2. Fundamental Theorem of Calculus, part I

Idea of FTC I:
we can compute integrals using antiderivatives

- Suppose \( f(x) \) is integrable; then:

\[
\text{Fundamental Theorem of Calculus, I}
\]

If \( F(x) \) is an antiderivative of \( f(x) \),
then \( \int_a^b f(x) \, dx = F(b) - F(a) \)

- This precisely the equality of the
10B version of displacement & the 10A version of displacement.
Let \( f(t) = v(t) \) velocity of object with position \( F(t) = s(t) \), then:

\[
\int_a^b v(t) \, dt = s(b) - s(a)
\]
Examples

\( \int_1^3 (x^3 - 5x^2 + 13) \, dx \)

An antiderivative for \( f(x) = x^3 - 5x^2 + 13 \)
looks like \( F(x) = \frac{x^4}{4} - \frac{5x^3}{3} + 13x + C \)
by the power rule.

Then, by FTC I:

\[
\int_1^3 (x^3 - 5x^2 + 13) \, dx = F(3) - F(1)
\]

\[
= \left( \frac{3^4}{4} - \frac{5(3)^3}{3} + 13(3) + C \right) - \left( \frac{1^4}{4} - \frac{5(1)^3}{3} + 13(1) + C \right)
\]

\[
= \frac{81}{4} - 45 + 39 + C - \frac{1}{4} - \frac{5}{3} - 13 - C
\]

\[
= \frac{80}{4} - \frac{5}{3} - 19 + C - C = \frac{80}{4} - \frac{5}{3} - 19
\]

\[
= \frac{80}{4} - \frac{5}{3} - 19 + C - C = \frac{80}{4} - \frac{5}{3} - 19
\]

\[
= \frac{20}{1} - \frac{5}{3} - 19 + \frac{11}{2} = 0
\]

Notice: The constant of integration always cancels!
Hence we can use any antiderivative \( F(x) \) of \( f(x) \).
\( 2 \int_0^{\ln(2)} (x + e^x) \, dx = ? \)

An antiderivative for \( f(x) = x + e^x \)
\[ 15 \quad F(x) = \frac{x^2}{2} + e^x \]

Now by FTC I:
\[
\int_0^{\ln(2)} (x + e^x) \, dx = F(\ln(2)) - F(0) \\
= \left( \frac{\ln(2)^2}{2} + e^{\ln(2)} \right) - \left( \frac{0^2}{2} + e^0 \right) \\
= \frac{\ln(2)^2}{2} + 2 - 0 - 1 \\
= \frac{\ln(2)^2}{2} + 1
\]

\( 3 \int_{\pi/2}^{2\pi} 5 \cos(x) \, dx = ? \)

\[
= 5 \left( \int_{\pi/2}^{2\pi} \cos(x) \, dx \right) = 5 \left( \sin(2\pi) - \sin(\frac{\pi}{2}) \right) \\
\text{linearity!} \\
= 5 \left( \sin(\pi) - \sin(\frac{\pi}{2}) \right) \\
\text{[\text{antiderivative}] F(x) = \sin(x)} = \boxed{-5}
\]
4) \[ \int_{2}^{7} |x^2 - 6x + 5| \, dx \]

First we need to find the zero points:

\[ f(x) = x^2 - 6x + 5 = (x-1)(x-5) \]

Hence \( f(x) = 0 \) for \( x = 1 \) and \( x = 5 \):

Since \( f(x) \) is negative on \( 1 \leq x \leq 5 \), we have that:

\[ |f(x)| = \begin{cases} 
  f(x) & x < 1, \ x > 5 \\
  -f(x) & 1 \leq x \leq 5
\end{cases} \]
So we want to compute the integral on the piece:

\[ \int_2^7 |f(x)| \, dx = \int_2^5 |f(x)| \, dx + \int_5^7 |f(x)| \, dx \]

\[ = \int_2^5 f(x) \, dx + \int_5^7 f(x) \, dx \]

\[ = -\int_2^5 (x^2 - 6x + 5) \, dx + \int_5^7 (x^2 - 6x + 5) \, dx \]

\[ C = -\left( F(5) - F(2) \right) + \left( F(7) - F(5) \right) \]

Apply FTC I with antiderivative

\[ F(x) = \frac{x^3}{3} - 6x^2 + 5x \]

\[ = -\left( \left( \frac{125}{3} - 75 + 25 \right) - \left( \frac{8}{3} - 12 + 10 \right) \right) + \left( \left( \frac{73}{3} - 34 + 35 \right) - \left( \frac{125}{3} - 75 + 25 \right) \right) \]

\[ = \left( \frac{8}{3} - 2 \right) + \left( \frac{73}{3} - 112 \right) - 2\left( \frac{125}{3} - 50 \right) \]
\textbf{Fundamental Theorem of Calculus, part II}

Idea of FTC II:
we can find antiderivatives using the integral.

- Suppose \( f(x) \) is integrable, then:

\[
\text{Fundamental Theorem of Calculus, II}
\]

For any value of \( a \), the area function

\[
F(x) = \int_a^x f(t) \, dt
\]

is an antiderivative of \( f(x) \).

\[
4 \quad \text{that is, } \frac{dF}{dx} = f(x) \quad \text{for any } a!
\]
What is an area function?

Example: 

\[ F(x) = \int_0^x f(t) \, dt \]

\[ = \left( \text{Area under the graph } y = f(t) \right) \]
\[ \text{over the interval } 0 \leq t \leq x \]

We can define such a function given any constant \( a \) and an integrable function \( f(x) \).

\( \text{FTC II} \) tells us that the antiderivatives of integrable functions are exactly the area functions!
Here the value $a$ plays the role of the "+C"; specifically, notice that:

$$F(0) = C \quad \mapsto \quad F(a) = 0$$

(usually) (integral over a zero-length interval)

That is to say, $a$ comes from the initial data.

In Leibniz notation:

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$
Examples

1. \( g(x) = \int_2^x e^t \sin(t) \, dt \); \( g'(x) = ? \)

By FTC II, \[ g'(x) = e^x \sin(x) \]

2. \( h(x) = \int_3^{x^2} \left( \frac{\ln(t)}{t} \right) \, dt \); \( h'(x) = ? \)

Notice that \( h(x) \) is a composition of a polynomial \( p(x) = x^2 + x \) and an area function \( F(x) = \int_3^x \left( \frac{\ln(t)}{t} \right) \, dt \).

Then \( h(x) = F(p(x)) \), and so by the chain rule:

\[ h'(x) = F'(p(x)) \cdot p'(x) \]
Now, by FTC II,

\[ F'(x) = \frac{\ln(x)}{x} \]

and by power rule,

\[ p'(x) = 2x + 1. \]

Therefore:

\[
\begin{align*}
    h'(x) &= F'(x^2 + x) \cdot (2x + 1) \\
    &= \frac{\ln(x^2 + x)}{x^2 + x} (2x + 1)
\end{align*}
\]
\[ \int_a^b f(t) \, dt = f(b) - f(a) \]

That is:
\[ \int_a^b \frac{df}{dt} \, dt = f(b) - f(a) \]

That is:
\[ \int_a^b \, df = f(b) - f(a) \]

\[ \rightarrow \text{This motivates the notation for antiderivatives:} \]

\[ f = \int df = \int \frac{df}{dt} \, dt \]
4. More examples w/ FTC

Recall

**FTC I**  
If \( \frac{dF}{dx} = f(x) \)  
then \( \int_a^b f(x) \, dx = F(b) - F(a) \)

**FTC II**  
If \( F(x) = \int_a^x f(t) \, dt \)  
then \( \frac{dF}{dx} = f(x) \)

\( \rightarrow \text{Remark: the given } F(x) \text{ is the specific antiderivative with } F(a) = 0 \)
\( \int_{0}^{27} (1 + \frac{3}{7}x) \, dx \)

- Let \( f(x) = 1 + \frac{3}{7}x = 1 + x^{1/3} \)
- \( F(x) = x + \frac{x^{4/3}}{4/3} = x + \frac{3x^{4/3}}{4} \)
- is an antiderivative of \( f(x) \).

- Now by FTC I:
\[
\int_{0}^{27} (1 + \frac{3}{7}x) \, dx = F(27) - F(0)
\]
\[
= \left[27 + \frac{3}{4}(27)^{4/3}\right] - \left[0 + \frac{3}{4}(0)^{4/3}\right]
\]
\[
= 27 + \frac{3 \cdot 5}{4}
\]

\( \int_{0}^{\frac{\pi}{2}} \frac{2\pi \, dt}{\sqrt{1-t^2}} \)

Recall \( \frac{d}{dt} (\arcsin(t)) = \frac{1}{\sqrt{1-t^2}} \)

Now:
\[
\int_{0}^{\frac{\pi}{2}} \frac{2\pi \, dt}{\sqrt{1-t^2}} = 2\pi \int_{0}^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-t^2}} = 2\pi (F(\frac{\pi}{2}) - F(0))
\]

\( \text{Linearity} \)
\[
= 2\pi (\arcsin(\frac{\pi}{2}) - \arcsin(0))
\]
\[
= 2\pi (\frac{\pi}{2} - 0) = \frac{\pi^2}{2}
\]
3) Consider two birds with the following vertical velocity functions:

\[ V_A(t) = 3 \cos(t) + 2t^2 \text{ m/s} \]

\[ V_B(t) = 5 \cos(t) \text{ m/s} \]

(a) What is \( A \)'s vertical displacement from \( t=0 \) to \( t=\pi \) seconds?

The question asks for \( \int_0^\pi V_A(t) \, dt \).

We can use FTC I, an antiderivative of \( V_A(t) \) is \( F_A(t) = 3 \sin(t) + \frac{2t^3}{3} + C \).

By FTC I:

\[ \int_0^\pi V_A(t) \, dt = F_A(\pi) - F_A(0) \]

\[ = (3 \sin(\pi) + \frac{2}{3} \pi^3 + C) - (0 + 0 + C) \]

\[ = 3 \cdot 0 + \frac{2}{3} \pi^3 + C - C \]

\[ = \frac{2}{3} \pi^3 \]
(b) What is B's total distance travelled over the same interval?

We sketch the graph:

\[ y = 5 \]
\[ y = v_B(t) \]
\[ y = |v_B(t)| \]
\[ t = 0 \]
\[ t = \pi \]

Recall the question is asking for \( \int_0^\pi |v_B(t)| \, dt \).

Since \( v_B(t) = 5 \cos(t) \) is negative for \( \frac{\pi}{2} < t < \pi \), we need to split the integral:

\[
\int_0^\pi |v_B(t)| \, dt = \int_0^{\pi/2} v_B(t) \, dt + \int_{\pi/2}^\pi v_B(t) \, dt
\]

\[
= \int_0^{\pi/2} 5 \cos(t) \, dt + \int_{\pi/2}^{\pi} -5 \cos(t) \, dt
\]

\[
\text{[Using FTC I with antiderivative } F(x) = \sin(x) \text{]} \quad = 5 \left[ \sin\left(\frac{\pi}{2}\right) - \sin(0) \right] + -5 \left[ \sin(\pi) - \sin\left(\frac{\pi}{2}\right) \right]
\]

\[
= 5[1-0-0+1] = 5(2) = 10
\]
(c) **Exercise**

If A starts at altitude 100 m and B starts at altitude 300 m, will the birds pass each other in the first 10π seconds?

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*Hint:* you need to find $s_A(t)$ and $s_B(t)$ and show that $s_A(t) = s_B(t)$ for some $t$ in $0 \leq t \leq 10\pi$.

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4. Let $f(x) = \int_0^x \sin(\cos(t)) \, dt$

What is $f'(x)$?

By FTC II,

$$f'(x) = \sin(\cos(x))$$
5) Let \( g(x) = \int_0^{\tan(x)} \sqrt{t + \tan(t)} \, dt \)

What is \( g'(x) \)?

We must again use the chain rule.

Let \( q(x) = \tan(x) \) and

\[ F(x) = \int_0^x \sqrt{t + \tan(t)} \, dt. \]

Then \( g(x) = F(q(x)) \)

and \( g'(x) = F'(q(x)) \cdot q'(x) \).

Now, \( q'(x) = \sec^2(x) \) and

\[ F'(x) = \sqrt{x + \tan(x)}. \]

So:

\[ g'(x) = F'(\tan(x)) \cdot \sec^2(x) \]

\[ = \frac{\sqrt{\tan(x) + \tan(x)}}{(\sqrt{\tan(x) + \tan(x)}) \cdot \sec^2(x)} \]
Let \( h(x) = \int_0^x \sin^3(t) \, dt \)

and \( h'(x) = ? \)

See:

\[
\frac{d}{dx} (h(x)) = \frac{d}{dx} \left( \int_0^x \sin^3(t) \, dt \right)
= \frac{d}{dx} \left( -\int_0^x \sin^3(t) \, dt \right)
= -\frac{d}{dx} \left( \int_0^x \sin^3(t) \, dt \right)
\]

Apply FTC II and chain rule

\[
= -\sin^3(e^x) \cdot \frac{d}{dx}(e^x)
= \left[ -e^x \sin^3(e^x) \right]
\]
Write down an area function that is the specific antiderivative $G(x)$ of $g(x) = \tan^3(x)$ such that $G(\pi) = 0$.

By FTC II we know $G(x) = \int_0^x \tan^3(x) \, dx$.

Note that if $\alpha = \pi$ then

$G(\pi) = \int_\pi^\pi \tan^3(x) \, dx = 0$

So $\int_\pi^x \tan^3(x) \, dx$