WEEK 1

1. Overview:

10A: Given position function \( s(t) \), find:

1. Displacement over time interval \( a \leq t \leq b \):

\[
\text{Geometric Interpretation} = \left( \frac{\text{slope of secant line}}{\text{through points} (a, s(a)), (b, s(b))} \right) \times \left( \frac{\text{change in time}}{b-a} \right)
\]

\[
\text{average velocity}
\]

2. Velocity at time \( t=a \):

\[
\text{Geometric Interpretation} = \text{slope of tangent line at} (a, s(a))
\]

\( s'(a) \) is the derivative

Let's find \( s'(a) \) as the limit of the slopes of secant lines!
10B: Given velocity function \( v(t) \), find:

1. **Displacement over time interval \( a \leq t \leq b \)**

   Geometric Interpretation
   
   \[
   \text{Area under } y = v(t) \text{ over } a \leq t \leq b = \int_a^b v(t) \, dt
   \]

   \( \Rightarrow \) We'll also find this quantity as a limit, like the derivative! (later)

2. **Position at time \( t = a \)**

   A function \( s(t) \) with \( s'(a) = v(a) \), that is, an antiderivative of \( v(t) \) at \( t = a \)

   \( \Rightarrow \) Problem: More than one possibility for \( s(t) \). To find specific antiderivative, we must also specify a starting point \( s(0) = k \)
e.g. 1: Constant velocity

Let \( v(t) = 5 \) m/s

\( Q \) What is the displacement aer \( 1 \leq t \leq 6 \) ?

\[
\text{displacement} = \int_{1}^{6} v(t) \, dt
\]

\[ = (6-1) \cdot (5) = 25 \]

\[ = \int_{1}^{6} v(t) \, dt = \int_{1}^{6} v(t) \, dt \]

\[ \text{What is the position function?} \]

See that both \( s_1(t) \) and \( s_2(t) \) are antiderivatives for \( v(t) \). Need to specify a starting point: "Initial value" problem.
• THIS WEEK

TODAY: [ANTIDERIVATIVES] → (Question 2)
  → Solving IVP's
  → Rules for finding general antiderivatives

WED: [INTEGRALS] → (Question 0)
  → Geometric definition and properties

FRI:
  → Finding \(\int_a^b f(x)\,dx\) at a limit
    (Motion from derivatives...)

• WEEK 2

FUNDAMENTAL THEOREM OF CALCULUS:

INTEGRALS → ANTIDERIVATIVES
② Antiderivative

- A function \( s(t) \) is \( \textit{an} \) antiderivative for \( v(t) \) if

\[
\frac{ds}{dt} = s'(t) = v(t)
\]

- An initial value problem asks what function \( s(t) \) is an antiderivative with a given initial value \( k \):

\[
\begin{cases}
\frac{ds}{dt} = v(t) \\
\quad \text{\( \Rightarrow \) } s(t) = ? \\
\quad \text{\( s(0) = k \)}
\end{cases}
\]
two objects with different positions and the same velocity

(a) Car 1 leaves San Diego for LA @ 60 mph.
Car 2 leaves LA for San Francisco @ 60 mph.
Write their position functions in miles north of San Diego

Note that \( v_1(t) = v_2(t) \) by definition.
Therefore we know that:

\[ s_1(t) = 60t + C_1 \]
\[ s_2(t) = 60t + C_2 \]

(check that:\[ \frac{ds_1}{dt}(s_1) = \frac{ds_2}{dt}(s_2) = 60 \])

Now, since car 1 started in SD,
it must have position 0 @ time 0. That is:

\[ s_1(0) = 0 \]

Car 2 started in LA (90 miles N of SD),
hence it must have:

\[ s_2(0) = 90 \]
Putting the general antiderivatives and initial data together, we get:

\[ 0 = s_1(0) = 60(0) + C_1 = C_1 \Rightarrow C_1 = 0 \]

\[ 90 = s_2(0) = 60(0) + C_2 \Rightarrow C_2 = 90 \]

So the position functions are

\[ s_1(t) = 60t \quad \text{and} \quad s_2(t) = 60t + 90 \]

(b) Suppose Car 3 leaves San Diego to Tijuana going 60 mph. What is its velocity function?

Since Car 3 is going south, its velocity is negative:

\[ v(t) = -60 \]

Exercise: find its position function as in part (a)
A car has constant velocity 60 mph North of San Diego.

(a) If the car is 15 miles North of San Diego after $\frac{1}{2}$ hour, what is its position function $s(t)$ in miles North of SD?

Since the car has constant velocity +60, we know $s'(t) = 60$.

And we've given that the car is at position +15 at $t = \frac{1}{2}$ hour, so:

$$s\left(\frac{1}{2}\right) = 15$$

Putting these together, we solve:

$$\begin{cases} s(t) = 60t + C \quad \text{(general solution)} \\ 15 = s\left(\frac{1}{2}\right) = 60\left(\frac{1}{2}\right) + C \\ \Rightarrow 15 = 30 + C \implies C = -15 \end{cases}$$

Hence $
\boxed{s(t) = 60t - 15}$

(check: $s'(t) = 60$. )
(b) What time will the car arrive in OC (60 miles north of SD) if it left the starting point at 12 noon?

The question asks, so what do we have $s(t) = 60$?

That is, $60t - 15 = 60$

$\Rightarrow$ $60t = 75$ $\Rightarrow$ $t = \frac{75}{60} = \frac{5}{4}$

$\Rightarrow$ $t = 1$ hour, 15 min

$\Rightarrow$ Arrive at 1:15 PM
An object is dropped from 1600 feet.

(a) What is its position function?
(b) When does it hit the ground?

(let $g = -32 \text{ ft/s}^2$)

(a) The position function must satisfy

$$\frac{d^2 s}{dt^2} = -32$$

since it is accelerating due to gravity.

Furthermore, we are given the initial data:

- $s'(0) = v(0) = 0$ (since the object is dropped it starts with no velocity)
- $s(0) = 1600$ (we are told the starting height)
Now we have two initial value problems

\[
\begin{align*}
\frac{d^2 s}{dt^2} &= -32 \\
\frac{dv}{dt} &= -32 \\
v(0) &= 0 \\
s(0) &= 1600 \\
\end{align*}
\]

\[\Rightarrow \begin{cases} 
1 \left\{ 
\frac{dv}{dt} = -32 \\
v(0) = 0 \\
\right. \\
2 \left\{ 
\frac{ds}{dt} = v(t) \\
s(0) = 1600 \\
\right. 
\end{cases}
\]

**Step 1**: \[\frac{dv}{dt} = -32 \rightarrow v = -32t + C\]

\[v(0) = 0 \rightarrow 0 = -32(0) + C \Rightarrow C = 0\]

So: \[s'(t) = v(t) = -32t\]

**Step 2**: \[\frac{ds}{dt} = -32t \rightarrow s = \frac{-32t^2}{2} + D\]

\[s(0) = 1600 \rightarrow 1600 = s(0) = D\]

So: \[s(t) = -16t^2 + 1600\] (a)
To answer (b), we again want it such that $s(t) = 0$, so we solve:

\[ 0 = s(t) = -16t^2 + 1600 \]
\[ \Rightarrow 16t^2 = 1600 \Rightarrow t^2 = 100 \Rightarrow t = 10 \text{ seconds} \]

The preceding examples illustrate the power rule for antiderivatives:

Recall:

<table>
<thead>
<tr>
<th>Power Rule for Derivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = x^n$ ( \Rightarrow ) ( \frac{ds}{dx} = nx^{n-1} )</td>
</tr>
</tbody>
</table>

Now:

<table>
<thead>
<tr>
<th>Power Rule for Antiderivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{ds}{dx} = x^n \Rightarrow s = \frac{x^{n+1}}{n+1} + C )</td>
</tr>
</tbody>
</table>
(3) Antiderivative ruler

Just like with derivatives, we start with a list of basic antiderivatives for familiar functions.

1. \( \frac{ds}{dx} = x^n \Rightarrow s = \frac{x^{n+1}}{n+1} + C \)
2. \( \frac{ds}{dx} = \frac{1}{x} \Rightarrow s = \ln|x| + C \)
3. \( \frac{ds}{dx} = e^x \Rightarrow s = e^x + C \)
4. \( \frac{ds}{dx} = \cos(x) \Rightarrow s = \sin(x) + C \)
5. \( \frac{ds}{dx} = \sin(x) \Rightarrow s = -\cos(x) + C \)
6. \( \frac{ds}{dx} = \sec^2(x) \Rightarrow s = \tan(x) + C \)
7. \( \frac{ds}{dx} = \sec(x) \tan(x) \Rightarrow s = \sec(x) + C \)
8. \( \frac{ds}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow s = \arcsin(x) + C \)
9. \( \frac{ds}{dx} = \frac{1}{1+x^2} \Rightarrow s = \arctan(x) + C \)
Furthermore, antiderivatives are linear like derivatives:

- **Derivatives**: 
  \[ \frac{d}{dx}(ku) = k \frac{du}{dx} \]
  \[ \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \]

- **Antiderivatives**: 
  \[ \frac{du}{dx} = f(x), \quad \frac{dv}{dx} = g(x) \]

  1. **ku is an antiderivative for kf(x)**:
     \[ \frac{d}{dx}(ku) = k \frac{du}{dx} = kf(x) \quad \checkmark \]

  2. **u+v is an antiderivative for (f+g)(x)**:
     \[ \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} = f(x) + g(x) \]
     \[ = (f+g)(x) \quad \checkmark \]
More examples

1) \[ \frac{ds}{dt} = 3t^4 + 2\cos(t) \]

By linearity we only need the rules for \( t^4 \) and \( \cos(t) \):

\[ S = \frac{3t^5}{5} + 2\sin(t) + C \]

\[ \frac{ds}{dt} = \frac{d}{dt} \left( \frac{3t^5}{5} + 2\sin(t) + C \right) = 5\left(\frac{3}{5}\right)t^4 + 2\cos(t) + 0 = 3t^4 + 2\cos(t) \checkmark \]

2) \[ \frac{dy}{dx} = e^x + \frac{1}{x^2} - \frac{x^3}{3} \]

See that \( \frac{dy}{dx} = e^x + x^2 - \frac{1}{3}x^3 \). So:

\[ y = e^x + \frac{x^{-1}}{-1} - \frac{1}{3}(x^4) + C \]

\[ \Rightarrow y = e^x - \frac{1}{x} - \frac{x^4}{12} + C \]
3) \[ \begin{align*}
\frac{du}{dx} &= 2x + 3x^2 + 4x^3 \\
u(1) &= 5
\end{align*} \]

\[ u = x^2 + x^3 + x^4 + C \]

and \[ 5 = u(1) = 1^2 + 1^3 + 1^4 + C = 3 + C \]

\[ \Rightarrow C = 2 \]

So \[ u = x^2 + x^3 + x^4 + 2 \]

4) \[ \frac{dg}{dt} = e^{2t} \]

\[ \Rightarrow \text{we don't yet have a rule for} \]

\[ \text{this, but notice:} \]

\[ \frac{d}{dt}(e^{2t}) = 2e^{2t} \]

\[ \Rightarrow \]

\[ \text{What did we use to compute the derivative?} \]

\[ \text{How can we use it to find the antiderivative?} \]
\[ \begin{align*}
\frac{d^2 u}{dx^2} &= e^x + x^3 \\
\Rightarrow \frac{du}{dx} &= e^x + \frac{x^4}{4} + C \\
\Rightarrow u &= e^x + \frac{x^5}{20} + Cx + D
\end{align*} \]

Apply antiderivative rules twice!

Next use initial data to find C, D:

\[ \begin{align*}
3 &= u'(0) = e^0 + \frac{0}{4} + C \\
\Rightarrow C &= 2 \\
2 &= u(0) = e^0 + \frac{0}{20} + (C(0) + D) \\
\Rightarrow D &= 1
\end{align*} \]

\[ u(x) = e^x + \frac{x^5}{20} + 2x + 1 \]
(6) Suppose a ball is tossed up in the air from height 192 feet at velocity 64 ft/s. (Recall \( g = -32 \) ft/s²)

(a) When does the ball hit the ground?

We know that \( s''(t) = -32 \)

So we have \( s'(t) = -32t + C \)

We're given that \( s'(0) = 64 \), so

\[
64 = -32(0) + C \Rightarrow C = 64
\]

\[
\Rightarrow s'(t) = -32t + 64
\]

so now \( s(t) = \frac{-32t^2}{2} + 64t + D \)

\[
= -16t^2 + 64t + D
\]

Then, with \( s(0) = 192 \) we get:

\[
192 = -16(0)^2 + 64(0) + D \Rightarrow D = 192
\]

and so \( s(t) = -16t^2 + 64t + 192 \)

Now, we want to find \( t \) so that \( s(t) = 0 \)
Solve for $t$:

$$0 = -16t^2 + 64t + 192$$

$$= -16(t-6)(t+2) \implies t = 6 \text{ seconds}$$

(b) If an insect flies directly upward from the ground with constant velocity 128 ft/s. When do the ball and insect pass each other?

We know that $v_{\text{bug}}(t) = 128 = s'_{\text{bug}}(t)$

$$\implies s_{\text{bug}}(t) = 128t + C$$

Since the bug starts on the ground, $s_{\text{bug}}(0) = 0$ and so $C = 0$.

$$\implies s_{\text{bug}}(t) = 128t$$

Then the question asks for what $t$ does $s_{\text{bug}}(t) = s(t)$?
That is:

\[ 128t = -16t^2 + 64t + 192 \]

\[ \Rightarrow 0 = -16t^2 - 64t + 192 \]

\[ = -16(t-2)(t+6) \]

\[ t = 2 \text{ seconds} \]
Revisiting our motivating question, given a velocity \( v(t) \) we want to find the displacement over a time interval \( a \leq t \leq b \). This is exactly the area under the graph, which is called the integral:

\[
\int_a^b v(t) \, dt = \text{area under the curve}
\]
Consider old examples:

1) A car has constant velocity 60 mph North of San Diego. How far does it travel in three hours?

The question is asking for the quantity \( \int_0^3 v(t) \, dt \)

\[
\begin{array}{c}
y \\
\hline
\hline
y=v(t) \\
\hline
\hline
(3-0) \\
\hline
\end{array}
\]

\[
\int_0^3 v(t) \, dt = \text{Area of square}
\]

\[
= (3-0) \cdot (60) = 180 \text{ miles}
\]

\[
\text{hours} \quad \text{mi/hour}
\]
2. An object is dropped from 20 meters. How far does it travel in 2 seconds?

\[ \text{Constant acceleration} \]

The question again asks for \( \int_0^2 v(t) \, dt \)

where \( v(t) \) is the object's velocity function.

Recall that we found \( v(t) = -9.81 t \) (where \( g = -9.81 \, \text{m/s}^2 \)), so consider the graph:

\[ y = v(t) \]

\[ t = 0 \quad 2 \quad 2.981 \]

\[ y = 2 \rightarrow 0 \]

So \( \int_0^2 v(t) \, dt = \text{area of the triangle} \)

\[ = - \frac{1}{2} (2-0)(2.981) = -19.62 \, \text{meters} \]

The negative means the displacement was in the down direction.
5) Properties of the integral

1) Linear:
\[ \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

2) Fig:

\[ (f+g)(x) = y \]

\[ a \quad b \]

3) Fig:

\[ y = f(x) \]

\[ a \quad b \]

\[ \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \]

4) Fig:

\[ y = 3f(x) \]

\[ a \quad b \]

5) Interval split:

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

6) Fig:
3. Integral direction

\[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]

4. Index variable does not matter

\[ \int_a^b f(x) \, dx = \int_a^b f(u) \, du \]

\[ \left[ \begin{array}{c}
\text{\(a\)} \\
\text{\(b\)}
\end{array} \right] = \left[ \begin{array}{c}
\text{\(a\)} \\
\text{\(b\)}
\end{array} \right] \]

→ These are both numbers

→ The variable of integration doesn't matter if the function are unchanged.
Examples

1. Suppose $\int_{2}^{3} f(x) = 3$ and $\int_{2}^{3} g(x) = 10$.
   What is $\int_{2}^{3} (2f(x) + 3g(x))\,dx$?

   $\int_{2}^{3} (2f(x) + 3g(x))\,dx = 2\int_{2}^{3} f(x)\,dx + 3\int_{2}^{3} g(x)\,dx$

   Linerarity gives
   
   $= 2(3) + 3(10)$

   $= 36$

2. Suppose $f(x) = \begin{cases} 1 & x < 1 \\ 2 & x \geq 1 \end{cases}$
   What is $\int_{0}^{3} f(x)\,dx$?

   Then $\int_{0}^{3} f(x)\,dx = \int_{0}^{1} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

   $= \text{(Area of small square)} + \text{(Area of large square)}$
Hence:
\[ \int_{3}^{5} f(x) \, dx = (1-0)(1) + (3-1)(2) \]
\[ = 1 + 4 = 5 \]

Consider the graph of \( f(x) \).
Suppose \( \text{Area}(A) = 1 \), \( \text{Area}(B) = \text{Area}(C) = 2 \)

(a) What is \( \int_{3}^{5} f(x) \, dx \)?

See that \( \int_{3}^{5} f(x) \, dx = -\int_{0}^{3} f(x) \, dx \)

\[ = -\left( \int_{0}^{1} f(x) \, dx + \int_{1}^{3} f(x) \, dx \right) \]

\[ = -\left( \text{Area}(A) + \text{Area}(B) \right) \]

\[ = -(1-2) = -(-1) = 1 \]
(b) What is \( \int_1^5 (2f(x)+1)\,dx \)?

See that
\[
\int_1^5 (2f(x)+1)\,dx = 2\int_1^5 f(x)\,dx + \int_5^1 d\,x
\]
\[
= 2\left(\int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx\right) + \int_5^1 d\,x
\]
\[
= 2\left(-\text{Area}(B) + \text{Area}(C)\right) + (5-1)(1)
\]
\[
= 2(-2+2) + 4(1)
\]
\[
= 4
\]

(4) Kind \( \int_1^3 x^2\,dx \)

\[
\text{hard without additional tools!}
\]
\[
\text{No formulae from elementary geometry for this area!}
\]
A bumble bee flying up and down has the following velocity function: \( v(t) = 6 - 2t \) m/s.

(a) What is the bee's displacement over time interval \( 1 \leq t \leq 5 \)?

(b) What is the total distance traveled?

Part (a) is asking for \( \int_1^5 v(t) \, dt \).

Consider the graph:

\[ \int_1^5 v(t) \, dt = \text{(Area of top triangle)} - \text{(Area of bottom triangle)} \]

\[ = \frac{1}{2} (3-1)(4) - \frac{1}{2} (5-3)(4) = \boxed{0} \]
hence the bee’s position is the same at time $t = 1$ and $t = 5$

(b) asks for total distance, or in how much distance did the bee cover.

This is precisely the integral of the absolute value of $v(t)$:

$$\int_{1}^{5} |v(t)| \, dt$$

Consider the graph of $y = |v(t)|$.

So here we have

$$\int_{1}^{5} |v(t)| \, dt = (\text{Area of left triangle}) + (\text{Area of right triangle}) = 8 + 8 = 16$$
\* Note that this answer is the same as if we treated all areas as positive (even under the x-axis) in our integral!

\* Warning: You must be wary of sign changes when integrating an absolute value!
6) **Integration as limit of Riemann Sums**

[10A] The derivative is defined as a limit of "the simplest derivatives", that is, slopes of lines: (i.e., derivatives of linear functions)

\[
\frac{d}{dx} (kx + b) = k
\]

[10B] The integral is likewise defined as a limit of "the simplest integrals", that is, areas of rectangles, which is the integral of a constant function:

\[
\int_{a}^{b} k \, dx = k \left( b - a \right)
\]
\( f'(x) = \frac{df}{dx} \)

\( f(x) = kx + b \)

\( \text{lines} \)

\( \text{antiderivative} \)

\( \text{constant} \)

This limit is obtained in the following way:

1. Chop up the interval \( a \leq x \leq b \) into \( N \) equal-length pieces (sub-intervals):

\[ N \text{ subintervals of length } \Delta x := \frac{b-a}{N} \]
2) Now can divide a new function $f_N(x)$ which is constant on each of the sub-intervals.

We pick the constant to be one of the values of $f(x)$ on the sub-interval. (More on this later)

3) The new function $f_N(x)$ is piece-wise constant, hence it is all rectangles and we can compute

$$\int_{a}^{b} f_N(x) \, dx$$

\[ y = f_N(x) \]
\[ y = f(x) \]
4) The value $\int_a^b f_N(x) \, dx$
   is called the N\textsuperscript{th} Riemann sum
   for $f(x)$ over $a \leq x \leq b$.

   → We have different kinds
   depending on how we build $f_N(x)$.

5) For good functions,
   we obtain $\int_a^b f(x) \, dx$
   by letting $N \to \infty$!

**Notation:**

$$
\Delta x = \frac{b-a}{N} = \text{width of intervals}
$$

$$
a = x_0, x_1, x_2, \ldots, x_N = b
$$

$$
x_n = a + n \Delta x = \text{endpoints of sub-intervals}
$$
Example: \( f(x) = x^2 \)

Consider \( \int_1^3 f(x) \, dx \)

Right Riemann sum: \( R_N \)

Define the constant with the value of \( f(x) \) on the right endpoints of each sub-interval.

\( \Delta x = \frac{3-1}{N} \)

\( x_n = 1 + n \cdot \Delta x \)

\[ R_4 = \int_1^3 f(x) \, dx = \int_1^{\frac{3}{2}} f\left(\frac{3}{2}\right) \, dx + \int_{\frac{3}{2}}^{2} f(2) \, dx + \int_{2}^{\frac{5}{2}} f\left(\frac{5}{2}\right) \, dx + \int_{\frac{5}{2}}^{3} f(3) \, dx \]

\[ = \left( \frac{3}{2} - 1 \right) \left( \frac{3}{2} \right)^2 + \left( 2 - \frac{3}{2} \right) (2)^2 + \left( \frac{5}{2} - 2 \right) \left( \frac{5}{2} \right)^2 + \left( 3 - \frac{5}{2} \right) (3)^2 \]

\[ = \Delta x \left( \frac{3}{2} \right)^2 + \Delta x (2)^2 + \Delta x \left( \frac{5}{2} \right)^2 + \Delta x (3)^2 \]

\[ = \Delta x \left( \frac{3^2}{2} + 2^2 + \frac{5^2}{2} + 3^2 \right) \]

\[ R_N = \Delta x \left( f(x_1) + \cdots + f(x_N) \right) \]
- Left Riemann sum: \( \sum \)

\[ L_4 = \int_1^{3/2} f(x) \, dx + \int_{3/2}^2 f(3) \, dx + \int_2^{5/2} f(2) \, dx + \int_5^{3} f(3) \, dx \]

\( \Rightarrow \) choose the left endpoint

\[ \Rightarrow \quad L_N = \Delta x (f(x_0) + \ldots + f(x_{N-1})) \]

- Defn: \( f(x) \) is called integrable over \( a \leq x \leq b \) if the limits as \( N \to \infty \) of any Riemann sum all agree.

\( \star \) - In fact, it must be true for any selection of constant and different values of subintervals.
Thus (Riemann)

If \( f(x) \) is integrable then

\[
\lim_{N \to \infty} R_N = \int_a^b f(x) \, dx
\]

(or be any other Riemann sum)

Thus

\( f(x) \) continuous \( \Rightarrow \) \( f(x) \) integrable

Return to example \( f(x) = x^2 \) is continuous, hence integrable.

It can be shown that \( R_N \) for

\[
\int_1^3 x^2 \, dx
\]

is given by

\[
R_N = 6 + \frac{4}{N} + \frac{8N^2 + 12N + 4}{3N^2}
\]
Hence,

\[
\int_0^3 x^2 \, dx = \lim_{{N \to \infty}} R_N = 6 + 0 + \frac{8}{3} = \left\lfloor \frac{26}{3} \right\rfloor
\]

Difficulty:

Where did \( R_N \) come from?

\[ V \]

This is a hard problem!

\( \rightarrow \) Requires 2 theorems about sums of integer powers due to Gauss!

\( \rightarrow \) Bonus problem on how to find \( R_N \) in a simple function!
For $N$: $f(x) = x^2$, $a = 1$, $b = 3$

$R_N = \Delta x \left( f(x_1) + \cdots + f(x_N) \right)$

$= \left( \frac{2}{N} \right) \left[ f\left(1 + \frac{2}{N}\right) + f\left(1 + 2\frac{2}{N}\right) + \cdots + f\left(1 + N\frac{2}{N}\right) \right]$

$= \frac{2}{N} \left[ \left(1 + \frac{2}{N}\right)^2 + \left(1 + 2\frac{2}{N}\right)^2 + \cdots + \left(1 + N\frac{2}{N}\right)^2 \right]$

$= \frac{2}{N} \left[ \left(1 + \frac{2}{N}\right)^2 + \left(1 + 2\frac{2}{N}\right)^2 + \cdots + \left(1 + N\frac{2}{N}\right)^2 \right]$

$\text{Expand and rearrange!}$

Then use Gauss's formulas:

$1 + 2 + \cdots + N = \frac{N(N+1)}{2}$

$1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$

(hard!)