I. Antiderivatives and position functions

1. A beaver is swimming 3 mph south in a river.

(a) Suppose the beaver started 5 miles north of his dam. Find the beaver's position function \( s(t) \) in miles north of the dam. (Here we take negative numbers to be south of the dam)

The beaver has constant velocity of 3 mph south, hence we have:

\[ v(t) = s'(t) = -3 \]

and therefore:

\[ s(t) = -3t + C \]

Now, the initial condition was \( s(0) = 5 \), so we have:

\[ 5 = s(0) = -3(0) + C = C \]

\[ \therefore 5 = C \Rightarrow s(t) = -3t + 5 \]
(b) What is the beaver's displacement from \( t=0 \) to \( t=5 \) hours? That is, what is:
\[
\int_0^5 s'(t) \, dt
\]

We have \( s'(t) = -3 \), so now:
\[
\int_0^5 s'(t) \, dt = \int_0^5 (-3) \, dt = -3 \int_0^5 dt
\]

Apply \( \text{FTC I} \):
\[
= -3(5-0) = -15
\]

Or - geometry:
\[
\Rightarrow [15 \text{ miles South}]
\]

(Note that also by \( \text{FTC I} \), \( \int_0^5 s'(t) \, dt = s(5) - s(0) \), so could also use answer to (a))

(c) What is the beaver's total distance traveled in the same time interval? That is, \( \int_0^5 |s'(t)| \, dt \)?

Now we have:
\[
\int_0^5 |s'| \, dt = \int_0^5 3 \, dt = 3 \int_0^5 dt
\]

\[
= 3(5-0) = 15
\]

\[
\Rightarrow [15 \text{ miles}]
\]

(Remark: note the easy relationship between displacement and total distance when velocity is constant; not true later!!)
Suppose we have a sea lion swimming in the ocean whose depth is given by the function \( s(t) \). (Take the surface of the water to be the origin and negative values to be under water.)

(a) Suppose the sea lion has constant acceleration \(-2 \text{ meters/s}^2\) down, and initial velocity of \(15 \text{ meters/s} \) up. Find the velocity as a function of \( t \).

We know \( v'(t) = -2 \) since the acceleration is constant in the downward direction.

So now we have:

\[ v(t) = -2t + C \]

and using the initial data, that is:

\[ +15 = v(0) = -2(0) + C \]

\[ \Rightarrow C = 15 \]

\[ \Rightarrow [v(t) = -2t + 15] \]
(b) Suppose the sea lion starts 54 meters below the surface. Find its position function $s(t)$.

We know $s'(t) = v(t) = -2t + 15$

part (a)

$s(t) = -2\left(\frac{1}{2}t^2\right) + 15t + C$

$-54 = s(0) = 0 + 0 + C$

$\uparrow$

$54$ meters below surface

$\Rightarrow C = -54$

$\Rightarrow s(t) = -t^2 + 15t - 54$

(c) What was its total distance travelled from $t = 3$ sec. to $t = 9$ sec.?

(That is, what is $\int_3^9 |s'(t)|\,dt$?)

See that $s'(t) = -2t + 15$ from part (a)/(b).

Consider the graph:

So we must treat $|s'(t)|$ as a piecewise function.
That is:

\[ |s'(t)| = \begin{cases} 
  s'(t) & \text{if } s'(t) \geq 0 \\
  -s'(t) & \text{if } s'(t) < 0
\end{cases} \]

See that 

\[ 0 = s'(t) = -2t + 15 \Rightarrow t = \frac{15}{2} \]

So we have:

\[ |s'(t)| = \begin{cases} 
  s'(t) & \text{if } t \leq \frac{15}{2} \\
  -s'(t) & \text{if } t > \frac{15}{2}
\end{cases} \]

So:

\[ \int_3^9 |s'(t)| \, dt = \int_3^{15/2} |s'(t)| \, dt + \int_{15/2}^9 |s'(t)| \, dt \]

Apply FTC and geometry:

\[ = \left[ -2 \left( \frac{t^2}{2} \right) + 15t \right]_{3}^{15/2} + \left[ 2 \left( \frac{t^2}{2} \right) - 15t \right]_{15/2}^9 \]

\[ = \left[ -2 \left( \frac{\left( \frac{15}{2} \right)^2}{2} \right) + 15 \cdot \frac{15}{2} \right] - \left( -2 \cdot \frac{3^2}{2} + 15 \cdot 3 \right) \]

\[ + \left[ 9^2 - 15 \cdot 9 - \left( \frac{15}{2} \right)^2 + 15 \cdot \frac{15}{2} \right] \]

\[ = \left[ \frac{15^2}{4} + \frac{15^2}{2} + (9 - 15) \right] + \left[ 9^2 - 9 \cdot 9 - \left( \frac{15}{2} \right)^2 + 15 \cdot \frac{15}{2} \right] \]
\[ = 15^2 + 9 - 45 + 9^2 - 9(15) \]
\[ = 15^2 + 9(-6 + 9 - 15) = 15^2 + 9(-10) \]
\[ = 15^2 + 90 \text{ meters} \]

II. Riemann sums/geometry

Consider the graph of \( y = f(x) \)

\[ y = 2 \]

1. What is \( \int_0^4 f(x) \, dx \)?
2. Which are over-estimates of the value of 1? \( M_7, T_3, T_7, M_3 \)
3. What is \( \int_0^6 f(x) \, dx \)?
4. Order the estimate of \( \int_4^6 f(x) \, dx \) from greatest to least: \( R_4, L_4, M_4, T_4 \)
⑤ Which is the best estimate of $\int_0^8 f(x) \, dx$? $M_4, T_4, R_{30}, L_{20}$

⑥ What is the value of $\int_0^8 f(x) \, dx$?

→ Answer:

① $\int_0^4 f(x) \, dx = \frac{1}{2} \cdot \text{(area of radius 2 circle)}$
   \[= \frac{1}{2} \left( \pi (2)^2 \right) = 2\pi \]

② $M_7, M_3$: These are the trapezoids from the tangent line, which are on top because the graph is concave down here.

③ $\int_0^6 f(x) \, dx = \int_0^4 f(x) \, dx + \int_4^6 f(x) \, dx$
   \[= 2\pi - \frac{1}{4} \cdot \text{(area of radius 2 circle)}
   = 2\pi - \left( \pi (2)^2 \right) = 2\pi - \pi = \pi \]
See that $y = |f(x)|$ over $4 \leq x \leq 6$ has the graph:

Now see a sketch of these Riemann sums:

- **L₄**: From left rectangles over each interval
- **T₄**: From secant line trapezoids over each interval
- **M₄**: From tangent line trapezoids over each interval
- **R₄**: From right rectangles over each interval

So:

$L₄ < T₄ < M₄ < R₄$
5. $R_{30}$ → more $\pi$ is better, no matter the sum type!

6. The two regions cancel!

III. Fundamental theorem of calculus

1. Let $F(x) = \int_{\pi}^{x} \ln(\sin(t)+2) \, dt$

(a) What is $F(x)$?

(b) Is $F(x)$ increasing or decreasing at $x=2\pi$?

(a) See that, by FTC 1,

$$F'(x) = \frac{d}{dx} \left( \int_{\pi}^{x} \ln(\sin(t)+2) \, dt \right)$$

$$= \frac{d}{dx} \left( \ln(\sin(x)+2) \right)$$

(b) See that $F'(2\pi) = \ln(\sin(2\pi)+2) = \ln(2) > 0$

so it is \underline{increasing}
\[ (2) \quad G(x) = \int_{\sqrt{2}}^{\sqrt{x}} e^{t^2-1} \, dt \]
(a) \( G'(x) = ? \)
(b) Increase or decrease at \( x = 1 \) ?

(3) See first that \( G(x) \) is a composition of the following functions:
\[ f(x) = \int_{\sqrt{2}}^{x} e^{-t^2} \, dt \]
\[ h(x) = \sqrt{x} \]
and then \( G(x) = f(h(x)) \).
Now apply the chain rule:
\[ G'(x) = f'(h(x)) \cdot h'(x) \]
Now by FTC2, \( f'(x) = e^{-x^2-1} \)
and by power rule, \( h'(x) = \frac{1}{2} x^{-\frac{1}{2}} \)
Then:
\[ G'(x) = f'(\sqrt{x}) \cdot \left(\frac{1}{2} x^{-\frac{1}{2}}\right) \]
\[ = e^{-\left(\sqrt{x}\right)^2-1} \cdot \frac{1}{2\sqrt{x}} \]
\[ = \frac{e^{-\sqrt{x}-1}}{2\sqrt{x}} \]

(b) Then we have
\[ g'(1) = \frac{e^{-\sqrt{1-1}}}{2\sqrt{1}} = \frac{e^0}{2} = \frac{1}{2} \geq 0 \]
so it is increasing.

\[ \int_{\pi/2}^{\pi} (2\sin(x) + x) \, dx = ? \]

See that:
\[ \int_{\pi/2}^{\pi} (2\sin(x) + x) \, dx = 2 \int_{\pi/2}^{\pi} \sin(x) \, dx + \int_{\pi/2}^{\pi} x \, dx \]

\[ \text{FTC} \rightarrow 2 \left[ -\cos(x) \right]_{\pi/2}^{\pi} + \left[ \frac{x^2}{2} \right]_{\pi/2}^{\pi} \]

\[ = 2 \left[ -\cos(\pi) + \cos(\frac{\pi}{2}) \right] + \left[ \frac{\pi^2}{2} - \frac{\pi^2}{2} \right] \]

\[ = 2 \left[ -(-1) + 0 \right] + \left[ \frac{\pi^2}{4} \right] = \sqrt{2 + \frac{\pi^2}{4}} \]
\[ \int \ln(x) e^{3x} dx \]

\[ \int e^{3x} dx = \frac{1}{3} e^{3x} + C \quad \text{(can do w/ sub)} \quad u = 3x \]

Now apply FTC\( \text{I} \):

\[ \int \ln(x) e^{3x} dx = \left[ \frac{1}{3} e^{3x} \right]^{\ln(x)}_{1} \]

\[ = \frac{1}{3} \left[ e^{3\ln(x)} - e^{3} \right] \]

\[ = \frac{1}{3} \left[ 2^{3} - e^{3} \right] = \frac{8 - e^{3}}{3} \]
IV. Substitution:

1. \( \int \cos(\sin(x)) \cos(x) \, dx = ? \)

Let \( u = \sin(x) \)

Then \( du = \cos(x) \, dx \), and so:

\[
\int \cos(\sin(x)) \cos(x) \, dx = \int \cos(u) \, du
\]

\[
= \sin(u) + C
\]

\( \text{re-sub } u = \sin(x) \)

\[
\int \cos(\sin(x)) \cos(x) \, dx = \sin(\sin(x)) + C
\]
\[ \int \frac{2 \, dx}{x \ln(x)} = ? \]

\[ \begin{align*}
\text{Let } u &= \ln(x) \\
\text{Then } du &= \frac{1}{x} \, dx \quad \Rightarrow \quad \text{apply sub:} \\
\int \frac{2 \, dx}{x \ln(x)} &= 2 \int \frac{1}{u} \, du \\
&= 2 \ln|u| + C \\
&= 2 \ln|\ln(x)| + C
\end{align*} \]
\( \int_0^1 x^2 e^{5x^3} \, dx = ? \)

We'll find \( \int x^2 e^{5x^3} \, dx \) and apply FTC.0.

Let \( u = 5x^3 \)

\[
\begin{align*}
\frac{du}{dx} &= 15x^2 \\
\frac{du}{15} &= x^2 \, dx
\end{align*}
\]

Then:

\[
\int x^2 e^{5x^3} \, dx = \int e^u \frac{du}{15}
\]

\[
= \frac{1}{15} e^u + C
\]

\[
= \frac{1}{15} e^{5x^3} + C
\]

Then:

\[
\int_0^1 x^2 e^{5x^3} \, dx = \left[ \frac{1}{15} e^{5x^3} \right]_0^1
\]

\[
= \frac{1}{15} [e^5 - e^0] = \frac{1}{15} [e^5 - 1]
\]

\[
= \frac{e^5 - 1}{15}
\]
$\int_1^2 x^3(x^4+1)^{3/2} \, dx$

Let $u = x^4 + 1$

$du = 4x^3 \, dx \quad \rightarrow \quad \frac{du}{4} = x^3 \, dx$

Then:

$\int x^3(x^4+1)^{3/2} \, dx$

$= \int u^{3/2} \frac{du}{4}$

$= \frac{1}{4} \int u^{3/2} \, du$

$= \frac{1}{4} \left( \frac{u^{5/2}}{5/2} \right) + C$

$= \frac{2}{4 \cdot 5} \cdot u^{5/2} + C$

resub $u = \int \frac{1}{10} (x^4 + 1)^{5/2} + C$

Then:

$\int_1^2 x^3(x^4+1)^{3/2} \, dx = \left[ \frac{1}{10} (x^4 + 1)^{5/2} \right]_1^2$

$= \frac{1}{10} \left[ 17^{5/2} - 2^{5/2} \right]$
**Integration by parts**

1. \( \int x^3 \ln(x) \, dx \)

Let \( u = \ln(x) \)
\[ du = \frac{1}{x} \, dx \]
\[ dv = x^3 \, dx \]
\[ v = \int x^3 \, dx = \frac{x^4}{4} \]

Now apply Int. by parts:

\[ \int x^3 \ln(x) \, dx = uv - \int v \, du \]

\[ = \frac{\ln(x) \cdot x^4}{4} - \int \left( \frac{x^4}{4} \right) \left( \frac{1}{x} \, dx \right) \]

\[ = \frac{\ln(x) \cdot x^4}{4} - \frac{1}{4} \int x^3 \, dx \]

\[ = \frac{\ln(x) \cdot x^4}{4} - \frac{1}{4} \left( \frac{x^4}{4} \right) + C \]

\[ = \left( \frac{\ln(x) \cdot x^4}{4} - \frac{1}{4} \left( \frac{x^4}{4} \right) \right) + C \]
\[ \int \arcsin(x) \, dx = ? \]

Let \( u = \arcsin(x) \)
\[ dv = dx \]
Then \( du = \frac{dx}{\sqrt{1-x^2}} \) and \( v = x \)

Apply part 1:
\[ \int \arcsin(x) \, dx = uv - \int v \, du \]
\[ = x \arcsin(x) - \int \frac{x \, dx}{\sqrt{1-x^2}} \]

Substitute:
\[ w = 1-x^2 \]
\[ dw = -2x \, dx \]
\[ \Rightarrow -\frac{dw}{2} = x \, dx \]

\[ \int \frac{x \, dx}{\sqrt{1-x^2}} = \frac{1}{2} \int \frac{dw}{\sqrt{w}} \]
\[ = x \arcsin(x) + \frac{1}{2} \left( 2 \sqrt{w} \right) + C \]
\[ = x \arcsin(x) + \sqrt{1-x^2} + C \]
\(3) \int \sqrt{x} e^{\sqrt{x}} \, dx\)

First we substitute: \(u = \sqrt{x}\).

Then \(du = \frac{dx}{2\sqrt{x}}\).

Now:

\[
\int \sqrt{x} e^{\sqrt{x}} \, dx = \int \frac{x}{\sqrt{x}} e^{\sqrt{x}} \, dx
\]

\[
= \int x e^{\sqrt{x}} \frac{dx}{\sqrt{x}}
\]

\[
= \int u^2 e^u (2du)
\]

\[
= 2 \int u^2 e^u \, du.
\]

Now we must use parts to integrate the resulting integral.

Let \(p = u^2\) and \(dq = e^u \, du\).

So \(dp = 2u \, du\) and \(q = e^u\).

Then:

\[
2 \int u^2 e^u \, du = 2 \int p \, dq = 2(pq - \int q \, dp)
\]

\[
= 2(u^2 e^u - \int 2e^u \, du)
\]
Now we must use parts a second time:

\[ 2 \int u^2 e^u \, du = 2 \left( u^2 e^u - 2 \int e^u \, du \right) \]

Let \[ s = u \quad \Rightarrow \quad ds = du \]
\[ at = e^u \quad \Rightarrow \quad t = e^u \]

Then \[ \int e^u \, du = \int s \, dt = st - \int t \, ds = ue^u - Se^u \, du = ue^u - e^u \]

So we finally have:

\[ \int xe^{-x} \, dx = 2 \int u^2 e^u \, du = 2 \left( u^2 e^u - 2 \int e^u \, du \right) + C \]
\[ = 2 \left( u^2 e^u - 2(ue^u - e^u) \right) + C \]

\[ \text{re-sub } (u = -x) \]
\[ = 2\left( (x)^2 e^{-x} - 4(\frac{x}{e^x} - e^{-x}) \right) + C \]
\[ = 2xe^{-x} - 4\frac{x}{e^x} + 4e^{-x} + C \]
\[ \int 5x^2 \sin(x) \, dx =? \]

Let \( u = 5x^2 \)
\( dv = \sin(x) \, dx \).

Then \( du = 10x \, dx \)
\( v = -\cos(x) \, dx \).

Now apply IBP:
\[
\int 5x^2 \sin(x) \, dx = uv - \int v \, du \\
= (5x^2)(-\cos(x)) - \int (-\cos(x)) (10x \, dx) \\
= -5x^2 \cos(x) + \int 10x \cos(x) \, dx \\
\]

Apply IBP again on this integral:

Let \( p = 10x \Rightarrow dp = 10 \, dx \)
\( dq = \cos(x) \, dx \Rightarrow q = \sin(x) \)

\[
= -5x^2 \cos(x) + \int p \, dq \\
= -5x^2 \cos(x) + (pq - \int q \, dp) \\
= -5x^2 \cos(x) + 10x \sin(x) - \int \sin(x) (10 \, dx) \\
\]
\[\begin{align*}
&= -5x^2 \cos(x) + 10x \sin(x) \\
&\quad - 10 \int \sin(x) \, dx \\
&= \left[ -5x^2 \cos(x) + 10x \sin(x) + 10 \cos(x) \right] + c
\end{align*}\]