109 Last Problem Set

2c. Claim. Every Cauchy sequence in $\mathbb{R}$ is bounded.

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. (That is, assume that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever m,$n \geq N$ we get $|x_n - x_m| < \epsilon$.) Firstly, we can take $\epsilon = 1$ to get that there is an $N \in \mathbb{N}$ such that every term in the sequence after x_N is within a distance of 1 from every other term after x_N. In particular, every term after x_N is within a distance 1 from x_N itself. This bounds all but finitely many terms of the sequence so we can say the sequence is bounded above by $\max\{x_i, x_N + 1\}$ for $i = 1, \ldots, N$. Similarly, it is bounded below by $\min\{x_i, x_N - 1\}$ for $i = 1, \ldots, N$. (Think about why!). Thus, every Cauchy sequence is bounded.

2d. Claim. Every sequence in $\mathbb{R}$ has a sub-sequence that is non-decreasing or non-increasing.

Proof. As per the hint, we let a peak be a point in the sequence x_n that is greater than or equal to all later elements in the sequence ($x_n \geq x_m$ whenever $m \geq n$). Then if we have infinitely many peaks, make the sequence (x_{n_i}) where the x_{n_i} are successive peaks. By the definition of peak, this is a non-increasing sequence. If there are not infinitely many peaks, on the other hand, we will construct a non-decreasing sequence.

If there are only finitely many peaks then either there are no peaks or there is a maximal N such that x_N is a peak (if there are no peaks, just define N to be 0). Then there are no peaks in the sequence for $n > N$ so I claim we can pick an infinite number of terms from the sequence, each greater than or equal to the previous. This is because if, at any point x_n we couldn't pick an x_m with $x_m \geq x_n$ and $m > n$, then x_n is a peak further along in the sequence than x_N, which is a contradiction.

2e. Claim. If every subset of $\mathbb{R}$ that is bounded from above has a supremum then every Cauchy sequence in $\mathbb{R}$ converges.

Proof. If (x_n) is a Cauchy sequence then, by (d), we can assume there is a sub-sequence (x_{n_i}) that is non-decreasing. (Technically, (d) guarantees the existence of either a non-decreasing or non-increasing sub-sequence, but if there were a non-increasing sequence, we could consider the sequence $(-x_n)$, which would then have a non-decreasing sub-sequence and which converges if and only if (x_n) does). By (c), now, (x_{n_i}) is bounded and, consequently, by (a), converges to the supremum of the set $\{x_{n_i}\}$, call it $x \in \mathbb{R}$. Now we have a sub-sequence of our Cauchy sequence, (x_n), converging to x and all that remains to be shown is that, in fact, (x_n) itself converges.

Given $\epsilon > 0$, because (x_{n_i}) converges, there is an $N_1 \in \mathbb{N}$ such that for every $n_i \geq N_1$, $|x - x_{n_i}| < \epsilon/2$. However, because (x_n) is

Cauchy there is an $N_2 \in \mathbb{N}$ after which $|x_{n_i} - x_n| < \epsilon/2$. Adding and subtracting x_{n_i}, and then applying the triangle inequality, we see $|x - x_n| = |x - x_{n_i} + x_{n_i} - x_n| \leq |x - x_{n_i}| + |x_{n_i} - x_n|$. Combining this with the above bounds, whenever $n \geq \max\{N_1, N_2\}$ we get $|x_n - x| < \epsilon/2 + \epsilon/2 = \epsilon$. And we're done.