Calculating \( \int_{0}^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx \)

M.R.Darafsheh, Hassan Jolany

Abstract:
In this paper we will extend the Lobachevsky works about Dirichlet integral, and we will find a method for calculating \( \int_{0}^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx \), where \( f(x+\pi) = f(x) \) and \( f(\pi-x) = f(x) \)

1. Introduction

Let

\[ f(x) = 1 - \frac{\sin x}{\sin \alpha} \quad \text{Where,} \quad \alpha \neq k\pi \quad (1) \]

By applying Taylor expansion on \( \sin x \), we get

\[ f(x) = 1 - \frac{x}{\sin \alpha} + \frac{x^3}{3! \sin \alpha} - \frac{x^5}{5! \sin \alpha} + ... \]
Now we assume the right hand side of the above equality as a polynomial of infinite degree. If \(a_1, a_2, ..., a_n, \ldots\) are the roots of \(f(x)\), then we can write (see [1-4])

\[
f(x) = (1 - \frac{x}{a_1})(1 - \frac{x}{a_2})(1 - \frac{x}{a_3})... (1 - \frac{x}{a_n})... = \prod_{k=1}^{\infty} \left(1 - \frac{x}{a_k}\right)
\]

One can see from (1), the roots of \(f(x)\) are

\[
x = \left(\frac{2n\pi + \alpha}{2n\pi - \alpha + \pi}\right)\quad n = 0, \pm 1, \pm 2, \ldots
\]

So

\[
f(x) = \prod_{n=0}^{\infty} \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi - \alpha + \pi}\right)
\]

Then we can write

\[
f(x) = \left(1 - \frac{x}{\alpha}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{(2n-1)\pi - \alpha}\right) \left(1 + \frac{x}{(2n-1)\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 + \frac{x}{2n\pi - \alpha}\right)
\]

By writing the right hand side of the equality (1) as a power series and equate the coefficients of both sides, the right hand side is infinite elementary symmetric functions. Now we use the Newton relations in infinite case. Put

\[
\sigma_m = \sum a_1 a_2 ... a_m \quad \text{and} \quad S_m = \sum_{i=0}^{m} a_i^n
\]

Then \(S_1 = \sigma_1, S_2 = \sigma_1^2 - 2\sigma_2, S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \ldots\) and \(\sigma_2 = \sigma_4 = \sigma_6 = \ldots = \sigma_{2n}\) \((n = 1, 2, 3, \ldots)\). So we get

\[
\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{\alpha - m\pi} + \frac{1}{\alpha + m\pi}\right)
\]

But we have

\[
\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} \left(\frac{1}{(2m-1)\pi - \alpha} - \frac{1}{(2m-1)\pi + \alpha} + \frac{1}{2m\pi + \alpha} - \frac{1}{2m\pi - \alpha}\right)
\]

and also

\[
\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left(\frac{1}{((2m-1)\pi - \alpha)^2} + \frac{1}{((2m-1)\pi + \alpha)^2} + \frac{1}{(2m\pi + \alpha)^2} + \frac{1}{(2m\pi - \alpha)^2}\right)
\]

Therefore we arrive

\[
\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left(\frac{1}{(\alpha - m\pi)^2} + \frac{1}{(\alpha + m\pi)^2}\right)
\]

**Theorem 1**: Suppose that \(f(x)\) satisfies in

\[
f(x + \pi) = f(x), f(\pi - x) = f(x), \quad 0 \leq x < \infty
\]

If the following integral exists:

\[
\int_{0}^{\infty} \frac{\sin^2 x}{x^2} f(x) dx
\]
Then
\[
\int_0^\infty \frac{\sin^2 x}{x^2} f(x)dx = \int_0^\infty \frac{x}{x} f(x)dx = \int_0^\infty f(x)dx
\]

Proof. We put
\[
I = \int_0^\infty \frac{\sin x}{x} f(x)dx
\]

So we can write I as follows
\[
I = \sum_{\nu=0}^{(\nu+1)\frac{\pi}{2}} \int_{\frac{\nu\pi}{2}}^{(\nu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x)dx.
\]

Where \(\nu = 2\mu - 1\) or \(\nu = 2\mu\).

By Changing \(x = \mu \pi + t\) or \(x = \mu \pi - t\) we get
\[
\int_{2\mu\pi}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x)dx = (-1)^\mu \int_{\frac{\pi}{2}}^{\mu \pi + t} \sin t f(t)dt
\]

and
\[
\int_{(2\mu-1)\frac{\pi}{2}}^{2\mu\pi} \frac{\sin x}{x} f(x)dx = (-1)^{\mu-1} \int_{\frac{\pi}{2}}^{\mu \pi - t} \sin t f(t)dt
\]

So we get
\[
I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t)dt + \sum_{\mu=1}^\infty \left( (-1)^\mu \int_{0}^{\frac{\pi}{2}} \frac{1}{t + \mu \pi} + \frac{1}{t - \mu \pi} \right) \sin t dt
\]

Consequently we can write I in the form of
\[
I = \int_0^{\frac{\pi}{2}} \sin \left( \frac{1}{t} + \sum_{\mu=1}^\infty (-1)^\mu \left( \frac{1}{t + \mu \pi} + \frac{1}{t - \mu \pi} \right) \right) f(t)dt
\]

So
\[
I = \int_0^{\frac{\pi}{2}} f(t)dt
\]

and proof is complete \(\square\)

Now will present a general form for calculating generalized Dirichlet integral, as follows
\[
I = \int_0^{\frac{\pi}{2}} \sin^{2n} x \frac{f(x)}{x^{2n}} dx \quad (n = 1, 2, 3, \ldots) \quad (2)
\]

where \(f(x + \pi) = f(x), f(\pi - x) = f(x)\) \(0 \leq x < \infty\).
We start with $n = 2$ in (2)

Let

$$ I = \int_{0}^{\infty} \frac{\sin^{4} x}{x^{4}} f(x) dx $$

we know

$$ \frac{1}{\sin^{4} \alpha} + \frac{2}{3 \sin^{2} \alpha} = \frac{1}{\alpha^{4}} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^{4}} + \frac{1}{(\alpha + m\pi)^{4}} \right) $$

Like previous method we get

$$ I = \int_{0}^{\infty} \sin^{4} t \left( \frac{1}{t^{4}} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha + m\pi)^{4}} + \frac{1}{(\alpha - m\pi)^{4}} \right) \right) f(t) dt $$

$$ = \int_{0}^{\frac{\pi}{2}} \sin^{4} t \left( \frac{1}{\sin^{4} t} + \frac{2}{3 \sin^{2} t} \right) f(t) dt $$

So

$$ I = \int_{0}^{\frac{\pi}{2}} f(t) dt + \frac{2}{3} \int_{0}^{\frac{\pi}{2}} \sin^{2} t f(t) dt $$

So by applying to this method we can calculate the following generalized Dirichlet integral

$$ \int_{0}^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx $$

Example 1: As a fast result we get

$$ \int_{0}^{\infty} \frac{\sin^{4} x}{x^{4}} dx = \frac{\pi}{3} $$

Remark 1: If $f(x)$ satisfies in assumption of Theorem 1 we can write

$$ I = \int_{0}^{\infty} \frac{\sin^{2n+1} x}{x} f(x) dx = \int_{0}^{\infty} \sin^{2n} x f(x) \sin x dx $$

So if we set $\sin^{2n} x f(x) = g(x)$ we get $g(x + \pi) = g(x), g(\pi - x) = g(x)$ therefore if we put $f(x) = 1$ by applying Theorem 1 we obtain

$$ \int_{0}^{\infty} \frac{\sin^{2n+1} x}{x} dx = \int_{0}^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)! \pi}{(2n)!} \frac{\pi}{2} $$

Remark 2: by applying this method for $n = 2, n = 4, n = 5$ we get

I) $\int_{0}^{\infty} \frac{\sin^{6} x}{x^{6}} dx = \frac{11\pi}{40}$

II) $\int_{0}^{\infty} \frac{\sin^{8} x}{x^{8}} dx = \frac{151\pi}{630}$
III) \[ \int_{0}^{\infty} \frac{\sin^{10} x}{x^{10}} \, dx = \frac{15619\pi}{72576} \]

References