

to a super semigroup map  $\Phi_E : \mathbb{R}_{>0}^{1|1} \rightarrow \text{HS}^{sa}(\mathcal{H})$ . The arguments of propositions 3.1.1 and 3.2.2 in [ST] show that a susy EFT is the same as a super semigroup of self-adjoint Hilbert-Schmidt operators, i.e. a susy EFT in the sense of Definition 4. However, we hope that a better treatment of this question will be found.

### 3.5 Appendix: Odd ordinary differential equations

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<sup>5</sup>As far as I can see, the same remark applies to the formulation given in [Mar] in terms of the dual 1-form  $dt + \theta d\theta$ .

Let  $X$  be a closed supermanifold and  $V$  a vector field on  $X$ . If  $V$  is even there exists an associated flow  $\phi : \mathbb{R} \times X \rightarrow X$  satisfying the flow property  $\phi_t \phi_s = \phi_{t+s}$  (see [DM], §3.7). If  $V$  is odd, the associated flow is a map

$$\phi : \mathbb{R}^{1|1} \times X \longrightarrow X,$$

and the corresponding flow property can be expressed as the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \times X & \xrightarrow{\text{id} \times \phi} & \mathbb{R}^{1|1} \times X \\ \mu \times \text{id} \downarrow & & \downarrow \phi \\ \mathbb{R}^{1|1} \times X & \xrightarrow{\phi} & X, \end{array}$$

where  $\mu$  is the twisted group structure on  $\mathbb{R}^{1|1}$  introduced in Section 2.1. In other words, an odd vector field defines an action of  $\mathbb{R}^{1|1}$  on  $X$  (of course, the converse is also true). The existence and uniqueness of the flow  $\phi$  associated with an odd vector field is proved in [Du].<sup>6</sup> Note that the ‘odd flow’  $\phi$  maps the right invariant vector field  $\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$  to the given vector field  $V$  on  $X$  (just like an ordinary flow maps  $\frac{\partial}{\partial t}$  to the given vector field for all times  $t$ ). For simplicity, we have considered closed supermanifolds here. In the non-closed case, there is the usual local existence theorem.

All of this can be generalized to families. Let  $\pi : F \rightarrow S$  be a family of closed supermanifolds, i.e.  $|\pi|^{-1}(s)$  is a closed manifold for all  $s \in |S|$ , and let  $V$  be an odd vertical vector field on  $F$ . Then there is a fiber-preserving flow

$$\mathbb{R}^{1|1} \times F \rightarrow F.$$

In particular, we have a (unique) family of integral curves

$$\varphi : \mathbb{R}^{1|1} \times S \rightarrow F$$

for each family of initial conditions, i.e. for each section  $S \rightarrow F$ .

**Lemma 9.** *Assume that in this situation the fibers of  $\pi$  are  $(1|1)$ -dimensional. Then the following conditions are equivalent:*

---

<sup>6</sup>This statement is a special case of the fact that an action of a super Lie group  $G$  on a supermanifold is the same as an action of the reduced group  $G_{\text{red}}$  and a compatible action of the super Lie algebra  $\text{Lie}(G)$  (see [DM], §3.8).

- $\varphi$  is a local isomorphism.
- $V$  generates a  $(0|1)$ -dimensional distribution as in Definition 5.
- The vector field  $(i^*[V, V])_{\text{red}}$  on the reduced pullback family  $(i^*F)_{\text{red}}$  is nowhere vanishing, where  $i : S_{\text{red}} \hookrightarrow S$  denotes the inclusion.

*Proof.* The statement is local in nature. Thus it suffices to consider an odd vertical vector field  $V = a \frac{\partial}{\partial t} + b \frac{\partial}{\partial \theta}$  on  $\mathbb{R}^{1|1} \times U \rightarrow U$  and an associated family of flow lines

$$\varphi : (-\epsilon, \epsilon)^{1|1} \times U \longrightarrow \mathbb{R}^{1|1} \times U, \quad \varphi(t, \theta, u) = (F(t, \theta, u), G(t, \theta, u), u)$$

satisfying, say,  $\varphi(0, 0, u) = (0, 0, u)$ . Write

$$F(t, \theta, u) = f_1(t, u) + \theta f_2(t, u) \quad \text{and} \quad G(t, \theta, u) = g_1(t, u) + \theta g_2(t, u).$$

We want to compare the three statements at a point  $(0, u) \in \mathbb{R} \times |U|$ . We observe:

- By the inverse function theorem, cf. [Lei], [Var],  $\varphi$  is an isomorphism near  $(0, u)$  if and only if  $\frac{\partial F}{\partial t}$  and  $\frac{\partial G}{\partial \theta}$ , or, equivalently,  $\frac{\partial f_1}{\partial t}$  and  $g_2$ , are invertible near  $(0, u)$ .
- $V$  generates a  $(0|1)$ -dimensional distribution as in Definition 5 exactly if  $b$  and  $\frac{\partial a}{\partial \theta}$  are invertible near  $(0, u)$ , cf. [LR], proof of Lemma 1.2. On the other hand, at the point  $(0, u)$  we have

$$a = \frac{\partial f_1}{\partial t} \theta + f_2 \quad \text{and} \quad b = -\frac{\partial g_1}{\partial t} \theta + g_2,$$

because  $\varphi$  maps  $\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}$  to  $V$ . Hence  $V$  generates a distribution as desired if and only if  $\frac{\partial f_1}{\partial t}$  and  $g_2$  are invertible near  $(0, u)$ .

- $(i^*[V, V])_{\text{red}}$  is obtained by first pulling  $[V, V]$  back to  $i^*F$  (i.e. setting the odd functions on  $S$  equal to zero) and then projecting onto  $(i^*F)_{\text{red}}$  (i.e. setting  $\theta = 0$ ).<sup>7</sup> An elementary computation shows that  $V^2 = b \frac{\partial a}{\partial \theta} \frac{\partial}{\partial t}$  modulo odd coefficients. Hence  $(i^*(V^2))_{\text{red}} = \frac{1}{2}(i^*[V, V])_{\text{red}}$  is non-zero at  $(0, u)$  if and only if  $b$  and  $\frac{\partial a}{\partial \theta}$  are invertible near  $(0, u)$ .

These observations directly imply the equivalence of the three conditions.  $\square$

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<sup>7</sup>More formally: since  $i^*F$  has only one odd dimension, an even vector field  $X$  on  $i^*F$  preserves  $\mathcal{O}_{i^*F}^{\text{red}} = \mathcal{O}_{i^*F}^{\text{ev}}$  and hence it induces a vector field  $X_{\text{red}}$  on  $(i^*F)_{\text{red}}$ .

## Chapter 4

# Supersymmetry and Conformal Field Theories with holomorphic Partition Function

We now turn to the role of supersymmetry in the CFT case. Our whole discussion is based on the approach of Stolz and Teichner and we will use definitions and notations from [ST], especially Section 3.3, without further mention.

### 4.1 Partition functions and the semigroup of toy annuli

As explained in the introduction, we would like to have a map

$$Z : \pi_0(\mathcal{CFT}_n) \longrightarrow \mathrm{wMF}_{n/2}^{\mathbb{Z}}$$

that is given by associating to a CFT  $E$  of degree  $n$  its *partition function*  $Z_E$ . Recall that the function  $Z_E : \mathfrak{h} \longrightarrow \mathbb{C}$  is given by associating to  $\tau \in \mathfrak{h}$  the value of  $E$  on the conformal spin torus  $\Sigma_\tau$  equipped with a certain canonical element in the  $-n^{\mathrm{th}}$  tensor power of the Pfaffian line of  $\Sigma_\tau$ , see [ST], 3.3.5. According to Theorem 3.3.4 in [ST], the function  $Z_E$  has the transformation property of a modular form of weight  $\frac{n}{2}$ . However, there is no reason why  $Z_E$  should be holomorphic, which is clearly necessary if we want it to be a modular form. Stolz and Teichner address this problem by only allowing a

certain class of CFTs whose partition functions are holomorphic. Let us recall their description of these theories.

Let  $E$  and  $Z_E$  be as above. According to (3.3.10) in [ST] we can compute  $Z_E(\tau)$  by taking the Clifford super trace of the Hilbert-Schmidt operator  $E(A_q)$  obtained by evaluating  $E$  on the annulus  $A_q$  associated with  $q = e^{2\pi i\tau}$ ,

$$Z_E(\tau) = \text{str}_{C_n}(E(A_q)).$$

In particular, the partition function  $Z_E$  only depends on the values of  $E$  on the surfaces  $A_q$ , which we will refer to as *toy annuli*.<sup>1</sup> Let us recall their definition in greater detail (see [ST], Definition 3.3.5):  $A_q$  is the conformal spin annulus

$$A_q = \{ z \in \mathbb{C} \mid |q| \leq |z| \leq 1 \}$$

equipped with the following boundary embeddings: the embedding  $l_q : S^1 \rightarrow |q| \cdot S^1$  of the incoming circle is given by multiplication by  $q$  and the outgoing boundary circle  $S^1 \subset \mathbb{C}$  is embedded identically. The spin structure on  $A_q \approx S^1 \times [|q|, 1]$  is the one corresponding to the periodic spin structure on  $S^1$ .

The annuli  $A_q$  form a semigroup under the gluing operation. It can be identified with the punctured open disc  $D_0 := D^2 \setminus \{0\}$  equipped with complex multiplication by sending  $A_q$  to  $q$ . This map is indeed a homomorphism: given  $A_{q_1}$  and  $A_{q_2}$  we identify

$$l_{q_1} : A_{q_2} \xrightarrow{\cong} A'_{q_2} := \{ z \mid |q_1 q_2| \leq |z| \leq |q_1| \}$$

(where the boundary embeddings of the latter are given by  $l_{q_1 q_2}$  and  $l_{q_1}$ ) to see that the annulus obtained by gluing  $A_{q_1}$  and  $A_{q_2}$  is isomorphic to  $A_{q_1} \cup A'_{q_2} = A_{q_1 q_2}$ .

The CFT axioms tell us that the map

$$\phi_E : D_0 \longrightarrow \text{HS}(\mathcal{H}), \quad q \mapsto E(A_q),$$

is a homomorphism of semigroups and that it is  $\mathbb{Z}_2$ -equivariant with respect to complex conjugation on  $D_0$  and taking adjoints on  $\text{HS}(\mathcal{H})$ . Note that the partition function  $Z_E$  only depends on the homomorphism  $\phi_E$ . In order to formulate the condition Stolz and

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<sup>1</sup>Whenever we write  $A_q$  it is understood that we equip  $A_q$  with the canonical vacuum vector in  $\text{Falg}(A_q)$ . We should also point out that in [ST] the notation  $A_\tau$  rather than  $A_q$  is used. We prefer to write  $A_q$ , since  $A_\tau$  only depends on  $q = e^{2\pi i\tau}$ .

Teichner impose on the theory  $E$  to force holomorphicity of  $Z_E$ , we use the following result that tell us that each homomorphism  $\phi$  can be written in terms of infinitesimal generators  $L_0$  and  $\bar{L}_0$ .<sup>2</sup>

**Lemma 10** (Stolz<sup>3</sup>). *Let  $\phi : D_0 \rightarrow \text{HS}(H)$  be a  $\mathbb{Z}_2$ -equivariant semigroup homomorphism. Then there exist a subspace  $W \subset H$  and self-adjoint operators  $L_0$  and  $\bar{L}_0$  on  $W$  such that*

$$\phi(q, \bar{q}) = q^{L_0} \bar{q}^{\bar{L}_0} \text{ on } W$$

*and  $\phi(q, \bar{q}) = 0$  on  $W^\perp$ . The operators  $L_0$  and  $\bar{L}_0$  commute and have compact resolvent. Furthermore, the spectrum of  $L_0 - \bar{L}_0$  is contained in the integers.*

Stolz and Teichner prove that the partition function  $Z_E$  of a CFT  $E$  of degree  $n$  is holomorphic provided the ( $C_n$ -linear) infinitesimal generator  $\bar{L}_0$  associated with  $\phi_E$  is the square of an *odd* ( $C_n$ -linear) operator  $\bar{G}_0$  with  $[L_0, \bar{G}_0] = 0$ , see Theorem 3.3.14 in [ST].<sup>4</sup> In fact, Stolz and Teichner show that under this assumption  $Z_E$  is a weak integral modular form.

Now, to simply consider CFTs  $E$  with the property that  $\bar{L}_0$  has an odd square root seems quite ungeometric and artificial. The condition  $\bar{L}_0 = \bar{G}_0^2$  clearly looks a lot like supersymmetry, and Stolz and Teichner suggest to introduce a notion of ‘super conformal structures’ on super surfaces so that for CFTs that are supersymmetric in the corresponding sense the existence of  $\bar{G}_0$  is automatic. This is formulated in Hypothesis 3.3.13 in [ST].

Note that the situation is quite similar to the EFT case, where we used supersymmetry to make the infinitesimal generator of the semigroup of operators  $\phi_E$  associated with an EFT  $E$  into the square of an odd operator. Recall that a supersymmetric extension of  $E$  was precisely a super semigroup homomorphism  $\Phi_E : \mathbb{R}_{>0}^{1|1} \rightarrow \text{HS}(\mathcal{H})$  with underlying reduced map  $\phi_E$ . In the CFT case  $\mathbb{R}_{>0}$  is replaced by  $D_0$ , and this raises the

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<sup>2</sup>The motivation for this notation is that the operator  $L_0$  belongs to a much bigger Lie algebra, the Virasoro algebra, whose generators are typically denoted by  $L_n$ . Similarly,  $\bar{L}_0$  belongs to a second copy of this algebra (and is *not* the complex conjugate of  $L_0$ ).

<sup>3</sup>Thanks to Stephan Stolz for showing me his personal notes containing the proof of this fact! The main tool in the proof is the spectral theorem for compact normal operators.

<sup>4</sup>Recall that, by definition of a CFT of degree  $n$ , the  $\phi_E(z, \bar{z})$ , and thus  $L_0$  and  $\bar{L}_0$ , are even operators on a  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H}$ .

question what the CFT analogue of  $\mathbb{R}_{>0}^{1|1}$  should be. As pointed out in the introduction, the appropriate super extension  $\bar{\mathfrak{A}}_{cs}$  of  $D_0$  is actually a  $cs$  semigroup. This is why we will introduce  $cs$  manifolds and  $cs$  (semi)groups in the next section. The  $cs$  semigroup  $\bar{\mathfrak{A}}_{cs}$  will be defined in Section 4.3.

Finally, let us point out that there is an important difference between the EFT and CFT case: while describing a (susy) EFT was essentially equivalent to specifying the corresponding (super) semigroup of operators, the CFT case is much more involved. The semigroup of operators  $\phi_E : D_0 \rightarrow \text{HS}(\mathcal{H})$  is only a small part of the data associated with a CFT  $E$ . For example,  $\phi_E$  extends to a representation of a much bigger (infinite-dimensional) semigroup, see [Se1], Chapter 2. However, as we have seen,  $\phi_E$  determines the partition function of  $E$ , and this is all we care about for now.

## 4.2 Basics about $cs$ manifolds

In this section we briefly introduce the basics about  $cs$  manifolds. More details will be given in 4.4. Roughly speaking, a  $cs$  manifold is a ringed space whose structure sheaf looks like smooth complex-valued functions tensored by some exterior algebra. More precisely, the local model for  $cs$  manifolds is  $\mathbb{R}^p$  equipped with the sheaf of functions

$$U \mapsto C^\infty(U, \mathbb{C})[\theta_1, \dots, \theta_q] = \mathcal{O}_{\mathbb{R}^p|q}(U) \otimes_{\mathbb{R}} \mathbb{C},$$

where the  $\theta_i$  are the usual odd quantities. We denote this ringed space by  $\mathbb{R}_{\otimes \mathbb{C}}^{p|q}$ .

**Definition 6.** A  $cs$  manifold  $M = (|M|, \mathcal{O}_M)$  of dimension  $(p|q)$  is a topological manifold  $|M|$  together with a sheaf of complex super algebras  $\mathcal{O}_M$  locally isomorphic to  $\mathbb{R}_{\otimes \mathbb{C}}^{p|q}$ . Morphisms between  $cs$  manifolds are maps of ringed spaces.

### Examples.

1. In the purely even case  $q = 0$ , a  $cs$  manifold  $M$  is just a usual smooth manifold. The point is that in this case  $\mathcal{O}_M = C^\infty(-, \mathbb{C})$  has a canonical involution given by (pointwise) complex conjugation. Hence we can recover the sheaf of real-valued smooth functions on  $M$  by looking at the fixed point set of this involution. Note

that such a canonical involution does not exist if  $q \neq 0$ . Hence *cs* manifolds are really different than usual (real) supermanifolds.

2. When  $p = 0$ , the notion of a *cs* manifold coincides with the notion of a complex supermanifold (of the same dimension).
3. Every (real) supermanifold  $M$  defines a *cs* manifold  $M_{\otimes\mathbb{C}}$  by complexification:

$$\mathcal{O}_{M_{\otimes\mathbb{C}}} := \mathcal{O}_M \otimes_{\mathbb{R}} \mathbb{C}$$

This defines a functor from (real) supermanifolds to *cs* manifolds. As already indicated, this is not an equivalence of categories. For example, there are more morphisms from  $M_{\otimes\mathbb{C}}$  to  $N_{\otimes\mathbb{C}}$  than from  $M$  to  $N$ .

4. Every complex supermanifold  $M$  of dimension  $(p|q)$  has an underlying *cs* manifold  $M_{cs}$  of dimension  $(2p|q)$ . This seems quite intuitive if one compares the two local models: holomorphic functions are replaced by smooth maps. The idea is that when one interprets a complex manifold  $M$  as a smooth manifold  $M_{C^\infty}$ , one proceeds in two steps: first, one tensors  $\mathcal{O}_M$  by smooth  $\mathbb{C}$ -valued functions and then obtains the sheaf of real-valued smooth functions  $\mathcal{O}_{M_{C^\infty}}$  as the subsheaf of functions fixed under complex conjugation. In the super case, we can still perform the first step, but not the second, hence ending up with a *cs* manifold. The construction will be explained in Section 4.4, where we take a more detailed look at the relation between complex supermanifolds and *cs* manifolds.

**Remarks.**

1. As in the case of usual supermanifolds, morphisms between *cs* manifolds can be written down naively in coordinates. This will be described in Section 4.4.
2. The *tangent sheaf*  $TM$  of a *cs* manifold  $M^{p|q}$  is the  $\mathcal{O}_M$ -module of  $\mathbb{C}$ -linear derivations on  $\mathcal{O}_M$ ; its sections are called *vector fields*. The basic facts about  $TM$  are the same as in the non *cs* case, e.g.  $TM$  is locally free of rank  $(p|q)$  over  $\mathcal{O}_M$ . However, one should note that not all vector fields on a *cs* manifold are (locally) integrable,

since they are naturally complex. For example, if  $M$  is a smooth manifold considered as a  $cs$  manifold, then  $TM$  is the same as the complexified (usual) tangent bundle of  $M$ .

3. A  $cs$  (semi)group  $G$  is a (semi)group object in the category of  $cs$  manifolds. The super Lie algebra  $\text{Lie}(G)$  of a  $cs$  group  $G$  consists of the left-invariant vector fields on  $G$ ; the subtleties of the super case are the same as for (real) super Lie groups, cf. [DM], §3.3.6, or [Var], 6.1. As is to be expected, the complex dimension of  $\text{Lie}(G)$  equals the dimension of  $G$ .
4. In the same way as we considered  $\mathbb{Z}_2$ -graded (real) Banach algebras as generalized super semigroups in Chapter 2, we can interpret  $\mathbb{Z}_2$ -graded complex Banach algebras as generalized  $cs$  semigroups: maps from a (finite-dimensional)  $cs$  domain  $U \subset \mathbb{R}_{\otimes \mathbb{C}}^{p|q}$  to a Banach algebra  $B$  are elements in the algebra  $B(U) := (C^\infty(U, B)[\theta_1, \dots, \theta_q])^{ev}$ . We will use this below to make sense of homomorphisms from a  $cs$  semigroup to Hilbert-Schmidt operators on a  $\mathbb{Z}_2$ -graded complex Hilbert space  $\mathcal{H}$ .

### 4.3 The $cs$ semigroup of $cs$ toy annuli

Let us now return to the situation we considered at the end of Section 4.1. We were looking for a super semigroup that contains the punctured disc  $D_0$  in such a way that a  $\mathbb{Z}_2$ -equivariant homomorphism  $\phi : D_0 \rightarrow \text{HS}(\mathcal{H})$ ,  $\phi(z, \bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0}$ , extends to this super semigroup if and only if  $\bar{L}_0$  is the square of an odd operator  $\bar{G}_0$ .<sup>5</sup> On the level of Lie algebras this means that  $\partial_{\bar{z}}$  should become the square of an odd vector field - at least near the identity element 1, which, of course, is not actually contained in  $D_0$ .

Instead of just looking at  $D_0$ , we might as well try to find a super extension of the group  $\mathbb{C}_0 := \mathbb{C}^\times$  whose super Lie algebra has the appropriate structure. Since we are trying to make the complex tangent vector  $\partial_{\bar{z}}$  into a square, it seems quite natural

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<sup>5</sup>Here and in the following we use the notation  $\phi(z, \bar{z})$  (instead of simply  $\phi(z)$ ) to indicate that the map  $\phi$  is only smooth and not necessarily holomorphic: it may depend on  $\bar{z}$ . This at first glance confusing convention will be very useful in the next section where we will deal with maps between  $cs$  manifolds that come from complex supermanifolds.

to work in the  $cs$  category. Playing around with formulas it is not too difficult to come up with a candidate for an appropriate super extension. Consider the  $(2|1)$ -dimensional  $cs$  manifold  $\mathbb{R}_{0 \otimes \mathbb{C}}^{2|1} = \mathbb{R}_{0 \otimes \mathbb{C}}^{2|0} \times \mathbb{R}_{\otimes \mathbb{C}}^{0|1} = \mathbb{C}_0 \times \mathbb{R}_{\otimes \mathbb{C}}^{0|1}$ . The map

$$(z_1, \bar{z}_1, \theta_1), (z_2, \bar{z}_2, \theta_2) \mapsto (z_1 z_2, \bar{z}_1 \bar{z}_2 + \theta_2 \theta_1, \bar{z}_1 \theta_2 + \bar{z}_2 \theta_1)$$

makes  $\mathbb{C}_0 \times \mathbb{R}_{\otimes \mathbb{C}}^{0|1}$  into a  $cs$  group, which we will denote by  $\bar{\mathfrak{C}}_{cs}$ .<sup>6</sup> The group properties are easy to check, but also follow in a more conceptual way from the considerations in Section 4.5. A basis for the super Lie algebra of  $\bar{\mathfrak{C}}_{cs}$  is given by

$$z\partial_z, \bar{z}\partial_{\bar{z}} + \theta\partial_\theta, \text{ and } \theta\partial_{\bar{z}} + \bar{z}\partial_\theta.$$

The Lie algebra structure is given by the relations  $[\theta\partial_{\bar{z}} + \bar{z}\partial_\theta, z\partial_z] = 0$  and

$$\frac{1}{2}[\theta\partial_{\bar{z}} + \bar{z}\partial_\theta, \theta\partial_{\bar{z}} + \bar{z}\partial_\theta] = \bar{z}\partial_{\bar{z}} + \theta\partial_\theta.$$

In particular, we see that  $\partial_{\bar{z}}$  becomes a square at the identity element  $(1, 0)$ .

Now, let  $\bar{\mathfrak{A}}_{cs}$  be the open sub  $cs$  semigroup of  $\bar{\mathfrak{C}}_{cs}$  defined by the inclusion  $D_0 \hookrightarrow \mathbb{C}_0$ . The structure of the super Lie algebra of  $\bar{\mathfrak{C}}_{cs}$  tells us that  $\bar{\mathfrak{A}}_{cs}$  is the  $cs$  semigroup we were looking for. This is made precise in the following proposition.

**Proposition 11.** *A  $\mathbb{Z}_2$ -equivariant semigroup homomorphism*

$$\phi : D_0 \rightarrow \text{HS}(\mathcal{H}), \phi(z, \bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0},$$

*extends to a  $cs$  semigroup homomorphism  $\Phi : \bar{\mathfrak{A}}_{cs} \rightarrow \text{HS}(\mathcal{H})$  if and only if  $\bar{L}_0$  is the square of an odd operator  $\bar{G}_0$  that commutes with  $L_0$ . If, in addition,  $\phi$  takes values in  $C_n$ -linear operators ( $\Rightarrow L_0, \bar{L}_0$  are  $C_n$ -linear), then  $\bar{G}_0$  can be chosen to be  $C_n$ -linear.*

*Proof.* Assume we are given an extension

$$\Phi : \bar{\mathfrak{A}}_{cs} \rightarrow \text{HS}(\mathcal{H}), \Phi(z, \bar{z}, \theta) = \phi(z, \bar{z}) + \theta B(z, \bar{z}).$$

Then the homomorphism property says that

$$\begin{aligned} \Phi(z_1, \bar{z}_1, \theta_1)\Phi(z_2, \bar{z}_2, \theta_2) &= (\phi(z_1, \bar{z}_1) + \theta_1 B(z_1, \bar{z}_1))(\phi(z_2, \bar{z}_2) + \theta_2 B(z_2, \bar{z}_2)) \\ &= \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2) + \theta_1 B(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2) \\ &\quad + \theta_2 \phi(z_1, \bar{z}_1)B(z_2, \bar{z}_2) - \theta_1 \theta_2 B(z_1, \bar{z}_1)B(z_2, \bar{z}_2) \end{aligned}$$

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<sup>6</sup>The notation  $\bar{\mathfrak{C}}_{cs}$  will become clear in Section 4.5

equals

$$\begin{aligned}
\Phi(z_1 z_2, \bar{z}_1 \bar{z}_2 + \theta_2 \theta_1, \bar{z}_1 \theta_2 + \bar{z}_2 \theta_1) &= \phi(z_1 z_2, \bar{z}_1 \bar{z}_2 + \theta_2 \theta_1) + (\bar{z}_1 \theta_2 + \bar{z}_2 \theta_1) B(z_1 z_2, \bar{z}_1 \bar{z}_2 + \theta_2 \theta_1) \\
&= \phi(z_1 z_2, \bar{z}_1 \bar{z}_2) + \theta_2 \theta_1 \frac{\partial \phi}{\partial \bar{z}}(z_1 z_2, \bar{z}_1 \bar{z}_2) \\
&\quad + \theta_1 \bar{z}_2 B(z_1 z_2, \bar{z}_1 \bar{z}_2) + \theta_2 \bar{z}_1 B(z_1 z_2, \bar{z}_1 \bar{z}_2).
\end{aligned}$$

Comparing the coefficients we find, in addition to the homomorphism property of  $\phi$ , that

$$\bar{z}_2^{-1} B(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) = B(z_1 z_2, \bar{z}_1 \bar{z}_2) = \bar{z}_1^{-1} \phi(z_1, \bar{z}_1) B(z_2, \bar{z}_2)$$

and that

$$\frac{\partial \phi}{\partial \bar{z}}(z_1 z_2, \bar{z}_1 \bar{z}_2) = B(z_1, \bar{z}_1) B(z_2, \bar{z}_2).$$

The first relation implies that the operators  $\phi(z_1, \bar{z}_1)$  and  $B(z_2, \bar{z}_2)$  commute for all  $z_1$  and  $z_2$ . Thus, if we denote by  $H_{\lambda, \mu}$  the subspace of  $\mathcal{H}$  on which  $\phi(z, \bar{z})x = z^\lambda \bar{z}^\mu x$ , then  $H_{\lambda, \mu}$  is also an invariant subspace for all  $B(z, \bar{z})$ . The second relation gives that on  $H_{\lambda, \mu}$  we have

$$\mu \bar{z}^{-2} z^{2\lambda} \bar{z}^{2\mu} = \frac{\partial \phi}{\partial \bar{z}}(z^2, \bar{z}^2) = B(z, \bar{z})^2.$$

This equation, together with  $\bar{L}_0 = \mu$  on  $H_{\lambda, \mu}$ , implies that if we define  $\bar{G}_0$  by requiring

$$\bar{G}_0 := \bar{z} z^{-\lambda} \bar{z}^{-\mu} B(z, \bar{z})$$

on  $H_{\lambda, \mu}$ , we automatically have  $\bar{L}_0 = \bar{G}_0^2$ . Furthermore, because  $B(z, \bar{z})$  is odd, the same holds for  $\bar{G}_0$ . Finally, since  $L_0 = \lambda$  on  $H_{\lambda, \mu}$  and since  $H_{\lambda, \mu}$  is an invariant subspace of  $\bar{G}_0$ , we see that  $[\bar{G}_0, L_0] = 0$ . This proves the first implication.

We should point out that the  $z$ -value chosen in the definition of  $\bar{G}_0$  is irrelevant. In fact,  $\bar{G}_0$  is independent of  $z$ . To see this, note that all  $\bar{G}_0(z, \bar{z})$  commute, because the  $B(z, \bar{z})$ 's do. This together with  $\bar{G}_0^2 = \bar{L}_0 = \mu$  implies that  $H_{\lambda, \mu}$  splits into simultaneous eigenspaces  $H_{\lambda, \mu}^+$  and  $H_{\lambda, \mu}^-$  with eigenvalues  $\pm \sqrt{\mu}$  for all  $\bar{G}_0(z, \bar{z})$ . Clearly, the signs in front of  $\sqrt{\mu}$  are the same for all  $z$ , since  $\bar{G}_0(z, \bar{z})$  depends continuously on  $z$ .

Let us now prove the converse. Assume we are given  $\bar{G}_0$ . Then we can define a  $cs$  semigroup homomorphism  $\Phi = \Phi_{\bar{G}_0}$  by letting

$$\Phi(z, \bar{z}, \theta) := \phi(z, \bar{z}) + \theta \bar{G}_0 \bar{z}^{-1} z^{L_0} \bar{z}^{\bar{L}_0}$$

It is immediate that  $\phi(z, \bar{z})$  and  $B(z, \bar{z}) = \bar{G}_0 \bar{z}^{-1} z^{L_0} \bar{z}^{\bar{L}_0}$  satisfy the relations we computed in the first part of the proof. Thus  $\Phi$  is indeed a  $cs$  semigroup homomorphism.

Finally, the remark about the  $C_n$ -linearity is clear from the definition of  $\bar{G}_0$ .  $\square$

**Remark.** It is possible to interpret  $\bar{\mathfrak{A}}_{cs}$  as a  $cs$  semigroup of *conformal  $cs$  toy annuli* endowed with an appropriate geometric structure. This leads to the approach suggested by Stolz and Teichner (unpublished). Roughly speaking, they define a *conformal  $cs$  surface* to be a  $(2|1)$ -dimensional  $cs$  manifold  $\Sigma$  equipped with two distributions  $\mathcal{D}_{ev}, \mathcal{D}_{odd} \subset T\Sigma$  of dimensions  $(1|0)$  and  $(0|1)$ , respectively, such that the Lie bracket induces an isomorphism

$$\mathcal{D}_{odd} \otimes \mathcal{D}_{odd} \longrightarrow T\Sigma / (\mathcal{D}_{odd} \oplus \mathcal{D}_{ev}).$$

We will describe these objects more in detail in Section 5.2. For now, suffice it to say that  $\mathbb{R}_0^{2|1} \otimes \mathbb{C}$  becomes a conformal  $cs$  surface when equipped with the distributions  $\mathcal{D}_{odd}$  and  $\mathcal{D}_{ev}$  coming from the left invariant vector fields of the  $cs$  group  $\bar{\mathfrak{C}}_{cs}$ ,

$$\mathcal{D}_{odd} = \langle \theta \partial_{\bar{z}} + \bar{z} \partial_{\theta} \rangle \text{ and } \mathcal{D}_{ev} = \langle z \partial_z \rangle.$$

Left translations in  $\bar{\mathfrak{C}}_{cs}$  can be used to associate to every morphism of  $cs$  manifolds  $\alpha : S \longrightarrow \bar{\mathfrak{A}}_{cs}$  a unique family  $A_\alpha \rightarrow S$  of ‘conformal  $cs$  toy annuli’ over  $S$ . We will describe the construction in detail for SUSY toy annuli in Section 5.1; the  $cs$  case works in the same way. As in the proof of Proposition 13 below, one checks that the super group structure on  $\bar{\mathfrak{A}}_{cs}$  corresponds exactly to gluing of super conformal annuli. This yields an interpretation of  $\bar{\mathfrak{A}}_{cs}$  in terms of conformal  $cs$  toy annuli.

We thus would expect to find  $\bar{\mathfrak{A}}_{cs}$  as a sub  $cs$  semigroup of the moduli  $cs$  semigroup of all conformal  $cs$  annuli. In particular, a definition of supersymmetric CFTs based on conformal  $cs$  surfaces would lead to a representation of the super semigroup  $\bar{\mathfrak{A}}_{cs}$  extending the homomorphism  $D_0 \rightarrow \text{HS}(\mathcal{H})$  that determines the partition function of the theory. As a consequence, our Proposition 11 shows that the super conformal structure suggested by Stolz and Teichner indeed gives rise to holomorphic partition functions.

As laid out in the introduction, what we have in mind is something slightly different: our goal is to connect the supersymmetry aspect in the Stolz-Teichner project

to SUSY curves. A first step in this direction is the observation that the  $cs$  group  $\bar{\mathfrak{C}}_{cs}$  is in fact complex analytic in the sense that it comes from a complex analytic super group  $\bar{\mathfrak{C}}$ . This will allow us to interpret  $\bar{\mathfrak{A}}_{cs}$  as a super semigroup of (antiholomorphic) SUSY annuli, see Chapter 5. Before we can say in what sense  $\bar{\mathfrak{C}}_{cs}$  comes from a complex super group, we have to investigate the relation between  $cs$  manifolds and complex supermanifolds.

## 4.4 Complex supermanifolds and $cs$ manifolds

In the definition of susy CFTs that we have in mind complex supermanifolds play an important role. However, we will also want to look at functions on complex supermanifolds that are not holomorphic, but merely smooth. When trying to define the ‘smooth supermanifold underlying a complex analytic supermanifold’ one is naturally led to consider  $cs$  manifolds.<sup>7</sup> In this section, we add more detail to the brief introduction to  $cs$  manifolds in 4.2 and investigate the relation between complex supermanifolds and  $cs$  manifolds. A good reference for the basic theory of complex supermanifolds is Chapter 4 in [Ma1]; for  $cs$  manifolds, we refer to [DM], §4.8.

### Coordinate representation of morphisms

Morphisms between supermanifolds can be described in terms of coordinates.<sup>8</sup> This works for real and complex supermanifolds, as well as for  $cs$  manifolds. We will explain the formalism, but omit proofs. For real and complex supermanifolds the proof is given in [Ma1], Chapter 4, §1.8; a more detailed exposition can be found in [Lei]. The case of  $cs$  manifolds is not treated in these sources, but the proof carries over without essential change.

Let  $U^{p|q}$  and  $V^{r|s}$  be super domains in any of the three categories of supermanifolds we are interested in, and let  $u^1, \dots, u^p, \theta^1, \dots, \theta^q$  and  $v^1, \dots, v^r, \eta^1, \dots, \eta^s$  be the standard coordinates on  $U$  and  $V$ , resp.<sup>9</sup> Then a morphism  $f = (|f|, f^*) : U \rightarrow V$  can

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<sup>7</sup>Every complex supermanifold  $M$  of dimension  $(p|q)$  has an underlying *real* analytic supermanifold of dimension  $(2p|2q)$ , see [DM], §4.6. The construction we have in mind is different and yields a  $cs$  manifold of dimension  $(2p|q)$ .

<sup>8</sup>By a morphism between supermanifolds we always mean a morphism of ringed spaces.

<sup>9</sup>A *super domain* is an open sub supermanifold of  $\mathbb{R}^{p|q}$ ,  $\mathbb{C}^{p|q}$ , or  $\mathbb{R}_{\otimes\mathbb{C}}^{p|q}$ .

be described by specifying the images  $f^k$  and  $\psi^l$  of the coordinate functions  $v^k$  and  $\eta^l$  in the algebra of global functions  $O_U := O_U(U)$  on  $U$ . This justifies the frequently used notation

$$f(u^1, \dots, u^p, \theta^1, \dots, \theta^q) = (f^1, \dots, f^r, \psi^1, \dots, \psi^s)$$

for morphisms between super domains. Note the components  $f^k$  and  $\psi^l$  are elements in  $O_U$  and hence are indeed functions of the  $u^i$  and  $\theta^j$  in the sense that they can be (uniquely) expressed in the form

$$f^k = \sum_I f_I^k(u^1, \dots, u^p) \theta^I \text{ and } \psi^l = \sum_I \psi_I^l(u^1, \dots, u^p) \theta^I.$$

Here the index  $I$  ranges over all subsets of  $\{1, \dots, q\}$  and the  $f_I^k$  and  $\psi_I^l$  are smooth functions on  $|U| \subset \mathbb{R}^p$ . We also write  $f_b^k$  for the ‘body part’  $f_\emptyset^k$  of  $f^k$ .

Since the algebra map  $f^* : O_V \rightarrow O_U$  preserves the grading, the  $f^k$  are even functions on  $U$ , whereas the  $\psi^l$  are odd. Furthermore, it can be checked that the induced map  $f_{\text{red}}^* : O_{V_{\text{red}}} \rightarrow O_{U_{\text{red}}}$  is necessarily given by pullback under  $|f|$ . This implies that

$$|f|(u) = (f_b^1(u), \dots, f_b^r(u)) \in |V| \text{ for all } u \in |U|.$$

The *fundamental fact* is that, conversely, any choice of  $f^k$  and  $\psi^l$  satisfying these necessary conditions determines a unique morphism  $f : U \rightarrow V$ .

We would like to point out that in the case of *cs* manifolds, the second condition contains implicitly that the  $f_b^k$  are *real*. The reason is that the standard coordinate functions  $v^k$  are real-valued and hence the same is true for their pullbacks under  $|f|$ . When studying the relation between complex supermanifolds and *cs* manifolds, it will be convenient to work with coordinate systems that are not real. In this case one should be careful not to forget about this reality condition. For example, if we choose the coordinate system  $(z, \bar{z}, \theta)$  on  $\mathbb{R}_{\otimes \mathbb{C}}^{2|1}$  and want to describe a map  $f : \mathbb{R}_{\otimes \mathbb{C}}^{2|1} \rightarrow \mathbb{R}_{\otimes \mathbb{C}}^{2|1}$  by, say,  $f(z, \bar{z}, \theta) = (f^1, f^2, f^3)$ , then we need to ensure that  $\frac{1}{2}(f_b^1 + f_b^2)$  and  $\frac{1}{2i}(f_b^1 - f_b^2)$  are real.

The above coordinate notation of morphisms only describes the images of the coordinates  $v^k$  and  $\eta^l$  under the algebra map  $f^* : O_V \rightarrow O_U$ . However, sometimes we will need to know what happens to an arbitrary element  $g$  in  $O_V$ . For example, this is necessary when one tries to compute the composition  $gf$  of two morphisms: the image

of a coordinate function  $w_i \in O_W$  under  $f^*g^*$  is  $f^*(g_i)$ , where  $g_i = g^*(w_i) \in O_V$ . Thus, let us write down what  $f^*$  does to an arbitrary element  $g = \sum_J g_J(v_1, \dots, v_r)\eta^J$ . Since  $f^*$  is an algebra map, we must have

$$f^*(g) = \sum_J f^*(g_J)\psi^J.$$

Thus it suffices to consider the case  $g = g_b$  of a purely even function on  $V$ . This is where the typical ‘Taylor expansion around the body part’ comes in:

$$f^*(g) = \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}} g}{\partial u^{\mathbf{k}}} (f_b^1, \dots, f_b^r) (f - f_b)^{\mathbf{k}}.$$

Here  $\mathbf{k} = (k_1, \dots, k_r)$  is a multi-index and  $(f - f_b)^{\mathbf{k}} = \prod_{1 \leq i \leq r} (f^i - f_b^i)^{k_i}$ . This formula might look surprising at first, but one sees quite quickly, for example by looking at monomials, that there is no other choice, since  $f^*$  is an algebra homomorphism. The formula is also the key point in the proof that every admissible choice of  $f^k$  and  $\psi^l$  defines an algebra map, cf. [Lei], 2.2.11, or [DM], §2.4.

## Complex supermanifolds and $cs$ manifolds

**Proposition 12.** *There is a forgetful functor  $\mathcal{F} = (\cdot)_{cs}$  from the category of complex supermanifolds to the category of  $cs$  manifolds. It associates to a complex supermanifold  $M$  of dimension  $(p|q)$  a  $cs$  manifold  $M_{cs}$  of dimension  $(2p|q)$ . It is determined by sending a super domain  $(|U|, \text{Hol}(\cdot)[\theta^1, \dots, \theta^q]) \subset \mathbb{C}^{p|q}$  to  $(|U|, C^\infty(\cdot, \mathbb{C})[\theta^1, \dots, \theta^q]) \subset \mathbb{R}_{\otimes \mathbb{C}}^{2p|q}$  and by the formula for morphisms  $g$  between super domains given below.*

*Proof.* It suffices to define  $\mathcal{F}$  on the full subcategory of complex super domains. The general case then follows by gluing, since functoriality on super domains implies that the cocycle condition for triple intersections is preserved under  $\mathcal{F}$ .

For a complex super domain  $U$  we define a  $cs$  domain  $U_{cs} = \mathcal{F}(U)$  by allowing smooth functions instead of only holomorphic ones: we endow  $|U|$  with the sheaf  $C^\infty(\cdot, \mathbb{C})[\theta^1, \dots, \theta^q]$ . Hence considering  $U$  as a  $cs$  domain might be thought of as doubling the number of even coordinates: in addition to the holomorphic coordinates  $z^1, \dots, z^p, \theta^1, \dots, \theta^q$  we have the antiholomorphic coordinates  $\bar{z}^1, \dots, \bar{z}^p$ .

Next, we have to define  $\mathcal{F}$  on morphisms. We will do so using the coordinate representation of morphisms described in the previous section. Let  $f : U^{p|q} \rightarrow V^{r|s}$  be given by

$$f(z^1, \dots, z^p, \theta^1, \dots, \theta^q) = (f^1, \dots, f^r, \psi^1, \dots, \psi^s).$$

We define by  $\mathcal{F}(f) = f_{cs}$  by

$$f_{cs}(z^1, \dots, z^p, \bar{z}^1, \dots, \bar{z}^p, \theta^1, \dots, \theta^q) := (f^1, \dots, f^r, \bar{f}_b^1, \dots, \bar{f}_b^r, \psi^1, \dots, \psi^s),$$

i.e. we extend  $f$  trivially in the ‘antiholomorphic soul directions’. Note that there is no choice possible in the ‘antiholomorphic body directions’, since these are already determined by the underlying topological map  $|f_{cs}| = |f|$ .

Of course, the crucial point is to check that this association is functorial. Let  $f : U^{p|q} \rightarrow V^{r|s}$  as before and consider, in addition,  $g : V \rightarrow W^{t|u}$  given by

$$g(w^1, \dots, w^r, \eta^1, \dots, \eta^s) = (g^1, \dots, g^t, \chi^1, \dots, \chi^u).$$

Write  $g^m = \sum_I g_I^m \eta^I$ . Then for  $1 \leq m \leq t$ , the  $m^{\text{th}}$  coordinate function of the composition  $gf$  is given by

$$f^*(g^m) = g^m(f^1, \dots, \psi^s) = \sum_{I, \mathbf{k}} \frac{1}{\mathbf{k}!} \frac{\partial g_I^m}{\partial w^{\mathbf{k}}} (f_b^1, \dots, f_b^r) (f - f_b)^{\mathbf{k}} \psi^I.$$

Similarly, when  $t < m \leq t + u$ , the  $m^{\text{th}}$  coordinate of  $gf$  is equal to the same expression with  $g$  replaced by  $\chi$ . According to the definition of  $\mathcal{F}$ , the same formulas also describe the first  $t$  and the last  $u$  coordinates of  $(gf)_{cs}$ . To get the  $t$  ‘antiholomorphic’ coordinates of  $(gf)_{cs}$ , we have to take the complex conjugate of the body part of  $g^m(f^1, \dots, \psi^s)$  which equals  $\bar{g}_b^m(f_b^1, \dots, f_b^r)$  as follows directly from the formula above.

What happens if we apply  $\mathcal{F}$  first and then compose? Let us begin with the first  $t$  coordinates: for  $1 \leq m \leq t$ , the  $m^{\text{th}}$  coordinate function of  $g_{cs} f_{cs}$  is

$$f_{cs}^*(g_{cs}^m) = g^m(f^1, \dots, f^r, \bar{f}_b^1, \dots, \bar{f}_b^r, \psi^1, \dots, \psi^s) = \sum_{I, \mathbf{k}} \frac{1}{\mathbf{k}!} \frac{\partial g_I^m}{\partial w^{\mathbf{k}}} (f_b^1, \dots, \bar{f}_b^r) (f - f_b)^{\mathbf{k}} \psi^I.$$

The crucial point here is that even though we write down the Taylor series in the  $cs$  category, the holomorphicity of  $g_I^m$  implies that all terms involving derivatives in the  $\bar{w}$  directions vanish and hence they do not show up in the sum. Thus, we see that the first

$t$  coordinates of  $(gf)_{cs}$  and  $g_{cs} f_{cs}$  agree. The argument for the last  $u$  coordinates is identical: simply replace  $g$  by  $\chi$ .

Now, look at the ‘antiholomorphic’ directions: the  $m^{\text{th}}$  ‘antiholomorphic coordinate function’ of  $g_{cs} f_{cs}$  is

$$f_{cs}^* (\bar{g}_b^m) = \bar{g}_b^m(f^1, \dots, \bar{f}_b^r, \psi^1, \dots, \psi^s) = \sum_{\mathbf{k}} \frac{1}{\mathbf{k}!} \frac{\partial \bar{g}_b^m}{\partial \bar{w}^{\mathbf{k}}} (f_b^1, \dots, \bar{f}_b^r) (f_b - (f_b)_b)^{\mathbf{k}} = \bar{g}_b^m(f_b^1, \dots, \bar{f}_b^r).$$

Since  $\bar{g}_b^m$  is antiholomorphic, no terms involving derivatives in the  $w$  directions occur. The last equality follows, since for  $\mathbf{k}$  non-trivial  $(f_b - (f_b)_b)^{\mathbf{k}} = (f_b - f_b)^{\mathbf{k}} = 0$ . We conclude that the  $t$  ‘antiholomorphic’ coordinates of  $(gf)_{cs}$  and  $g_{cs} f_{cs}$  also agree. This completes the proof.  $\square$

**Remark.** It would be more satisfactory to write down the sheaf of functions  $\mathcal{O}_{M_{cs}}$  of the  $cs$  manifold  $M_{cs}$  underlying a complex supermanifold  $M$  explicitly, rather than work locally as in our proof. It was pointed out to me by Stephan Stolz and Peter Teichner that it should be possible to construct  $\mathcal{O}_{M_{cs}}$  directly as a completion of the tensor product (over  $\mathbb{C}$ ) of the original sheaf of holomorphic functions  $\mathcal{O}_M$  and the quotient of its complex conjugate by its nilradical  $\bar{\mathcal{O}}_M/\text{Nil}$ . A similar completion is used to explicitly construct the product of two supermanifolds in [BBH].

## 4.5 The $cs$ semigroup of toy annuli is complex analytic

We will now use Proposition 12 to identify  $\bar{\mathfrak{C}}_{cs}$  as a  $cs$  group underlying a complex super Lie group. Let us look at some examples of complex super Lie groups.

**Examples.**

1. The addition on  $\mathbb{C}$  extends to the structure of a complex super Lie group on  $\mathbb{C}^{1|1}$ ,

$$\mathbb{C}^{1|1} \times \mathbb{C}^{1|1} \longrightarrow \mathbb{C}^{1|1}, (z_1, \theta_1), (z_2, \theta_2) \mapsto (z_1 + z_2 + \theta_1 \theta_2, \theta_1 + \theta_2).$$

The Lie algebra of this super group is freely generated by the odd vector field  $-\theta \partial_z + \partial_\theta$ . The right invariant vector field  $\theta \partial_z + \partial_\theta$  will come up again in the next chapter, when we consider the local normal form of a (family of) SUSY curves.

2. The multiplication on  $\mathbb{C}_0 = \mathbb{C}^\times$  also has a super extension; it is given by

$$\mathbb{C}_0^{1|1} \times \mathbb{C}_0^{1|1} \longrightarrow \mathbb{C}_0^{1|1}, (z_1, \theta_1), (z_2, \theta_2) \mapsto (z_1 z_2 + \theta_1 \theta_2, z_1 \theta_2 + z_2 \theta_1).$$

The Lie algebra of this group is freely generated by  $-\theta \partial_z + z \partial_\theta$ ; this vector field squares to  $-(z \partial_z + \theta \partial_\theta)$ . Furthermore, we have the odd right invariant vector field  $\theta \partial_z + z \partial_\theta$ . The relation between  $\mathbb{C}_0^{1|1}$  and the first example is given by the super extension of the exponential function, see [Ma2], Chapter 2, Lemma 8.2; it induces an isomorphism of the super Lie algebras  $\text{Lie}(\mathbb{C}^{1|1}) \xrightarrow{\cong} \text{Lie}(\mathbb{C}_0^{1|1})$ .

We assert that the *cs* group  $\mathbb{C}_{0\text{cs}}^{1|1}$  underlying  $\mathbb{C}_0^{1|1}$  is closely related to the the *cs* group  $\bar{\mathfrak{C}}_{cs}$  we considered in conjunction with the semigroup of *cs* toy annuli in 4.3. To see this, denote by  $\mathfrak{C}$  the opposite of the group of  $\mathbb{C}_0^{1|1}$ .<sup>10</sup> From the construction of the forgetful functor  $\mathcal{F}$  it is clear that the multiplication on the *cs* group  $\mathfrak{C}_{cs}$  underlying  $\mathfrak{C}$  is given by

$$(z_1, \bar{z}_1, \theta_1), (z_2, \bar{z}_2, \theta_2) \mapsto (z_1 z_2 + \theta_2 \theta_1, \bar{z}_1 \bar{z}_2, z_1 \theta_2 + z_2 \theta_1).$$

This is almost the multiplication on  $\bar{\mathfrak{C}}_{cs}$  - only the roles of  $z$  and  $\bar{z}$  are switched. In fact, the *cs* groups  $\mathfrak{C}_{cs}$  and  $\bar{\mathfrak{C}}_{cs}$  are isomorphic via the map

$$\mathfrak{C}_{cs} \longrightarrow \bar{\mathfrak{C}}_{cs}, (z, \bar{z}, \theta) \mapsto (\bar{z}, z, \theta).$$

The motivation for using the notation  $\bar{\mathfrak{C}}_{cs}$  lies in the hope that passing to the complex conjugate  $\bar{\mathfrak{C}}$  of  $\mathfrak{C}$  before applying the functor  $\mathcal{F}$  will result in a *cs* group that is even *canonically isomorphic* to  $\bar{\mathfrak{C}}_{cs}$ . Unfortunately, I was not able to formulate this properly, since I do not yet fully understand the relation between complex conjugates and the functor  $\mathcal{F}$ . The reason we would prefer to work with  $\bar{\mathfrak{C}}_{cs}$  instead of  $\mathfrak{C}_{cs}$  is that it gives supersymmetry in  $\bar{z}$ -direction (rather than in  $z$ -direction), when  $\mathbb{C}$  is embedded in the natural way in  $(\mathbb{C}, C^\infty(-, \mathbb{C})[\theta])$ , the *cs* manifold underlying both groups  $\mathfrak{C}_{cs}$  and  $\bar{\mathfrak{C}}_{cs}$ .

From this discussion we immediately see that  $\bar{\mathfrak{A}}_{cs}$  is isomorphic to the *cs* semigroup underlying the obvious complex super semigroup  $\mathfrak{A} \subset \mathfrak{C}$ . Of course, the question arises whether  $\mathfrak{A}$  can be interpreted as a super semigroup of *complex* super annuli. The answer is yes. More precisely, it will turn out that  $\mathfrak{A} \subset \mathfrak{C}$  is a super semigroup of *SUSY*

<sup>10</sup>The multiplication on the opposite  $G^{op}$  of a group  $G$  is defined by  $g_1 \cdot_{op} g_2 := g_2 g_1$ .

*toy annuli*, where we mean SUSY in the sense of Manin, see [Ma2]. Again, we hope that passing to the complex conjugate  $\bar{\mathfrak{A}}$  (or ‘antiholomorphic SUSY curves’) will give the desired  $\bar{z}$ -supersymmetry in a more natural way.

This interpretation of  $\bar{\mathfrak{A}}_{cs}$  in terms of SUSY annuli is our main motivation for the definition of supersymmetric conformal field theories in terms of SUSY curves in the next chapter.

## Chapter 5

# SUSY Curves and supersymmetric CFTs

In this chapter we describe our proposal for the definition of supersymmetric CFTs in terms of SUSY curves. A fundamental point is the introduction of the bordism category the definition is based upon; this is the content of Section 5.1. In 5.2 we compare our bordism category of SUSY curves to the category suggested by Stolz and Teichner (unpublished). In the last section we outline our definition and conclude with some remarks on possible further developments.

### 5.1 The bordism category of SUSY curves

The objects of our category  $\mathcal{C}$  of bordisms of SUSY curves are disjoint unions of copies of the circle  $S^1$  equipped with an orientation and a spin structure. There are two possibilities for the latter: the periodic (non-bounding) or the anti-periodic (bounding) spin structure. Morphisms in  $\mathcal{C}$  are, of course, SUSY bordisms. Since our ultimate goal is the definition of a category  $\underline{\mathcal{C}}$  enriched over complex supermanifolds, we will now define, more generally, families of SUSY bordisms. A good reference for SUSY curves (a.k.a. super Riemann surfaces) is Manin's book [Ma2]. The moduli problem of closed SUSY curves is treated in [LR]; for a more intuitive introduction avoiding the language of sheaves, see [CR].

**Definition 7.**

1. A *family of SUSY curves* over a complex supermanifold  $S$  is a family of complex supermanifolds  $\pi : F \rightarrow S$  of dimension  $(1|1)$  equipped with a  $(0|1)$ -dimensional distribution  $\mathcal{D} \subset T_{F/S}$  such that the Lie bracket induces an isomorphism of sheaves of  $\mathcal{O}_F$ -modules

$$\mathcal{D} \otimes \mathcal{D} \longrightarrow T_{F/S}/\mathcal{D},$$

where  $T_{F/S}$  denotes the relative tangent sheaf of  $F$  over  $S$ .

2. Now, let  $Y = S_{per}^1 \amalg \dots \amalg S_{per}^1 \amalg S_{ap}^1 \amalg \dots \amalg S_{ap}^1$ . Here ‘*per*’ and ‘*ap*’ indicate the choice of a spin structure on each component. A *family of SUSY zero bordisms*  $(\pi, \hat{F}, \bar{e})$  of  $Y$  is a family of SUSY curves  $\pi : F \rightarrow S$  together with a subfamily  $\hat{F} \subset F_{\text{red}}$  of compact surfaces with boundary and a germ of boundary embeddings<sup>1</sup>

$$\bar{e} = (|\bar{e}|, \bar{e}^*) : \text{nb}(Y) \times S \hookrightarrow F,$$

(over  $S$ ) such that  $|\bar{e}|$  restricts to a homeomorphism  $Y \times |S| \xrightarrow{\cong} \partial\hat{F}$ . Here  $\text{nb}(Y)$  denotes a ‘standard thickening’ of  $Y$  defined as follows. For each component  $S^1$  of  $Y$  the thickening  $\text{nb}(S^1)$  is an open sub supermanifold of  $\mathbb{C}_0^{1|1}$  such that  $S^1 \subset |\text{nb}(S^1)|$ . Depending on the spin structure on  $S^1$ , we equip  $\text{nb}(S^1)$  with the SUSY structure  $\mathcal{D} = \langle \theta\partial_z + z\partial_\theta \rangle$  (in the case of  $S_{per}^1$ ) or  $\mathcal{D} = \langle \theta\partial_z + \partial_\theta \rangle$  (in the case of  $S_{ap}^1$ ). Of course, we require the map  $\bar{e}$  to preserve the SUSY structure.

3. Using the collars, we obtain an induced orientation on the boundary exactly as in the EFT case, see Definition 5. We choose our convention so that the boundary of the standard disc inherits a positive orientation. *Families of (oriented) SUSY bordisms* between objects  $Y_1$  and  $Y_2$  in  $\mathcal{C}$  can now be defined in the same way as in Definition 5. Isomorphisms of bordisms are also defined as in the EFT case.

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<sup>1</sup>We should, more accurately, call  $\bar{e}$  a *boundary collar* rather than a boundary embedding. It seems to be possible to formulate the boundary embeddings for SUSY bordisms in the same way as in the case of Euclidian super bordisms, starting with an honest embedding  $e$  rather than a collar  $\bar{e}$ . However, this is technically more complicated, since we would have to consider the *cs* family underlying  $F$  ( $S^{1|1}$  is certainly not a complex supermanifold!). Note that, even though the geometric structure  $\mathcal{D}$  allows many more (local) automorphisms than the vector field  $V$  in the Euclidian case, we would still get unique (germs of) collars by an analytic continuation argument.

**Remarks.**

1. The fundamental example of a SUSY curve is  $\mathbb{C}^{1|1}$  equipped with the SUSY structure defined by the vector field  $\theta\partial_z + \partial_\theta$ . In fact, every family of SUSY curves locally looks like the product family  $\mathbb{C}^{1|1} \times S \rightarrow S$  equipped with the distribution  $\langle \theta\partial_z + \partial_\theta \rangle$  and hence can be obtained by gluing such trivial families, cf. Lemma 1.2 in [LR].
2. A family of SUSY curves over a point is the same as a conformal spin surface, cf. [LR], §1. In particular, we see that our category  $\mathcal{C}$  is equivalent to the category of conformal spin bordisms, as desired. Note that this also shows that SUSY structures satisfy the first condition of Hypothesis 3.3.13 in [ST].
3. The previous remark explains the appearance of the SUSY structures on  $\mathbb{C}_0^{1|1}$  given by the vector fields  $\theta\partial_z + \partial_\theta$  and  $\theta\partial_z + z\partial_\theta$  in the definition of families of zero bordisms above: the SUSY structures on  $\mathbb{C}_0^{1|1}$  (up to isomorphism) correspond to spin structures on  $\mathbb{C}_0$  and hence there are precisely two choices. Clearly, the distribution generated by  $\theta\partial_z + \partial_\theta$  extends over the origin; it defines the SUSY structure corresponding to the bounding spin structure on  $S^1$ .
4. The geometric collars ensure a well-defined gluing operation of families of SUSY bordisms. It is possible to allow slightly more general boundary parametrizations that still admit gluing without significant technical difficulties. Namely, we could use one-sided instead of two-sided collars and instead of gluing the two-sided collars of two bordisms directly, we could glue each of the one-sided collars to the standard neighborhood of the circle along which we join them. It is an interesting question whether or not there is a gluing theorem for families of SUSY curves with more general boundary parametrizations (such as *cs* diffeomorphisms), analogous to the ‘sewing theorem’ for ordinary Riemann surfaces (see [LV]). One possible idea is to try to imitate the proof of the sewing theorem using the solvability of the Beltrami equation. However, even though quite some theory of super Beltrami equations has been developed, this is a subtle question, since it would require considering discontinuous super maps.

## The super semigroup of SUSY toy annuli

Recall the fundamental role that the super semigroup of Euclidian super intervals played for susy EFTs. The analogue in the context of susy CFTs should be a super extension of the semigroup of annuli with parametrized boundaries that appeared in [Se1], §2. In fact, taking the spin structures into account we obtain two semigroups, one for each spin structure of the circle. Using SUSY families, we can easily define these as *generalized* complex super semigroups: gluing of families defines a semigroup structure on the set of isomorphism classes of connected genus zero SUSY bordisms from  $S_{per}^1$  to  $S_{per}^1$  over a fixed base  $S$  (similarly for  $S_{ap}^1$ ). It seems that these generalized objects can be represented as honest (infinite-dimensional) complex super semigroups, but we will not attempt to do this here. See, however, the remark about this in Section 5.3.

Instead, we will only consider the super semigroup of SUSY toy annuli. These are particularly simple SUSY annuli that occur as subfamilies of the product family  $\pi : \mathbb{C}^{1|1} \times S \rightarrow S$  equipped with the SUSY structure  $\mathcal{D} = \langle \theta \partial_z + z \partial_\theta \rangle$ . Recall from Section 4.5 that  $\theta \partial_z + z \partial_\theta$  is an odd left invariant vector field of the complex super group  $\mathfrak{C} = (\mathbb{C}_0^{1|1})^{op}$ . In particular, left translations in  $\mathfrak{C}$  preserve the distribution  $\mathcal{D}$ . Hence, for every morphism  $\alpha : S \rightarrow \mathfrak{A} \subset \mathfrak{C}$  we have a unique germ of maps of SUSY families

$$l_\alpha : \text{nb}(S_{per}^1) \times S \longrightarrow \mathbb{C}^{1|1} \times S$$

given by left translation with by  $\alpha$ . Note that  $l_\alpha(|S^1| \times |S|) \subset D_0^2 \times S$ , since  $\alpha$  maps to  $\mathfrak{A}$ . We denote by  $A_\alpha$  the family of SUSY bordisms  $(\pi, \hat{F}, \bar{e})$  from  $S_{per}^1$  to itself defined by the trivial SUSY family  $\pi$ ,

$$\hat{F} := \{ (z, s) \in \mathbb{C} \times |S| \text{ such that } |l_\alpha(s)| \leq |z| \leq 1 \},$$

and the boundary embedding  $\bar{e}$  defined by  $l_\alpha$  (incoming boundary) and the identity (outgoing boundary). We refer to the families  $A_\alpha$  as *SUSY toy annuli*. The following lemma says that the super semigroup of SUSY toy annuli is nothing but  $\mathfrak{A}$  itself.

**Lemma 13.** *The semigroup-valued functor that associates to a complex supermanifold  $S$  the set of isomorphism classes of families of SUSY toy annuli over  $S$  is represented by the complex super semigroup  $\mathfrak{A}$ . The universal family of toy annuli is  $A_{\text{id}_{\mathfrak{A}}}$ .*

*Proof.* From our definition of SUSY toy annuli over  $S$  it is almost obvious that SUSY toy annuli are classified by maps to  $\mathfrak{A}$ . The only thing to check is that two maps  $\alpha_i : S \rightarrow \mathfrak{A}$  define isomorphic families only if  $\alpha_1 = \alpha_2$ . The reason is that an isomorphism between  $A_{\alpha_1}$  and  $A_{\alpha_2}$  is necessarily the identity near the outgoing boundary and hence must be the identity by analytic continuation.<sup>2</sup> It remains to verify that gluing of annuli gives the semigroup structure on  $\mathfrak{A}$ . This amounts to checking that given two families  $A_{\alpha_i}$ , gluing the outgoing boundary of  $A_{\alpha_2}$  to the incoming boundary of  $A_{\alpha_1}$  yields  $A_{\alpha_1\alpha_2}$  (recall our convention for the composition law from Section 1.3!). This follows by the same argument that we used in the proof of Proposition 8. The only difference is that we now use left translations instead of right translations, because for the annuli  $A_\alpha$  the outgoing boundary is normalized as opposed to the incoming in the case of the intervals  $I_\omega$ . The universality of  $A_{\text{id}_{\mathfrak{A}}}$  is obvious.  $\square$

The lemma also tells us that we will find a copy of  $\mathfrak{A}$  inside of the super semigroup of all SUSY annuli with the non-bounding spin structure. Hence, if we define a supersymmetric CFT as an operator-valued function on the  $cs$  manifold  $\bar{\mathcal{M}}_{cs}$  underlying the complex conjugate of the moduli space  $\mathcal{M}$  of SUSY curves (see Section 5.3 for details) we obtain, among other things, a representation of  $\bar{\mathfrak{A}}_{cs}$ . As a consequence, Proposition 11 implies that our proposed notion of susy CFTs satisfies the second condition demanded by Hypothesis 3.3.13.

In the same way as we interpreted  $\mathfrak{A}$  as a super semigroup of SUSY annuli, we can consider  $\bar{\mathfrak{A}}_{cs}$  (which is isomorphic to  $\mathfrak{A}_{cs}$ ) as a  $cs$  semigroup of conformal  $cs$  toy annuli à la Stolz and Teichner (recall the remark at the end of Section 4.3). Altogether, we see that *the  $cs$  semigroup of conformal  $cs$  toy annuli is isomorphic to the  $cs$  semigroup underlying the super semigroup of SUSY toy annuli.* The main goal of the next section is to convince the reader that this is no coincidence and that there is an intrinsic relation between SUSY curves and conformal  $cs$  surfaces.

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<sup>2</sup>The usual analytic continuation arguments also work in the super case: simply apply the non-super argument to each coefficient  $f_I$  of a function  $\sum_I f_I \theta^I$ .

## 5.2 SUSY curves and conformal $cs$ structures

In this section we compare SUSY curves to the super conformal surfaces suggested by Stolz and Teichner. Recall from 4.3 that they consider  $cs$  manifolds  $\Sigma$  of dimension  $(2|1)$  equipped with two distributions  $\mathcal{D}_{ev}$  and  $\mathcal{D}_{odd}$  of dimensions  $(1|0)$  and  $(0|1)$  such that the Lie bracket induces an isomorphism

$$\mathcal{D}_{odd} \otimes \mathcal{D}_{odd} \longrightarrow T\Sigma / (\mathcal{D}_{odd} \oplus \mathcal{D}_{ev}).$$

We will refer to such objects as *conformal  $cs$  surfaces*. Just as before, we can define families of  $cs$  surfaces over a  $cs$  manifold  $S$ , possibly with boundary. We require the following normal form for families of conformal  $cs$  surfaces: locally, every family is assumed to be isomorphic to the trivial family  $\mathbb{R}_{\otimes\mathbb{C}}^{2|1} \times U \rightarrow U$  equipped with the distributions

$$\mathcal{D}_{odd} = \langle \theta \partial_{\bar{z}} + \partial_{\theta} \rangle \text{ and } \mathcal{D}_{ev} = \langle \partial_z \rangle.$$

The appearance of  $\bar{z}$  in the definition of  $\mathcal{D}_{odd}$  is due to the desire for ‘supersymmetry in  $\bar{z}$ -direction’. We could equally well have switched the roles of  $z$  and  $\bar{z}$  in the definition: the isomorphism

$$\mathbb{R}_{\otimes\mathbb{C}}^{2|1} \rightarrow \mathbb{R}_{\otimes\mathbb{C}}^{2|1}, (z, \bar{z}, \theta) \mapsto (\bar{z}, z, \theta),$$

maps  $\theta \partial_{\bar{z}} + \partial_{\theta}$  to  $\theta \partial_z + \partial_{\theta}$  and  $\partial_z$  to  $\partial_{\bar{z}}$ . The resulting local model with  $\mathcal{D}_{odd} = \langle \theta \partial_z + \partial_{\theta} \rangle$  and  $\mathcal{D}_{ev} = \langle \partial_{\bar{z}} \rangle$  is better suited for comparing conformal  $cs$  surfaces and SUSY curves and we will work with it for now.

### Coordinate changes of SUSY curves and conformal $cs$ surfaces

One way to compare SUSY curves and conformal  $cs$  surfaces is by looking at the admissible chart changes between the local models. Let us review the case of SUSY curves. A coordinate change for a family of SUSY curves is an isomorphism

$$\varphi : V \times U \longrightarrow W \times U, (z, \theta, u) \mapsto (F(z, \theta, u), H(z, \theta, u), u),$$

over a super domain  $U$  that respects the distribution  $\mathcal{D}$  generated by  $\theta \partial_z + \partial_{\theta}$  on the open complex super domains  $V, W \subset \mathbb{C}^{1|1}$ . Choosing the basis  $\{\partial_z, \partial_{\theta}\}$  for the vector

fields on  $V$  and  $W$ , the condition that  $\mathcal{D}$  is preserved amounts to the identity

$$\begin{pmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial \theta} \\ -\frac{\partial H}{\partial z} & \frac{\partial H}{\partial \theta} \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = A \begin{pmatrix} H \\ 1 \end{pmatrix}$$

for some invertible function  $A$  on  $V \times U$ . If we write

$$F = f(z, u) + \theta \tilde{f}(z, u) \text{ and } H = h(z, u) + \theta \tilde{h}(z, u),$$

then the matrix identity above yields that

$$\begin{aligned} F(z, \theta, u) &= f + \theta h \sqrt{\frac{\partial f}{\partial z}} \\ H(z, \theta, u) &= h + \theta \sqrt{\frac{\partial f}{\partial z} + h \frac{\partial h}{\partial z}}. \end{aligned}$$

This is the classical form of a ‘super conformal map’, cf. [CR].

Now, let us consider the case of conformal  $cs$  surfaces. Here a coordinate change is given by a morphism of  $cs$  manifolds

$$\varphi : V \times U \longrightarrow W \times U, (z, \bar{z}, \theta, u) \mapsto (F(z, \bar{z}, \theta, u), G(z, \bar{z}, \theta, u), H(z, \bar{z}, \theta, u), u)$$

that preserves the distributions  $\mathcal{D}_{odd} = \langle \theta \partial_z + \partial_\theta \rangle$  and  $\mathcal{D}_{ev} = \langle \partial_{\bar{z}} \rangle$ . The condition for  $\mathcal{D}_{odd}$  amounts to

$$\begin{pmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial \bar{z}} & \frac{\partial F}{\partial \theta} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial \bar{z}} & \frac{\partial G}{\partial \theta} \\ -\frac{\partial H}{\partial z} & -\frac{\partial H}{\partial \bar{z}} & \frac{\partial H}{\partial \theta} \end{pmatrix} \begin{pmatrix} \theta \\ 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} H \\ 0 \\ 1 \end{pmatrix},$$

where we used the basis  $\{\partial_z, \partial_{\bar{z}}, \partial_\theta\}$  for the vector fields on  $\mathbb{R}_{\otimes \mathbb{C}}^{2|1}$ . The first and last row of this matrix equation are exactly the conditions that appeared in the SUSY case and consequently lead to the same equations for  $F$  and  $H$  in terms of an even function  $f(z, \bar{z}, u)$  and an odd function  $h(z, \bar{z}, u)$ . Furthermore, the middle component yields

$$\frac{\partial g}{\partial \bar{z}} = 0 \text{ and } \tilde{g} = 0$$

if we write  $G = g(z, \bar{z}, u) + \theta \tilde{g}(z, \bar{z}, u)$ . Recall from Section 4.4 that the body part of  $g$  is necessarily equal to  $\bar{f}_b$ . The condition that  $\mathcal{D}_{ev}$  is preserved gives

$$\begin{pmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial \bar{z}} & -\frac{\partial F}{\partial \theta} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial \bar{z}} & -\frac{\partial G}{\partial \theta} \\ \frac{\partial H}{\partial z} & \frac{\partial H}{\partial \bar{z}} & \frac{\partial H}{\partial \theta} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which is equivalent to

$$\frac{\partial F}{\partial \bar{z}} = 0 \text{ and } \frac{\partial H}{\partial \bar{z}} = 0.$$

This means that  $f$  and  $h$  depend holomorphically on the  $z$  coordinate. Note:

- Naturally, in the SUSY case the dependence of  $F$  and  $H$  on  $u$  is holomorphic, whereas in the  $cs$  case the  $u$ -dependence of  $F$ ,  $G$ , and  $H$  is merely smooth. Our computation shows that the  $z$ -dependence of  $F$  and  $H$  is holomorphic in both cases. Furthermore, if the base is purely even, we necessarily have  $g = g_b = \bar{f}_b$ , so that in this case  $G$  is completely determined by  $F$ . Hence, over a reduced (complex) base, a family of SUSY curves is the same as a family of conformal  $cs$  surfaces whose coordinate change functions depend holomorphically on the base.
- The computation shows that every family of SUSY curves  $\pi : F \rightarrow S$  has an underlying family of  $cs$  surfaces with super conformal structure: it is given by using the same coordinate changes as for the SUSY family, just considered as  $cs$  maps using the forgetful functor  $\mathcal{F}$  from Section 4.4.
- The above observations suggest a close relationship between SUSY curves and conformal  $cs$  surfaces. In fact, we *conjecture* that the moduli  $cs$  manifold of conformal  $cs$  surfaces will turn out to be isomorphic to the  $cs$  manifold underlying the moduli supermanifold of SUSY curves. The point is that we expect the higher number of families of super conformal surfaces over  $S_{cs}$  (compared to SUSY families over  $S$ ) to be compensated for by the fact that there are more morphisms from  $S_{cs}$  to the  $cs$  moduli space than from  $S$  to the complex moduli space.

### 5.3 Outlook: the definition of supersymmetric CFTs

As explained in the introduction, we want to think of a supersymmetric conformal field theory as an operator-valued function  $E$  on the moduli supermanifold  $\mathcal{M}$  of SUSY curves with parametrized boundaries such that the gluing operation on  $\mathcal{M}$  is compatible with composition of operators. Recall from Section 1.3 that our formal way to say this was that a susy CFT should be a functor between enriched categories from

SUSY bordisms  $\underline{\mathcal{C}}$  to Hilbert spaces. Now, if everything works out as planned, the notion of SUSY families from 5.1 will lead to a *complex* moduli supermanifold  $\mathcal{M}$ , i.e. to a category  $\underline{\mathcal{C}}$  enriched over (infinite-dimensional) complex supermanifolds. In particular, it should be possible to apply the forgetful functor  $\mathcal{F}$  from Section 4.4 to the (complex) morphism supermanifolds and hence to obtain a category enriched over *cs* manifolds. On the other hand, in the same way as in the EFT case we can consider the category of complex Hilbert spaces and bounded operators to be a category  $\underline{\mathcal{V}}$  enriched over complex supermanifolds or *cs* manifolds. This leads to two notions of supersymmetric CFTs. If we choose to look at functors  $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{V}}$  of categories enriched over complex supermanifolds, we obtain a notion of *holomorphic supersymmetric CFTs*. Choosing enrichment over *cs* manifolds instead, we get (*not necessarily holomorphic*) *supersymmetric CFTs*. The latter is our proposed notion for the Stolz-Teichner project: recall from Section 1.2 that holomorphic dependence of the operators on the parameters of moduli spaces is exactly what we wished to avoid.

Since what we really want is ‘supersymmetry in the antiholomorphic directions’, we should first pass to the complex conjugate  $\bar{\mathcal{M}}$  of the complex moduli space and consider theories defined on  $\bar{\mathcal{M}}_{cs}$ . The same remark as in the case of toy annuli (see 4.5) applies: since  $\mathcal{M}$  has a canonical real structure (cf. [DM], Example 4.9.4), we have  $\mathcal{M} \cong \bar{\mathcal{M}}$  so that both choices essentially lead to the same definition. However, our feeling is that taking the complex conjugate will ensure that the copy  $\mathcal{M}_{\text{red}}$  of the moduli space of conformal spin surfaces that we find as a sub supermanifold of  $\mathcal{M}$  will carry its usual complex structure. Note that, in our language, extending a usual CFT, i.e. a map  $E$  from the moduli space of conformal spin surfaces  $(\mathcal{M}_{cs})_{\text{red}}$  to operators, to a supersymmetric theory amounts to give an extension of  $E$  to  $\mathcal{M}_{cs}$ .

As pointed out earlier, there are some difficulties with making our ideas into a rigorous mathematical theory. A central problem is the construction of the moduli spaces involved. Recall that in the non-super case, the moduli space of Riemann surfaces with parametrized boundaries is an infinite-dimensional complex manifold, see [Se1], §4. We hope that it will turn out that, in a similar way,  $\mathcal{M}$  is an infinite-dimensional complex supermanifold, at least for surfaces with non-empty boundary. It might be possible to follow Segal’s strategy to understand these spaces by trying to write the moduli space of

SUSY curves with parametrized boundaries as a super fiber bundle over the moduli space of closed SUSY curves with punctures. In the case of closed components we get finite-dimensional objects, but they are stacks rather than supermanifolds, see [LR]. Another possibly useful tool is the deformation theory for complex super spaces developed by Vaintrob, see [Va1].

Despite these obstacles, it seems to us that developing a mathematical theory of supersymmetric CFTs along these lines should be a very worthwhile thing to do.

### Comparison to the definition proposed by Stolz and Teichner

The strategy currently pursued by Stolz and Teichner is to work without representing objects. Roughly speaking, they suggest that a supersymmetric CFT should associate to each family of conformal  $cs$  surfaces over a  $cs$  manifold  $S$  a family of operators parametrized by  $S$  (in a way that is compatible with base changes). This approach seems to have some technical advantages over ours. Maybe the most important one is that it is possible to make sense of the definition without constructing the representing objects (i.e. moduli spaces) for the family functor first. It seems that once we get our definition to work, it should return a notion that is essentially equivalent to the one of Stolz and Teichner, the connection being given by the universal family of conformal  $cs$  surfaces over the moduli space  $\bar{\mathcal{M}}_{cs}$ : a theory of their type would be determined by its values on such a universal family, hence giving an operator-valued map on  $\bar{\mathcal{M}}_{cs}$ , i.e. precisely a susy CFT in our sense. Of course, this argument only works if the conjecture concerning the relation between the moduli spaces of SUSY bordisms and super conformal bordisms formulated at the end of 5.2 is true.

The theory we propose is certainly not developed enough to admit any serious comparison with the alternative definition suggested by Stolz and Teichner. However, we would like to mention some reasons why we think that the additional effort necessary to make our approach work is worthwhile.

1. Our approach connects the supersymmetry aspect in the Stolz-Teichner project to SUSY curves. These have been studied extensively and it seems quite likely that the existing theory could be useful for our understanding of supersymmetry in the context of elliptic cohomology.

For example, a central task in the Stolz-Teichner project will be to determine the spaces  $\mathcal{CFT}_n$ , in some way. It seems that a good knowledge of the moduli spaces of bordisms will be necessary to do so. The general theory of SUSY curves and, more specifically, results concerning moduli spaces of SUSY curves (cf. [LR], [Ma2]) might come handy in this context. Altogether, it seems that our approach may be suited better to do computations.

2. Another useful aspect could be the well-known relation between vector fields on SUSY curves and the Neveu-Schwarz and Ramond algebras, cf. [Ma2]. This might also help to connect the theory to the usual physical point of view on supersymmetric CFTs in which representations of superconformal (Lie) algebras play a central role. For the relation between superconformal algebras and moduli spaces, see [Va2].
3. A benefit of working in the complex analytic category is that the moduli supermanifolds automatically come with a complex structure. We hope that this additional structure will be helpful for constructing an embedding of (non-supersymmetric) holomorphic field theories into the larger class of supersymmetric theories. The idea is that for a holomorphic field theory finding a supersymmetric extension should be trivial, since in this case the derivatives in the  $\bar{z}$  directions are zero. Constructing this embedding is very important, since it would provide us with many examples of CFTs that are supersymmetric in our sense.

As we have seen, each of the two approaches to supersymmetric conformal field theory has its advantages and drawbacks. In order to make best use of each of the two standpoints, it would be very desirable to view them as two sides of the same coin. Our hope, of course, is that the relation between conformal  $cs$  surfaces and SUSY curves conjectured in 5.2 can be employed to unite the two points of view. We think that developing both definitions thoroughly and establishing connections between them is a very interesting prospect for future research.

## Yet more open questions

We want to conclude our (admittedly quite incomplete) discussion of supersymmetry in the Stolz-Teichner project with some more remarks, open questions, and suggestions for future research.

1. The reader might wonder why we insist on ‘geometric supersymmetry’ rather than simply asking for supersymmetric extensions of usual CFTs on the level of Lie algebras. This is technically much easier and seems particularly natural in view of the fact that a homomorphism  $f : G \rightarrow H$  between super Lie groups is nothing but a map  $f_{\text{red}} : G_{\text{red}} \rightarrow H_{\text{red}}$  between the underlying Lie groups together with an extension of the induced Lie algebra map  $df_{\text{red}}$  to the whole super Lie algebra of  $G$ , cf. [DM], §3.8. The main reason we are not content with such ‘algebraic supersymmetry’ is that we would like to understand the geometric significance of supersymmetry more completely.

For example, it would be very interesting to relate our notion of supersymmetry (motivated by the desire for holomorphic partition functions) and the supersymmetry appearing in Witten’s treatment of the ‘Dirac operator on loop space’, see [Wi4]. An encouraging coincidence is that in Witten’s considerations concerning the genuine (untwisted) Dirac operator (see pages 509 – 511) only one supersymmetry appears, just as in our approach.

Another argument for ‘geometric supersymmetry’ is that when one tries to unite the supersymmetry and the 3-tier aspect, one is lead to the question what the basic building blocks of such theories are. Recall that in the EFT case the basic items are Euclidian super intervals, since arbitrary bordisms can be obtained from them by gluing, disjoint unions, etc. As far as I understand, Stolz and Teichner expect the smallest building units for 3-tier susy CFTs to be super conformal triangles. These should form some sort of ‘super 2-semigroup’ that plays a role analogous to the role of  $\mathbb{R}_{>0}^{1|1}$  in the EFT case. In this context, it is not clear what supersymmetry on the Lie algebra level should mean; in fact, it seems that the geometric understanding may have to come first here.

2. It is not known what the role of surfaces of genus 2 and higher is in the context

of elliptic cohomology. It would be quite interesting to have a better understanding of the homotopy-type of certain ‘approximations’ to the spaces  $\mathcal{CFT}_n$  that are obtained by forgetting part of the data of a CFT. For example, the space of  $cs$  semigroup homomorphisms from  $\bar{\mathfrak{A}}_{cs} \rightarrow \text{HS}(\mathcal{H})$  appears to yield a kind of ‘un-modular elliptic cohomology’ like  $K[[q]][q^{-1}]$ . A better approximation to  $\mathcal{CFT}_n$  might be the space of smooth representations of the  $cs$  semigroup(s) of all SUSY annuli.<sup>3</sup>

3. In the homotopy-theoretic description of TMF, the integrality properties of the modular forms involved are forced by considering elliptic curves over  $\mathbb{Z}$ , rather than over the complex numbers. In the Stolz-Teichner approach, integrality is implied by supersymmetry, in a somewhat mysterious way. It would be interesting to know whether or not (and if so, how) these two mechanisms are related. Maybe via some kind of ‘super modular forms’?
4. In order to develop our proposed relation between SUSY curves and supersymmetry in the Stolz-Teichner project further, a better understanding of the connection between complex supermanifolds and  $cs$  manifolds is necessary. For example, it would be useful to have a good notion of complex structures on  $cs$  manifolds. Also, we would like to know how our forgetful functor  $\mathcal{F}$  from Section 4.4 behaves with respect to complex conjugates. Another not too difficult task seems to be to understand Dirac operators on conformal spin surfaces completely in the language of super geometry and to extend the theory to the case of families over supermanifolds. Finally, in order to deal with the appearing moduli spaces, a theory of infinite-dimensional supermanifolds would be very useful. The same holds true for a good notion of ‘super stacks’. For example, it might be possible to avoid some representability issues if one had a forgetful functor from complex analytic super stacks to  $cs$  stacks extending our functor  $\mathcal{F}$ .

We find it most appropriate to end our discussion with the following quote by Marcel Reich-Ranicki, based on a line from a play by Bertolt Brecht:

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<sup>3</sup>These comments and suggestions are due to S. Stolz, P. Teichner, C. Pries, and N. Kitchloo.

*Und so sehen wir betroffen / Den Vorhang zu und alle Fragen offen.*<sup>4</sup>

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<sup>4</sup>Roughly: *And thus we stand struck / The curtain closed and all questions open.*

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