1. **Sequences**

**Definition**: A sequence is an infinite ordered list of numbers.

**Notation**: 
\[ \{a_n\} = \{a_1, a_2, a_3, a_4, a_5, \ldots\} \]

(may sometimes start at \(n=0\))

**Examples**

1. \(a_n = \frac{1}{n}\)
   
   \[ \{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\} \]

2. \(b_n = \frac{1}{n^2}\)
   
   \[ \{b_n\} = \{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\} \]

3. \(c_n = \cos(n\pi)\)
   
   \[ \{c_n\} = \{-1, 1, -1, 1, -1, 1, -1, \ldots\} \] (note that: \(c_n = (-1)^n\))
\[ d_n = f^{(n)}(1) \text{ for } f(x) = x^4 + x^2 \]

\[
\begin{align*}
    f'(x) &= 4x^3 + 3x^2 \Rightarrow f'(1) = 7 \\
    f''(x) &= 12x^2 + 6x \Rightarrow f''(1) = 18 \\
    f'''(x) &= 24x + 6 \Rightarrow f'''(1) = 30 \\
    f^{(4)}(x) &= 24 \Rightarrow f^{(4)}(1) = 24 \\
    f^{(n)}(x) &= 0 \Rightarrow f^{(n)}(x) = 0 \text{ for all } n \geq 5
\end{align*}
\]

Hence \( \{d_n\} = \{7, 18, 30, 24, 0, 0, 0, 0, 0, \ldots \} \)

\[ s_n = s^{(n)}(0) \text{ where } s(x) = \sin(x) \]

\[
\begin{align*}
    s'(x) &= \cos(x) = s^{(3)}(x) \\
    s''(x) &= -\sin(x) \\
    s'''(x) &= -\cos(x) \\
    s^{(4)}(x) &= \sin(x) \\
    s'(0) &= 1 = s^{(5)}(0) \\
    s''(0) &= 0 = s^{(6)}(0) \\
    s'''(0) &= 0 = s^{(7)}(0) \\
    s^{(4)}(0) &= 0 = s^{(8)}(0), \text{ etc.}
\end{align*}
\]

\[ \{s_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, 1, 0, \ldots \} \]

Periodic with period 4

\[ R_2 \text{ for } f(x) = x^2 \text{ over } [0, 1] \text{ (Right Riemann)} \]

\[
\begin{align*}
    R_1 &= f(1) \cdot (1-0) = 1, \\
    R_2 &= [f(\frac{1}{2}) + f(1)] \cdot (\frac{1}{2}) = [\frac{1}{4} + 1] \cdot (\frac{1}{2}) = \frac{5}{8}, \\
    R_3 &= [f(\frac{1}{4}) + f(\frac{3}{4}) + f(1)] \cdot (\frac{1}{4}) = [\frac{1}{16} + \frac{3}{4} + 1] \cdot (\frac{1}{4}) = \frac{14}{32}, \\
    R_4 &= [f(\frac{1}{8}) + f(\frac{7}{8}) + f(\frac{3}{4}) + f(1)] \cdot (\frac{1}{4}) = (\frac{1}{64} + \frac{7}{16} + \frac{3}{4}) \cdot (\frac{1}{4}) = \frac{20}{64} = \frac{5}{16}.
\end{align*}
\]
*Important fact*

Sequences can have limits:

\[ \lim_{n \to \infty} a_n := \text{the value } a_n \text{ approaches as } n \text{ gets very large} \]

→ Limits of sequences follow most of the same rules of limits of functions; if the sequence is given by a function that has a limit, then the limit of the sequence will agree with the limit of the function:

**FACT #1** If \( \lim_{x \to \infty} f(x) = L \) (i.e., exists), then \( \lim_{n \to \infty} f(n) = L \) → sequence limit
Examples:

1. \[ \lim_{n \to \infty} \frac{1}{n} = ? \]
   
   \[ \lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{since we know} \quad \lim_{x \to \infty} \frac{1}{x} = 0 \]

2. \[ \lim_{n \to \infty} \frac{1}{n^2} = ? \]
   
   Similarly, \[ \lim_{n \to \infty} \frac{1}{n^2} = 0 \]

FACT #1.5 Sequences given by rational factors

have the following limits:

\[ \lim_{n \to \infty} \frac{a_n k + a_{n-1} k^{-1} + \cdots}{b n^m + b_0 n^{m-1} + \cdots} \]

\[ = \begin{cases} 
\frac{a}{b} & \text{if} \quad k = m \\
0 & \text{if} \quad k < m \\
(\pm \infty) & \text{if} \quad k > m 
\end{cases} \]

This follows directly from [FACT #1] and the theorem on infinite limits of rational factors (a L'Hôpital's rule) from [TOA].
\[\lim_{n \to \infty} \frac{3n^2 + 2n + 1}{5n^2 + 16} = \frac{3}{5}\]

sincd \deg (\text{top}) = \deg (\text{bottom})

\[\lim_{n \to \infty} \frac{3n^3 + 1}{n^2 - n - 1} = +\infty\] since \deg (\text{top}) > \deg (\text{bottom}) and \(3, 1 > 0\)

\[\lim_{n \to \infty} \frac{16n^4}{n^2 + 5} = 0\] since \deg (\text{top}) < \deg (\text{bottom})

---

Note that a convergent sequence can still come from a divergent function

\[\lim_{n \to \infty} \sin(2\pi n)\]

Note that here we have \(\lim_{x \to \infty} \sin(2\pi x)\) does not exist since the function fluctuates (x includes many fractions, and even \(1/2\), etc.)

But: \(\sin(2\pi n) = 0\) for every positive integer \(n\) !! hence the sequence...
7. \( \lim_{n \to \infty} \cos(\pi n) \)

Recall \( \cos(\pi n) = (-1)^n \), and 
\( \lim_{n \to \infty} (-1)^n \) does not exist because the sequence bounces between 1 and -1 for every n.

8. \( \lim_{n \to \infty} d_n \) (see eq 4 above)

since \( \{d_n\} = \{7, 18, 30, 24, 0, 0, 0, 0, 0, 0, \ldots\} \)
we can see visually that \( \lim_{n \to \infty} d_n = 0 \)

9. \( \lim_{n \to \infty} s_n \) (see eq 5 above)

since \( \{s_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, 1, 0, \ldots\} \)
it is periodic and must continue cycling through the 4 values and will not get close to only one of them.

so \( \lim_{n \to \infty} s_n \) does not exist
9. Let \( t_n = S_{n+2} \); \( \lim_{n \to \infty} t_n = ? \)

Here we have

\[ t_n = S^{(n+2)}(0) \text{ for } S(x) = \sin(x) \]

We saw before that \( S^{(n+2)}(x) = \pm \sin(x) \)

Hence \( S^{(n+2)}(0) = \pm \sin(0) = 0 \)

Hence \( t_n = 0 \) for all \( n \), hence

\[ \lim_{n \to \infty} t_n = 0 \]

10. Geometric sequence: \( r_n = \left( \frac{1}{2} \right)^n \) (start at \( n = 0 \))

\[ \{ r_n \} = \{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \} \]

Notice that this is getting close to 0!

Indeed,

\[ \lim_{n \to \infty} r_n = \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0 \]
FACT #2: The geometric sequence $r^n$

has the following limit:

$$\lim_{n \to \infty} r^n = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{if } -1 < r < 1 \\ \text{dregonent if } r \text{ is otherwise} \\ \end{cases}$$

(11) $\lim_{n \to \infty} \left( \frac{2^{n+1}}{3^{n-1}} \right)$

$$= \lim_{n \to \infty} \frac{2 \cdot 2^n}{3^n \cdot 3^{-1}} = \frac{2}{3^{-1}} \lim_{n \to \infty} \frac{2^n}{3^n}$$

$$= 6 \cdot \lim_{n \to \infty} \left( \frac{2}{3} \right)^n$$

$$= 0 \text{ by geometric sequence fact}.$$
\[ 0 = 0 \text{ } + \text{ } 2 \lim_{n \to \infty} \frac{1}{n} \]
\[ = \lim_{n \to \infty} \left( \frac{1}{n} \right) \quad \text{continuity of positive numbers} \]

\[ \lim_{n \to \infty} \frac{3^{n+2}}{7^n} = \lim_{n \to \infty} \frac{3^n \cdot 2}{7^n} = \lim_{n \to \infty} \frac{2}{7} \text{ } 
\]

FACT 3:
Continuous function can pass in and out of limits.

\[ \lim_{n \to \infty} f(an) = f(\lim_{n \to \infty} an) \]
13) Suppose $\lim_{n \to \infty} l_n = 3$

What is $\lim_{n \to \infty} e^{l_n+1} =$ ?

Then $\lim_{n \to \infty} e^{l_n+1} = \lim_{n \to \infty} e^{l_n} \cdot e$

$= e \lim_{n \to \infty} e^{l_n}$

$= e \cdot e^3 = e^3 + 1 = e^4$
2. Series:

**Defn:** Given a sequence \( \{a_n\} \),

the series is its corresponding sum:

\[
a_0 + a_1 + a_2 + a_3 + a_4 + \ldots = \sum_{n=0}^{\infty} a_n
\]

\(\Sigma\) notation:

see notes on Riemann sums

**Remark:**

The series is the limit of the sequence of partial sums:

Let \( S_n = a_0 + a_1 + a_2 + a_3 + \ldots + a_n \)

Then

\[
\lim_{n \to \infty} S_n = a_0 + a_1 + \ldots = \sum_{n=0}^{\infty} a_n
\]

limit of sequence = the series for \( \Sigma a_n \)
**Fact #1** Divergence Theorem

The series \( \sum a_n^3 \) is divergent if \( \lim_{n \to \infty} a_n \neq 0 \)

We'll apply this theorem to our examples from section 1:

1. \( a_n = \frac{1}{n} \): \( \lim_{n \to \infty} a_n = 0 \), so no conclusion

2. \( b_n = \frac{1}{n^2} \): \( \lim_{n \to \infty} b_n = 0 \), so no conclusion

3. \( \frac{3n^2 + 2n + 1}{5n^2 + 16} \): \( \lim_{n \to \infty} \frac{3n^2 + 2n + 1}{5n^2 + 16} = \frac{3}{5} \) so Divergent serie

4. \( \frac{5n^2 + 1}{n^2 - n - 1} \): \( \lim_{n \to \infty} \frac{5n^2 + 1}{n^2 - n - 1} = +\infty \) so Divergent serie

5. \( \frac{16n^4}{n^4 - n^6 + 5} \): \( \lim_{n \to \infty} (.) = 0 \) so no conclusion

6. \( \sin(2\pi n) \): \( \lim_{n \to \infty} (.) = 1 \), so Divergent serie

7. \( \cos(\pi n) \): \( \lim_{n \to \infty} (.) \) doesn't exist, so Divergent serie
Notice that if the sequence's limit is 0, then we can't make a conclusion about the convergence of the series.

**EXAMPLE**

Note that \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \),

but:

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = +\infty
\]

is divergent.

and

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}
\]

is convergent.

\( \text{Note: you don't need to know how to compute these zeta functions} \)
For this class we'll only consider one example of a series where we can compute the sum if it exists. The:

**Geometric series:**

\[ a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n \]

\[ \frac{1}{3} = 0.33333 \ldots = 3(10)^{-1} + 3(10)^{-2} + 3(10)^{-3} \]

\[ = 3\left(\frac{1}{10}\right) + 3\left(\frac{1}{10}\right)^2 + 3\left(\frac{1}{10}\right)^3 + \cdots \]

\[ = \sum_{n=0}^{\infty} 3\left(\frac{1}{10}\right)^{n+1} \]

\[ \text{sigma notation} \]
The above is a special case of the following theorem:

**FACT #2** (Geometric series)

1. \( a + ar + ar^2 + ar^3 + \ldots \) is convergent if and only if \(-1 < r < 1\)

2. If it is convergent then

\[
\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}
\]

\(\rightarrow\) Note that part 1 is from the theorem on Geometric series. \(\lim_{n \to \infty} ar^n = 0\) if and only if \(-1 < r < 1\).

\(\rightarrow\) We can get the above example with part 2:

\[
\sum_{n=0}^{\infty} 3 \left(\frac{1}{10}\right)^n = \frac{3}{10} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{3}{10} \times \frac{1}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}
\]
More examples:

1. $2 + 2(\frac{1}{3}) + 2(\frac{1}{3})^2 + 2(\frac{1}{3})^3 + \ldots$

   See that $a = 2$ and $ar = 2(\frac{1}{3})$,
   
   hence $r = \frac{ar}{a} = \frac{2(\frac{1}{3})}{2} = \frac{1}{3}$

   Therefore since $-1 < r < 1$, the series is convergent, and it is equal to:

   $$\frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = \frac{3}{2}$$

2. $\frac{1}{2} + \frac{1}{3}(\frac{3}{2}) + \frac{1}{2}(\frac{3}{2})^2 + \frac{1}{2}(\frac{3}{2})^3 + \ldots$

   Here $a = \frac{1}{2}$, $ar = \frac{1}{2}(\frac{3}{2})$, hence $r = \frac{ar}{a} = \frac{3}{2}$

   Now $\frac{3}{2} > 1$, hence the series is DIVERGENT.
Remark:

We get the Geometric Series Theorem from the following fact:

If \( a_n = ar^n \)
then \( S_n = a_0 + a_1 + a_2 + \ldots + a_n \)

\[ S_n = \frac{a(1-r^{n+1})}{1-r} \]

See page 566-568 for how this is derived; pretty cool.

Hence:

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \ldots = \lim_{n \to \infty} \frac{a(1-r^{n+1})}{1-r} \]

\[ = \lim_{n \to \infty} \frac{a}{1-r} + \lim_{n \to \infty} \frac{-ar^{n+1}}{1-r} \]

Since \( 1 < r < -1 \)

\[ = \frac{a}{1-r} + 0 \]

\[ = \frac{a}{1-r} \]
examples, cont'd

3) \[ 5 + 3 + \frac{9}{5} + \frac{27}{25} + \frac{81}{125} + \cdots = ? \]

See that \[ \frac{a}{1}, \frac{ar}{1} \]

\[
\begin{align*}
\text{Term 0} & \quad \text{Term 1} \\
\hline
5 & \quad 3 \\
\end{align*}
\]

So \[ r = \frac{ar}{a} = \frac{3}{5} \] \[ \Rightarrow \text{series is convergent} \]

And the sum is \[
\frac{5}{1 - \frac{3}{5}} = \frac{5}{\frac{2}{5}} = \frac{25}{2}
\]

4) \[ \frac{1}{2} + \frac{5}{8} + \frac{25}{32} + \frac{125}{128} + \cdots = ? \]

\[ a = \frac{1}{2}, \quad ar = \frac{5}{8} \]

\[ \Rightarrow r = \frac{ar}{a} = \frac{\frac{5}{8}}{\frac{1}{2}} = \frac{5}{4} \]

\[ = \frac{10}{8} > 1, \quad \text{hence divergent series} \]
(3) Power series

Just as a number can be represented by a decimal series, e.g.
\[ \sum_{n=0}^{\infty} a_n(10)^n = 12596 \]

For sequence \( \{a_n\} = \{6, 9, 5, 2, 1, 0, 9, 9, \ldots\} \)

We can represent a function with a power series; the geometric series gives our first example:

\[ \text{e.g. The function } f(x) = \frac{1}{1-x} \]

is given by
\[ \sum_{n=0}^{\infty} x^n = f(x) \]

In general, a nice enough function \( f(x) \) will have a representation like
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \]
This is like representing a given function as an infinitely long polynomial:

\[ f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \]

For very nice functions we can compute the power series using Taylor's theorem:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

For \( x \) near \( x=a \)

hence the power series is often called the Taylor series.

Recall: \( n! = n(n-1)(n-2)\cdots(2)(1) \) is the factorial
By taking finite sub-sums of the Taylor series we can estimate the value of a function.

In fact, you’ve already seen the degree 1 Taylor polynomial (i.e. the sum of the $n=0$ and $n=1$ terms):

It is the linear approximation:

$$f(x) \approx f^{(0)}(a) + f^{(1)}(a)(x-a)$$
More examples:

1. Since \( \frac{1}{3} = \sum_{n=0}^{\infty} \left( \frac{3}{10} \right)^n \), we can approximate to degree 1 by taking the sum of the \( n=0 \) and \( n=1 \) terms:

\[
\frac{1}{3} \approx \left( \frac{3}{10} \right)^0 + \left( \frac{3}{10} \right)^1 = 0.33
\]

2. Write the linear approximation of \( f(x) = \frac{3}{10(1-x)} \) at \( x=0 \)

\[
f''(x) = \frac{3}{10(1-x)^2} \Rightarrow f'(0) = \frac{3}{10}
\]

So:

\[
f(x) \approx f(0) + f'(0)(x-0) = \frac{3}{10} + \frac{3}{10}x
\]

Hence, \( \frac{6}{5} = f(\frac{1}{2}) \approx \frac{3}{10} + \frac{3}{20} = \frac{9}{20} \)
We'll call the linear approximation \( T_1(x) \) and the higher Taylor polynomials

\[
T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

\[\text{sum of the } n=0, \ldots, N \text{ terms}\]

These are the higher degree analogs of approximations corresponding to the linear one. (E.g. \( T_2(x) \) is the quadratic approximator near \( x=a \), etc.)
Examples.

3. Find the 3rd degree Taylor approximate for $f(x) = \sin(x)$ about $x = 0$

\[
f'(x) = \cos(x) \\
f''(x) = -\sin(x) \\
f'''(x) = -\cos(x)
\]

Now, $T_3(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!} (x-0) + \frac{f''(0)}{2!} (x-0)^2 \\
+ \frac{f'''(0)}{3!} (x-0)^3$

\[
= \frac{\sin(0)}{1} + \frac{\cos(0)}{1} (x) + \frac{-\sin(0)}{2} (x)^2 \\
+ \frac{-\cos(0)}{6} (x)^3
\]

\[
= 0 + x + 0 - \frac{1}{6} x^3
\]

\[
= \boxed{\left[x - \frac{1}{6} x^3\right]}
\]

(hence $f(x) \approx T_3(x)$ for $x$ near 0)
4. Find the 4th degree Taylor polynomial for \( f(x) = e^x \) about \( x = 0 \)

\[
\begin{align*}
 f'(x) &= e^x \\
 f''(x) &= e^x \\
 f'''(x) &= e^x \\
 f^{(4)}(x) &= e^x \\
\end{align*}
\]

Then:

\[
T_4(x) = f(0) + \frac{f'(0)(x-0)}{1!} + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \frac{f^{(4)}(0)(x-0)^4}{4!}
\]

\[
= 1 + \frac{e^0}{1!} (x) + \frac{e^0}{2!} (x)^2 + \frac{e^0}{3!} (x)^3 + \frac{e^0}{4!} (x)^4
\]

\[
= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}
\]

(hence \( f(x) \equiv T_4(x) \) for \( x \) near 0)
Find the 2nd degree Taylor approximation for \( f(x) = \cos(x) \) about \( x = \pi \)

\[
T_2(x) = f(\pi) + \frac{f'(\pi)}{1!}(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2
\]

\[
= \cos(\pi) + \frac{-\sin(\pi)}{1}(x-\pi) + \frac{-\cos(\pi)}{2!}(x-\pi)^2
\]

\[
= -1 + 0 - \frac{1}{2}(x-\pi)^2
\]

\[
= \left\{-1 + \frac{1}{2}(x-\pi)^2\right\}
\]

(and \( T_2(x) \approx f(x) \) for \( x \) near \( x = \pi \))
Find $T_3(x)$ for $f(x) = x^3 + x + 1$ about $x = 1$

$$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{6}$$

Now, $f'(x) = 3x^2 + 1$, so $f'(1) = 4$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

and so:

$$= 3 + 4(x-1) + \frac{6}{2}(x-1)^2 + \frac{6}{6}(x-1)^3$$

$$= 3 + 4(x-1) + 3(x-1)^2 + (x-1)^3$$

$$= 4x - 1 + 3(x^2 - 2x + 1) + (x-1)(x^2 - 2x + 1)$$

$$= 4x - 1 + 3x^2 - 6x + 3 + (x^3 - 2x^2 + x - x^2 + 2x - 1)$$

$$= 4x + 2 + 3x^2 + x^3 - 3x^2 + 3x - 1$$

$$= x^3 + x + 1$$

$\text{Note that } T_3(x) = f(x) \text{ here}$
In fact, if \( p(x) \) is deg \( N \) polynomial then \( T_n(x) = p(x) \) for any \( a \).

7. Find \( T_4(x) \) for \( \ln(x) \) near \( x = 1 \)

See that \( f''(x) = \frac{1}{x} \implies f''(1) = 1 \)
\[ f''(x) = \frac{-1}{x^2} \implies f''(1) = -1 \]
\[ f''''(x) = \frac{2}{x^3} \implies f''''(1) = 2 \]
\[ f^{(4)}(x) = \frac{-6}{x^4} \implies f^{(4)}(1) = -6 \]

Then
\[
T_4(x) = \ln(1) + \frac{\ln'(1)}{1!} (x-1) + \frac{\ln''(1)}{2!} (x-1)^2
+ \frac{\ln'''(1)}{3!} (x-1)^3 + \frac{\ln^{(4)}(1)}{4!} (x-1)^4
\]
\[= 0 + 1(x-1) + \frac{-1}{2} (x-1)^2 + \frac{2}{6} (x-1)^3 + \frac{-6}{24} (x-1)^4 \]
\[
= \sqrt{(x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{6} (x-1)^3 - \frac{1}{24} (x-1)^4} \]