2. **Area between curves**

Recall the definition of integration:

\[ \int_a^b f(x) \, dx = \text{(AREA)} \]

The area under the curve \( y = f(x) \) is given by a limit of Riemann sums. This can be depicted pictorially:

\[ \int_a^b f(x) \, dx = \int_a^b \left[ \sum_{i=1}^n f(x_i) \Delta x \right] \]

\[ \text{area} = f(x) \, dx \]
We should interpret the right-hand side as summing \( (S) \) up the areas of many infinitesimally thin rectangles of width \( dx \) and height \( f(x) \).

Using this same pictorialism we can write down the area between two curves as an integral:

\[
\int_a^b [f(x) - g(x)] \, dx
\]

That is, the area we're interested in is the sum of the rectangles depicted!
Now we can write the area as an integral; see from the picture that our rectangle are:
- height = $f(x) - g(x)$
- width = $dx$

Hence the area of one rectangle is:

$$(f(x) - g(x)) \, dx$$

and the total area is:

$$\int_{a}^{b} (f(x) - g(x)) \, dx$$
Examples:

1. What is the area enclosed by
   \[ y = e^x, \quad y = x, \quad x = 0, \quad x = 1 \]?

Consider the graphs with the area shaded.

\[ \int_0^1 (e^x - x) \, dx \]

\[ = \left[ e^x - \frac{x^2}{2} \right]_0^1 \]

\[ = e - \frac{1}{2} - [1 - 0] = \sqrt{\frac{e - 3}{2}} \]
2) What is the area enclosed by \( y = x^3 \) and \( y = x^2 \)?

Consider the graphs and the shaded area:

\[ y = x^3, \quad y = x^2 \]

\( x = 0 \quad x = 1 \)

→ Note that we must determine the [intersection points] to determine the endpoints of our area integral.

→ To find intersections, just set the functions equal and solve:

\[ x^3 = x^2 \Rightarrow x = 0 \]

Suppose \( x \neq 0 \). Then \( \frac{x^3}{x^2} = \frac{x^2}{x^2} = 1 \)

\[ \Rightarrow x = 1 \]

(These points are marked in the picture above.)
Now we set up the area integral:

\[
\text{Area} = \int_0^1 (x^2 - x^3) \, dx
\]

\[
= \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left[ \frac{1}{3} - \frac{1}{4} \right]
\]
3. What is the area enclosed by $y = 4x - x^2$ and $y = x$?

See the graph:

Step 1: Find the intersection points:

$$4x - x^2 = x \quad \Rightarrow \quad x = 0$$

Suppose $x 
eq 0$; then

$$x(4-x) = 4x - x^2 = x$$

$$\Rightarrow \quad (4-x) = 1$$

$$\Rightarrow \quad x = 3$$

Step 2: Set up the area integral:

$$\text{Area} = \int_{0}^{3} \left( \frac{x}{4x-x^2} - x \right) \, dx$$
\[ \int_0^3 \left( (4x-x^2) - x \right) \, dx \]
\[ = \int_0^3 (3x-x^2) \, dx \]
\[ = \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \left[ \frac{27}{2} - \frac{27}{3} \right] \]

\[ \text{What is the area enclosed by } \]
\[ y = 2x^2 - 10 \text{ and } y = -3x^2 + 10 \]

Consider the graphs:
**Step 1:** Find intersection points:

$$2x^2 - 10 = -3x^2 + 10$$

$$\Rightarrow 5x^2 = 20$$

$$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

**Step 2:** Set up area integral:

$$\int_{-2}^{2} \left( \frac{-5x^2}{3} + 20x \right) dx$$

Notice: this negative, so no need to worry about sign.

$$= \int_{-2}^{2} \left( -3x^2 + 10 \right) - \left( 2x^2 - 10 \right) dx$$

$$= \int_{-2}^{2} \left( -5x^2 + 20 \right) dx$$

$$= \left[ \frac{-5x^3}{3} + 20x \right]_{-2}^{2}$$

$$= \left[ \left( \frac{-5(2)^3}{3} + 20(2) \right) - \left( \frac{-5(2)^3}{3} + 20(-2) \right) \right] = \frac{-10(2)^3}{3} + 80$$
5) Find the area of the region enclosed by the curves

\[ x = y^2 - 4y + 4 = (y-2)^2 \]
\[ x = -y^2 + 8y - 7 = (y-1)(7-y) \]

Notice these curves are of the form \( x = f(y) \) and \( x = g(y) \), hence we are better off integrating along the \( y \) axis.

Let's see the picture:
Now consider a rectangle of height dy:

\[ (y-2)^2 \quad \text{height} \quad dy \]

\[ (y-1)(7-y) - (y-2)^2 \quad \text{width} \]

This rectangle has area

\[ [(y-1)(7-y) - (y-2)^2] \cdot dy \]

\[ \frac{\text{width}}{\text{height}} \]

hence the total area is

\[ \int_a^b [(y-1)(7-y) - (y-2)^2] \, dy \]

where \( y=a \) and \( y=b \) are the intersection points (see picture above)
To find the intercepts a, b we set the function equal:

\[(y-2)^2 = (y-1)(7-y)\]

\[\Rightarrow y^2 - 4y + 4 = -y^2 + 8y - 7\]

\[\Rightarrow 2y^2 - 12y + 11 = 0\]

\[\Rightarrow y = \frac{12 \pm \sqrt{(-12)^2 - 4(2)(11)}}{2(2)}\]

\[= \frac{12 \pm \sqrt{144 - 88}}{4}\]

\[= \frac{12 \pm \sqrt{56}}{4}\]

\[= 3 \pm \frac{\sqrt{14}}{2}\]

\[\Rightarrow a = 3 - \frac{\sqrt{14}}{2} \quad \text{and} \quad b = 3 + \frac{\sqrt{14}}{2}\]
So the total area is:

\[
\int_{3 - \sqrt{4}}^{3 + \sqrt{4}} \left[ (y-1)(7-y) - (y-2)^2 \right] \, dy
\]

\[
= \int_{3 - \sqrt{4}}^{3 + \sqrt{4}} \left[ -y^2 + 8y - 7 - y^2 + 4y - 4 \right] \, dy
\]

\[
= \int_{3 - \sqrt{4}}^{3 + \sqrt{4}} \left[ -2y^2 + 12y - 11 \right] \, dy
\]

\[
= \left[ -\frac{2y^3}{3} + 6\frac{y^2}{2} - 11y \right]_{3 - \sqrt{4}}^{3 + \sqrt{4}}
\]

\[
= \left( -\frac{2}{3} (3 + \sqrt{4})^3 + 6 (3 + \sqrt{4})^2 - 11 (3 + \sqrt{4}) \right) - \left( -\frac{2}{3} (3 - \sqrt{4})^3 + 6 (3 - \sqrt{4})^2 - 11 (3 - \sqrt{4}) \right)
\]