Midterm 2 for Math 10C

Fall Quarter 2007, UCSD | Henning Hohnhold

Name: **Section:**

#1: **#2:** **#3:** **#4:** **Total:**

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(1) **Partial derivatives and the gradient.** Consider the function

$$f(x,y) = x^3y^3 - x^2\ln(y).$$

(a) (*4 points*) Please compute the gradient of $f(x,y)$.

$$\nabla f(x,y) = (3x^2y^3 - 2x\ln(y))\vec{i} + (3x^3y^2 - \frac{x^2}{y})\vec{j}$$

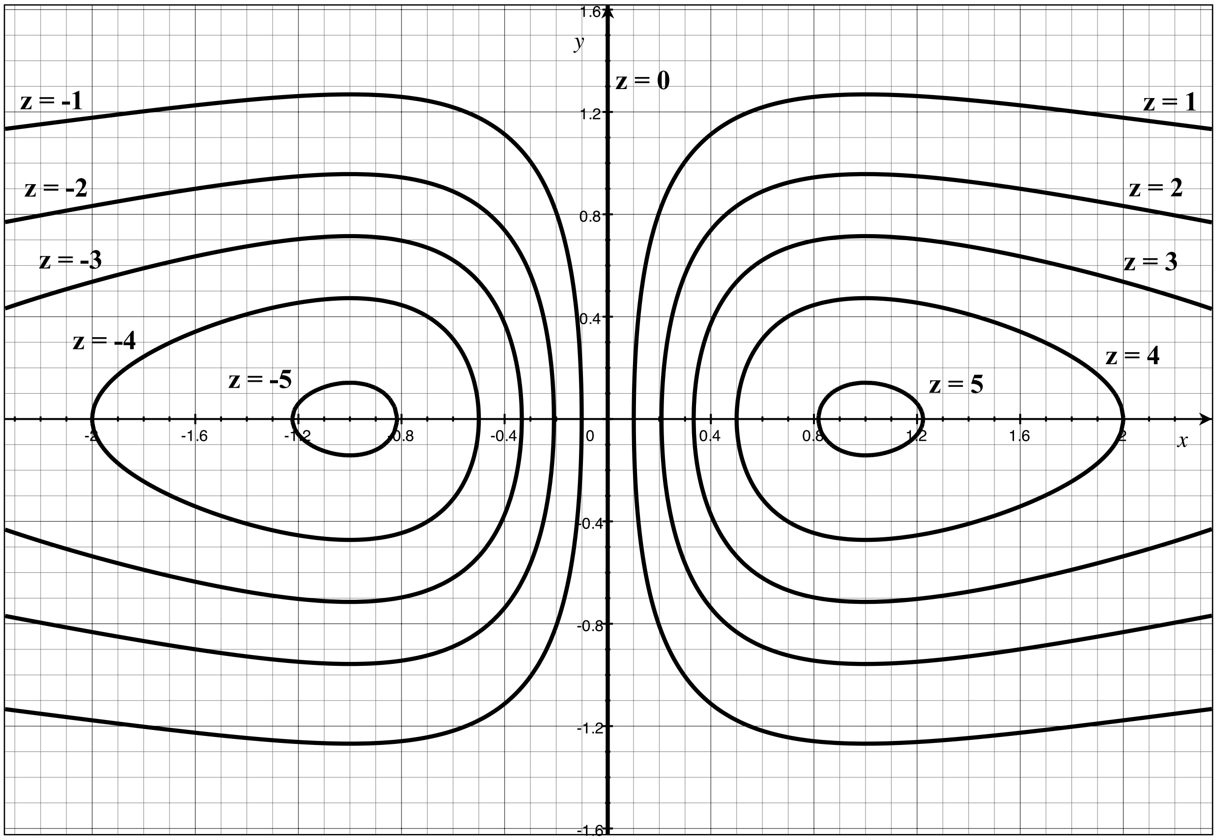
(b) (*4 points*) Find the rate of change $f_{\vec{u}}(2,1)$ of $f(x,y)$ at the point $(2,1)$ in the direction of the (unit) vector $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.

$$f_{\vec{u}}(2,1) = (12\vec{i} + 20\vec{j}) \cdot (\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}) = \frac{1}{5}(36+80) = \frac{116}{5}$$

(c) (*4 points*) Find a vector in whose direction the rate of change is maximal at the point $(1,1)$. Find this maximal rate of change.

Such a vector is given by the gradient $\nabla f(1,1) = 3\vec{i} + 2\vec{j}$. The maximal rate of change is given by its magnitude $|\nabla f(1,1)| = \sqrt{13}$.

(2) **Contour diagrams and gradient vectors.** Assume the following picture is the contour diagram of a function $f(x, y)$.



Note: The side length of one of the small squares is $0.1 = \frac{1}{10}$.

(a) (*3 points*) Use the contour diagram to estimate $f_x(-0.1, 0)$ and $f_y(0, 0)$.

We read off $f_x(-0.1, 0) \approx \frac{0-(-1)}{0.1} = 10$ and $f_y(0, 0) = 0$.

(b) (*3 points*) Use the contour diagram to estimate the directional derivative of $f(x, y)$ at the point $(-0.4, 0.4)$ in the direction of the vector $-2\vec{i} - \vec{j}$.

We read off $f_{-2\vec{i}-\vec{j}}(-0.4, 0.4) \approx \frac{-4-(-3)}{\sqrt{0.2^2+0.1^2}} = \frac{-10}{\sqrt{5}}$.

(c) (*2 points*) Is the directional derivative of $f_{\vec{i}+\vec{j}}(0.8, 0.7)$ positive or negative? Explain!

The vector $\vec{i} + \vec{j}$ points towards a level curve with a lower z value, hence the directional derivative is negative.

(d) (*2 points*) Do you think the magnitude of the gradient $\nabla f(x, y)$ is bigger at $(0.2, 0.8)$ or at $(2, 0)$? Explain!

The level curves are much closer together near $(0.2, 0.8)$ than at $(2, 0)$, so the magnitude of the gradient is larger at $(0.2, 0.8)$.

(e) (*3 points*) Please draw the gradient vectors of $f(x, y)$ at the points $(-0.3, 1)$ and $(-0.2, 0.8)$ in the contour diagram. Please indicate which of the two vectors is longer! *Draw the vectors in the contour diagram on page 2!*

Both vectors are othogonal on the level curves and point towards the level curve with $z = 0$. The gradient is longer at $(-0.2, 0.8)$, since the level curves are closer together there.

(3) **Normal vectors and the cross product.**

(a) *(9 points)* Use the cross product to set up the equation of the plane containing the points

$$P = (0, 1, -3), \;\; Q = (1, -2, 1), \;\; \text{and } R = (-1, 1, -2).$$

The vectors displacement vectors $\vec{v} = \vec{i} - 3\vec{j} + 4\vec{k}$ from P to Q and $\vec{w} = -\vec{i} + \vec{k}$ from P to R lie in the plane. We take their cross product to get a normal vector for the plane:

$$\vec{n} = \vec{v} \times \vec{w} = -3\vec{i} - 5\vec{j} - 3\vec{k}.$$

We can now set up the equation using the formula from the formula sheet:

$$-3x + (y-1)(-5) + (z+3)(-3) = 0$$

or

$$-3x - 5y - 3z = 4$$

(b) *(3 points)* Find the angle between the plane in part (a) and the plane given by

$$x + y + z = 1$$

Hint: The angle between two planes is the angle between their normal vectors. There are two possible solutions, depending on the choice of the normal vectors.

The normal vector of the plane $x + y + z = 1$ is $\vec{i} + \vec{j} + \vec{k}$. We find the angle θ between the normal vectors (and hence the planes) using the formula from the formula sheet:

$$\theta = \arccos \frac{(-3\vec{i} - 5\vec{j} - 3\vec{k}) \cdot (\vec{i} + \vec{j} + \vec{k})}{|-3\vec{i} - 5\vec{j} - 3\vec{k}| \cdot |\vec{i} + \vec{j} + \vec{k}|} = \arccos \frac{-11}{\sqrt{43}\sqrt{3}} \approx 14.4$$

degrees.

(4) **Vectors.** A plane is heading north-west at a speed of $300 \cdot \sqrt{2} \approx 424$ mph. Furthermore, wind is blowing a speed of 100 mph to the north.

(a) *(3 points)* Sketch (and label) the vectors representing the velocity of the plane, the wind, and the resulting total velocity of the airplane.

Sorry, sketch is missing here.

(b) *(7 points)* Express the velocity vectors of the plane and the wind in terms of the standard vectors $\vec{i}$ and $\vec{j}$ and use them to compute the total velocity and the total speed of the airplane. Explain in words why the result you get makes sense.

The vector $\vec{v}$ representing the velocity of the airplane is $\vec{v} = -300\vec{i} + 300\vec{j}$ and the vector $\vec{w} = 100\vec{j}$. The resulting total velocity of the airplane is $\vec{v}+\vec{w} = -300\vec{i} + 400\vec{j}$. The total speed is given by its magnitude, so it is $\sqrt{(-300)^2 + 400^2} = 500$. It makes sense that the total speed of the airplane is faster than its initial speed, since it experiences tail wind!

(c) *(3 points)* How much does the wind change the direction in which the plane is flying? (Your answer should be an angle! However, it is sufficient if you find the cosine of that angle.)

The wind changes the direction of the plane by the angle between the vectors $\vec{v}$ and $\vec{v} + \vec{w}$

$$\theta = \arccos \frac{(\vec{v} + \vec{w}) \cdot \vec{v}}{|\vec{v} + \vec{w}| \cdot |\vec{v}|} = \arccos \frac{(-300) \cdot (-300) + 400 \cdot 300}{500 \cdot 424} \approx 7.9$$

degrees.

Good Luck!

Formula sheet

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(1) **Displacement vectors**. The *displacement vector* $\vec{v}$ from $P = (x_1, y_1, z_1)$ to $Q = (x_2, y_2, z_2)$ is given by

$$\vec{v} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}.$$

(2) **Magnitude.** The *magnitude* of $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

(3) **Dot product.** Consider two vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$. The *dot product* of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta,$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. Dividing by $|\vec{v}| \cdot |\vec{w}|$ and applying arccos we get the following formula for the angle between $\vec{v}$ and $\vec{w}$:

$$\theta = \arccos \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}.$$

(4) **Cross product.** The *cross product* of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}.$$

The direction of the vector $\vec{v} \times \vec{w}$ is determined by the right hand rule and its magnitude is $|\vec{v}| \cdot |\vec{w}| \cdot \sin\theta$.

(5) **The equation of a plane, using a normal vector.** The plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$, containing the point $P = (x_0, y_0, z_0)$ is given by

$$(x - x_0)a + (y - y_0)b + (z - z_0)c = 0.$$

(6) **The volume of a parallelepiped.** The volume of the parallelepiped spanned by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is given by

$$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|.$$

(7) **Tangent planes.** The *tangent plane* of a function $f(x,y)$ at a point $(a,b,f(a,b))$ is given by

$$z = f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b).$$

(8) **Gradient and directional derivatives.** The *gradient vector* of a function $f(x,y)$ at (a,b) is

$$\nabla f(a,b) = f_x(a,b) \cdot \vec{i} + f_y(a,b) \cdot \vec{j}.$$

The *directional derivative* in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is given by

$$f_{\vec{u}}(x,y) = \nabla f(x,y) \cdot \vec{u} = f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$$