1. Motivation:

10A Q: Given a position function \( s(t) \),
what is:

1. displacement over time interval \( a \leq t \leq b \) ?
2. average velocity over time interval \( a \leq t \leq b \) ?
3. (instantaneous) velocity at time \( t \) ?

A: 1. \( s(b) - s(a) \), the change in \( s(t) \)
2. \( \frac{s(b) - s(a)}{b - a} = \text{slope of the secant line} \)
3. \( s'(t) = \frac{ds}{dt} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{t+h-t} = \text{slope of tangent line} \)
Given a velocity function \( v(t) \), what is:

1. Displacement?
2. Average velocity?
3. Position at time \( t \)?

Consider a simple example:

Constant velocity \( v(t) = 5 \text{ m/s} \).

1. Displacement from \( t = 1 \text{ sec} \) to \( t = 6 \text{ sec} \)?

Since the velocity is constant we need only find how many seconds elapsed: \( 6 - 1 = 5 \).

So now:

\[
\text{displacement} = (5 \text{ m/s}) \times (5 \text{ sec}) = 25 \text{ m}
\]
See the graph of \( v(t) = 5 \):

![Graph showing a constant velocity of 5 units over time from 1 to 6.]

The displacement is exactly the area of the shaded rectangle!

This area is denoted by:

\[
\int_{1}^{6} v(t) \, dt
\]

and it is called the (definite) integral of \( v(t) \) over \( 1 \leq t \leq 6 \).

(2) Average velocity from \( t=1 \) to \( t=6 \) ?

Remember average velocity = \( \frac{\text{displacement}}{\text{change in time}} \),

hence avg velocity = \( \frac{25 \text{ m}}{(6-1) \text{ s}} = \sqrt{5} \text{ m/s} \),
(3) Position function for $v(t) = 5$ ?

We know from 10A that the position function $s(t)$ must satisfy:

$$s'(t) = v(t)$$

That is, $s(t)$ is an antiderivative of $v(t)$.

One possibility is $s_1(t) = 5t$;

see that $s_1'(t) = \frac{d}{dt}(5t) = 5 = v(t)$

But there are others! See that:

$$s_2(t) = 5t + 3$$

also has $s_2'(t) = \frac{d}{dt}(5t + 3) = 5 = v(t)$

Which antiderivative is correct?

They differ by their starting points!

$s_1(0) = 0$ \(\triangleright\) different positions at time $t = 0$

$s_2(0) = 3$

To answer Q3, we must have both velocity function + starting point.
Two objects with different positions may have the same velocities!

Example:

(a) Suppose Car 1 leaves San Diego for Los Angeles going 60 mph, and Car 2 leaves LA for San Francisco going 60 mph. Write functions for their positions in miles from San Diego.

See that both cars have the same velocity function:

\[ v_1(t) = 60 \]
\[ v_2(t) = 60 \]

Hence their positions must be of the form:

\[ s_1(t) = 60t + C_1 \]
\[ s_2(t) = 60t + C_2 \]

(i.e. they are both antiderivatives of \( v(t) \) since \( \frac{ds_1}{dt} = 60 = \frac{ds_2}{dt} \))

But, Car 1 starts in SD, hence its position satisfies:

\[ s_1(0) = 0 \]
On the other hand, Car 2 starts in LA, which is about 90 miles from SD.

Hence its position must satisfy:

\[ s_2(0) = 90 \]

Using the starting point (initial data), we can find the constants \( C_1 \) and \( C_2 \):

\[ 0 = s_1(0) = 60(0) + C_1 = C_1 \]

\[ \Rightarrow C_1 = 0 \]

\[ 90 = s_2(0) = 60(0) + C_2 = C_2 \]

\[ \Rightarrow C_2 = 90 \]

So the position functions are

\[
\begin{align*}
    s_1(t) &= 60t \\
    s_2(t) &= 60t + 90
\end{align*}
\]
(b) Suppose Car 3 leaves SD for Tijuana going 60 mph. What is its velocity function?

Note that Car 3 is travelling South, hence its direction is negative and the constant velocity function is:

\[ v(t) = -60 \]

4 Exercise: Find the position function for Car 3.

The preceding examples show us how to answer the questions posed in (10b) for a general velocity function \( v(t) \):

1. **Displacement** = area under the graph \( y = v(t) \) over interval \( a \leq t \leq b \)

\[ : = \int_a^b v(t) \, dt \] (definite integral)

of \( v(t) \) over \( a \leq t \leq b \)

2. **Position** = antiderivative of \( v(t) \), i.e.

\[ \frac{ds}{dt} = v(t) \], with specified starting point

\[ s(0) = k \] (the "initial conditions")
First major theorem of 10B (later) \[ \frac{1}{8} \]

**Fundamental Theorem of Calculus**

If \( s(t) \) has \( s'(t) = v(t) \), then

\[
\int_{a}^{b} v(t) \, dt = s(b) - s(a)
\]

The theorem states that those two definitions are equal, which is, in fact, a highly non-trivial observation!

Thank you, Newton & Leibniz!
2) Antiderivatives

Given a velocity function, we want to find the position function.

Examples:

1. A car has constant velocity of 60 mph heading north of San Diego.
   (a) If the car is 15 miles north of SD after 1/2 hour, what is the position function $s(t)$ in miles north of SD?
   (b) What time will the car arrive in Orange County (60 miles away) if it left SD at 12 noon?

Since the car has constant velocity, we can write its velocity function:

$$v(t) = 60$$

Now, we've seen that the antiderivative is the form $s(t) = 60t + C$
Since the car is 15 miles north at $t = \frac{1}{2}$ hours we have that:

$$15 = s\left(\frac{1}{2}\right) = 60\left(\frac{1}{2}\right) + C$$

$$\Rightarrow 15 = 30 + C \Rightarrow C = -15$$

So the position function is

$$\text{(a) } s(t) = 60t - 15$$

Check: $s'(t) = 60$ $\equiv v(t)$

Now, since Orange Co. is 60 miles north, we want to know the time $t$ so that $s(t) = 60$, that is:

$$60t - 15 = 60$$

and solve for $t$:

$$\Rightarrow 60t = 75 \Rightarrow t = \frac{75}{60} = \frac{5}{4}$$

Hence it takes 1 hour, 15 minutes, and so it arrives at $1:15$ PM.
(2) An object is dropped from 20 meters. What is its position function? When does it hit the ground?

The object this time has constant acceleration (due to gravity), hence we know that

\[ \frac{d^2s}{dt^2} = -9.81 \text{ m/s}^2 \]

(here we take down to be negative)

This time we must find \( s(t) \) from the second derivative, hence we need two points of initial data.

We are told the object is dropped, hence initial velocity is 0:

\[ 0 = v(0) = \frac{ds}{dt}(0) \]
We are also told that the initial position is 20 meters:

\[ 20 = s(0) \]

So now we combine the three pieces of info to find \( s(t) \):

\[ \frac{d^2 s}{dt^2} = -9.81 \]

\[ \Rightarrow \frac{ds}{dt} = (-9.81)t + C \]

\[ \Rightarrow 0 = \frac{ds}{dt}(0) = -9.81(0) + C \]

\[ \Rightarrow C = 0 \]

\[ \Rightarrow s = \frac{(-9.81)t^2}{2} + Ct + D \]

\[ \Rightarrow 20 = s(0) = \frac{-9.81(0)^2}{2} + C(0) + D \]

\[ \Rightarrow D = 20 \]

So we have found:

\[ s(t) = \frac{(-9.81)t^2}{2} + 20 \]
To determine when the object hits the ground, we need to know

time \( t \) such that \( s(t) = 0 \).

So:

\[
0 = s(t) = \frac{(-9.81)t^2}{2} + 20
\]

\[
\Rightarrow \quad (-20)(2) = (-9.81)t^2
\]

\[
\Rightarrow \quad \frac{40}{9.81} = t^2
\]

\[
\Rightarrow \quad t = \sqrt{\frac{40}{9.81}} \text{ seconds}
\]
These examples illustrate our first rule for antiderivatives:

\[
\frac{ds}{dx} = x^n \quad \Rightarrow \quad s = \frac{x^{n+1}}{n+1} + C
\]

Compare this with the power rule for differentiation:

\[
s = x^n \quad \Rightarrow \quad \frac{ds}{dx} = nx^{n-1}
\]

We see then that:

\[
\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{(n+1)x^n}{(n+1)} + 0
\]

\[= x^n\]
Just as in 10A we had derivative rules, here we have a list of antiderivative rules:

<table>
<thead>
<tr>
<th>$\frac{ds}{dx}$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$s = \frac{x^{n+1}}{n+1} + C$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$s = \ln</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$s = e^x + C$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$s = \sin(x) + C$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$s = -\cos(x) + C$</td>
</tr>
<tr>
<td>$\sec^2(x)$</td>
<td>$s = \tan(x) + C$</td>
</tr>
<tr>
<td>$\sec(x)\tan(x)$</td>
<td>$s = \sec(x) + C$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$s = \arcsin(x) + C$</td>
</tr>
<tr>
<td>$\frac{1}{1+x^2}$</td>
<td>$s = \arctan(x) + C$</td>
</tr>
</tbody>
</table>
Antiderivatives are also linear, just like derivatives:

\[ \frac{d}{dx}(ku) = k \frac{du}{dx} \]
\[ \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \]

**Antiderivative**

If \( \frac{du}{dx} = f(x) \) and \( \frac{dv}{dx} = g(x) \) then:

- \( ku \) is an antiderivative of \( kf(x) \)
- \( u+v \) is an antiderivative of \( f+g)(x) \)

We can see this because:

\[ \frac{d}{dx}(ku) = k \frac{du}{dx} = kf(x) \]

\[ \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} = (f+g)(x) \]
More examples

1) \( \frac{ds}{dt} = 3t^4 + 2\cos(t) \)

By linearity we can find an antiderivative for each summand:

\[
\int s = \frac{3t^5}{5} + 2\sin(t) + C
\]

Check: \( \frac{ds}{dt} = \frac{d}{dt} \left( \frac{3t^5}{5} + 2\sin(t) + C \right) \)

\[
= \frac{3}{5} \frac{d}{dt}(t^5) + 2\frac{d}{dt}(\sin(t)) + \frac{d}{dt}(C)
\]

\[
= \frac{3}{5}(5t^4) + 2\cos(t) + 0
\]

\( = 3t^4 + 2\cos(t) \) \( \checkmark \)
(2) \( \frac{dy}{dx} = e^x + \frac{1}{x^2} - \frac{x^3}{3} \)

\[
\frac{dy}{dx} = e^x + x^{-2} - \frac{1}{3}x^3
\]

so: \( y = e^x - x^{-1} - \frac{1}{3} \left( \frac{x^4}{4} \right) + C \)

\[
\Rightarrow \quad y = e^x - \frac{1}{x} - \frac{x^4}{12} + C
\]

check: \( \frac{dy}{dx} = \frac{d}{dx} \left( e^x - \frac{1}{x} - \frac{x^4}{12} + C \right) \)

\[
= e^x - \frac{-1}{x^2} - \frac{4x^3}{12} + 0
\]

\[
= e^x + \frac{1}{x^2} - \frac{x^3}{3} \quad \checkmark
\]
③ \( \frac{du}{dx} = 2x + 3x^2 + 4x^3 \)

such that \( u(1) = 5 \)

\[
u = x^2 + x^3 + x^4 + C
\]

using the anti-power rule.

Now, \( u(1) = 5 \), so:

\[
5 = u(1) = 1^2 + 1^3 + 1^4 + C
\]

\[
\Rightarrow 5 = C + 3 \Rightarrow C = 2
\]

hence \( u(x) = x^2 + x^3 + x^4 + 2 \)

④ \( \frac{dg}{dt} = e^{2t} \)

We don't have a rule for this one!

But we know by chain rule that \( \frac{d}{dt}(e^{2t}) = 2e^{2t} \), hence \( g(t) = \frac{e^{2t}}{2} + C \).

*We'll show how to do this with a lot of other functions later!*
\[
\frac{d^2u}{dx^2} = e^x + x^3 \\
u(0) = 2 \\
u'(0) = 3
\]

[Notice: we need two initial data points to find \(u(x)\) given its second derivative.]

\[
\frac{d}{dx} \left( \frac{du}{dx} \right) = \frac{d^2u}{dx^2} = e^x + x^3
\]

\[
\Rightarrow \frac{du}{dx} = e^x + \frac{x^4}{4} + C
\]

\[
\Rightarrow u = e^x + \frac{x^5}{20} + Cx + D
\]

Now:
\[
2 = u(0) = e^0 + \frac{0^5}{20} + C(0) + D \\
\Rightarrow 2 = 1 + D \Rightarrow D = 1
\]

\[
3 = u'(0) = e^0 + \frac{0^4}{4} + C \\
\Rightarrow 3 = 1 + C \Rightarrow C = 2
\]

So the antiderivative is \(u = e^x + \frac{x^5}{20} + 2x + 1\).
(6) Suppose a ball is tossed straight up from a height of 192 ft. at velocity 64 ft/s. (let g = 32 ft/s²)

(a) When does the ball hit the ground, i.e., when is it at height 0?

(b) If an insect flies directly upward with constant velocity 128 ft/s, when do the ball and insect pass each other?

(a) First we'll write the velocity function:

\[ v(t) = 64 - 32t \]

and then its antiderivative as the form:

\[ s(t) = 64t - \frac{32t^2}{2} + C \]

Now we know the ball starts at 192 ft, hence:

\[ 192 = s(0) = 64(0) - \frac{32(0)^2}{2} + C \]

\[ \Rightarrow C = 192 \]

So the position function is:

\[ s(t) = 64t - 16t^2 + 192 \]
Now we want to know for which \( t > 0 \) we have \( s(t) = 0 \):

\[
0 = s(t) = -16t^2 + 64t + 192 \\
= -16(t^2 - 4t + 12) \\
= -16(t - 6)(t + 2)
\]

\[ \Rightarrow \boxed{t = 6 \text{ seconds}} \]

(b) The insect has velocity function

\[ v_{\text{bug}}(t) = 128 \]

and its antiderivative is

\[ s_{\text{bug}}(t) = 128t + C \]

The insect starts on the ground, so \( s_{\text{bug}}(0) = 0 \), which means \( C = 0 \) (check!)

So we have \( s_{\text{bug}}(t) = 128t \)
The question asks: for what $t$ do we have $s(t) = s_{bug}(t)$?

We can solve:

$$-16t^2 + 64t + 192 = 128t$$

$$\Rightarrow -16t^2 + 64t + 192 = 0$$

$$\Rightarrow -16(t^2 + 4 - 12) = 0$$

$$\Rightarrow -16(t - 2)(t + 6) = 0$$

$$\Rightarrow \boxed{t = 2 \text{ seconds}}$$
Two balls are thrown from a height of 1000 meters, one up at 5 m/s and the other down at 2 m/s.

(a) What are the balls' velocities when they hit the ground?

(b) Do the balls pass each other?

The initial velocity of ball #1 is +5 m/s and the initial velocity of ball #2 is -2 m/s. Both are accelerating down due to gravity, i.e., at -9.81 m/s². So we can write their velocity functions:

\[ v_1(t) = 5 - (9.81)t \]
\[ v_2(t) = -2 - (9.81)t \]
Now we’ll find the positions:

\[ s_1(t) = 5t - \frac{(9.81)t^2}{2} + C_1 \]

\[ s_2(t) = -2t - \frac{(9.81)t^2}{2} + C_2 \]

Both balls start at 1000 m, so:

\[ 1000 = s_1(0) = 5(0) - \frac{(9.81)(0)^2}{2} + C_1 \]

\[ \Rightarrow C_1 = 1000 \]

\[ 1000 = s_2(0) = -2(0) - \frac{(9.81)(0)^2}{2} + C_2 \]

\[ \Rightarrow C_2 = 1000 \]

So we have:

\[ s_1(t) = 5t - \frac{(9.81)t^2}{2} + 1000 \]

\[ s_2(t) = -2t - \frac{(9.81)t^2}{2} + 1000 \]
(a) Using these we find the times when each hits the ground; i.e. \( t_1 \) 

So that \( s_1(t_1) = 0 \) and \( t_2 \) s.t. \( s_2(t_2) = 0 \):

\[
\begin{align*}
0 &= s_1(t_1) = 5t_1 - \frac{(9.81)t_1^2}{2} + 1000 \\
t_1 &= -5 \pm \sqrt{25 - 4\left(-\frac{9.81}{2}\right)(1000)} \\
&= \frac{-9.81 \pm \sqrt{25 + (9.81)(2000)}}{9.81} \\
&= 5 \pm \sqrt{25 + (9.81)(2000)} \\
&\approx 5 + \frac{\sqrt{25 + (9.81)(2000)}}{9.81} \\
&\approx 5,1
\end{align*}
\]
\[ t_2 \quad 0 = s_2(t_2) = -2t_2 - \left(\frac{9.81}{2}\right) t_2^2 + 1000 \]

\[ \Rightarrow \quad t_2 = \frac{2 \pm \sqrt{4 - 4\left(\frac{-9.81}{2}\right)(1000)}}{2 \left(\frac{-9.81}{2}\right)} \]

\[ = \frac{2 \pm \sqrt{4 + (9.81)(2000)}}{-9.81} = t_2 \]

Now, to find the velocities we can plug these times into the velocity functions:

\[ v_1(t_1) = 5 - (9.81)t_1 \]

\[ = 5 - (9.81)\left(\frac{5 + \sqrt{25 + (9.81)(2000)}}{9.81}\right) \]

\[ = 5 - 5 - \sqrt{25 + (9.81)(2000)} \]

\[ = -\sqrt{25 + (9.81)(2000)} \text{ m/s} \]
\[ V_2(t_2) = -2 - (9.81)t_2 \]
\[ = -2 - (9.81) \left( -2 + \frac{\sqrt{4 + (9.81)(2000)}}{9.81} \right) \]
\[ = -2 + 2 - \sqrt{4 + (9.81)(2000)} \]
\[ = \left[ -\sqrt{4 + (9.81)(2000)} \right] \text{ m/s} \]

(b) The question is asking if there is a value of \( t \) such that
\[ s_1(t) = s_2(t) \]
We will try to solve for \( t > 0 \):

\[ 5t - \frac{(9.81)t^2}{2} + 1000 = -2t - \frac{(9.81)t^2}{2} + 1000 \]
\[ \Rightarrow 5t = -2t \Rightarrow 7t = 0 \]
\[ \Rightarrow t = 0, \]

Hence there are no times \( t > 0 \) with this equality, so the balls do not pass each other.
3. Area and integration:

Given a velocity function \( v(t) \), we want to find the displacement (change in position) over a time interval \( a \leq t \leq b \).

Recall that this is equal to the area under the graph \( y = v(t) \) over the interval \( a \leq t \leq b \), and we call this quantity the:

\[
\text{integral} = \int_a^b v(t) \, dt
\]
Consider some old examples:

1) A car has constant velocity of 60 mph heading North of SD.
   How far does it travel in 3 hours?

The question is asking for the quantity $\int_{0}^{3} v(t)\,dt$ where $v(t) = 60$
See the graph:

\[
\text{Area = } \int_{0}^{3} v(t)\,dt = (3-0)(60) = \boxed{180 \text{ miles}}
\]
2) An object is dropped from 20 meters. How far does it travel in 2 seconds?

The question wants to know
\[ \int_0^2 v(t) \, dt. \]

Recall we found that \( v(t) = -9.81/t \); consider the graph:

\[ y = v(t) \]

\[ t=0 \quad t=2 \]

area of triangle = \[ \int_0^2 v(t) \, dt \]

= \[ - \frac{1}{2} (2-0)(2 \cdot 9.81) \]

negative because base height area is below the t-axis

negative because the object travels down!

= \boxed{-19.62 \text{ meters}}
Properties of the integral

1. Linear:
   \[ \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

   ![Diagram](linear.png)

   \[ \int_a^b k \cdot f(x) \, dx = k \int_a^b f(x) \, dx \]

   ![Diagram](linear.png)

2. Interval Split:
   \[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

   ![Diagram](interval_split.png)

   This area = sum of these areas
③ Interval direction:

\[ \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx \]

\[ \text{direction of integration} \quad \Rightarrow \quad \text{direction of integration} \]

negative!

④ Index variable doesn't matter

\[ \int_a^b f(x) \, dx = \int_a^b f(u) \, du \]

- these are both numbers since they give areas
- the variable of integration doesn't matter if the functions are the same (i.e. same f)
(3) Suppose $\int_2^3 f(x) = 3$ and $\int_2^3 g(x) = 10$.

What is $\int_2^3 (2f(x) + 3g(x)) \, dx$?

\[ \int_2^3 (2f(x) + 3g(x)) \, dx = 2 \int_2^3 f(x) \, dx + 3 \int_2^3 g(x) \, dx \]

[\text{Linearity}]

\[ = 2(3) + 3(10) = 36 \]

(4) Suppose $f(x) = \begin{cases} 1 & x < 1 \\ 2 & x \geq 1 \end{cases}$

What is $\int_0^3 f(x) \, dx$?

Consider the graph:

\[ \text{area} = \int_0^3 f(x) \, dx \]
Clearly the area is the sum of the areas of the two rectangles. This is just interval splitting!

\[ \int_0^3 f(x) \, dx = \int_0^1 f(x) \, dx + \int_1^3 f(x) \, dx \]

\[ \Rightarrow \int_0^3 f(x) \, dx = \int_0^1 (1) \, dx + \int_1^3 (2) \, dx \]

\[ = (1-0)(1) + (3-1)(2) \]

\[ = 1 + 4 = 5 \]
Suppose Area of A is 1 and Area of B and Area of C are both 2.

(a) What is \( \int_{3}^{0} f(x) \, dx \) ?

(b) What is \( \int_{1}^{5} (2f(x) + 1) \, dx \) ?

(a) See that \( \int_{3}^{0} f(x) \, dx = -\int_{0}^{3} f(x) \, dx \)

\[
= - \left( \int_{0}^{1} f(x) \, dx + \int_{1}^{3} f(x) \, dx \right)
\]

\[
= - \left( \text{Area}(A) + \frac{1}{2} \text{Area}(B) \right)
\]

\[
= -(1 - 2) = -(-1) = 1
\]
(b) \[ \int_{1}^{5} (2f(x) + 1) \, dx \]

\[ = \int_{1}^{5} 2f(x) \, dx + \int_{1}^{5} 1 \, dx \]

\[ = 2 \int_{1}^{5} f(x) \, dx + \int_{1}^{5} 1 \, dx \]

\[ = 2 \left( \int_{1}^{3} f(x) \, dx + \int_{3}^{5} f(x) \, dx \right) + \int_{1}^{5} 1 \, dx \]

\[ = 2 \left( - \text{Area (B)} + \text{Area (C)} \right) + (5-1)(1) \]

\[ = 2 \left( -2 + 2 \right) + (4)(1) \]

\[ = 4 \]
(6) Find \( \int^3_1 x^2 \, dx \)

We don't have any shapes from elementary geometry to help us here.

We need a definition for integration that will let us compute difficult areas like this!
7. A bumble bee flying up and down has the following velocity function: \( v(t) = 6 - 2t \) m/s.

(a) What is the bee's displacement over time interval \( 1 \leq t \leq 5 \)?

(b) What is the bee's total distance travelled over time interval \( 1 \leq t \leq 5 \)?

(a) Asks us to find \( \int_{1}^{5} v(t) \, dt \).

Look at the graph:

\[
\int_{1}^{5} v(t) \, dt = \int_{1}^{3} v(t) \, dt + \int_{3}^{5} v(t) \, dt
\]

\( = \frac{1}{2} (3-1)(4) + \frac{1}{2} (5-3)(4) \)
\[ = \frac{1}{2}(2)(4) - \frac{1}{2}(2)(4) = 0 \]

(b) Asks us to find the total distance travelled, that is we should count movement up and down both as positive; in other words, we want:

\[
\int_1^5 |v(t)| \, dt
\]

Consider the graph \( y = |v(t)| = 16 - 2t \)

See that here:

\[
\int_1^5 |v(t)| \, dt = \int_1^3 |v(t)| \, dt + \int_3^5 |v(t)|
\]

\[
= \int_1^3 v(t) \, dt + \int_3^5 -v(t) \, dt
\]
\[ = \int_{1}^{3} (6-2t) \, dt - \int_{3}^{5} (6-2t) \, dt \]

\[ = \frac{1}{2} (3-1)(4) - \left( -\frac{1}{2} (5-3)(4) \right) \]

\[ \text{found above in part (a)} \]

\[ = 4 + 4 = 8 \]

\[ \]

Notice: All integrals of absolute values must be split up at the sign changes.

Recall that for \( f(x) \) we have: the piecewise definition of \( |f(x)| \):

\[ |f(x)| = \begin{cases} 
  f(x) & \text{for } x \text{ with } f(x) \geq 0 \\
  -f(x) & \text{for } x \text{ with } f(x) < 0
\end{cases} \]