1. Trigonometric integrals

- These are just integrals of products of trig functions, e.g.: 

  $ \int \sin(x) \cos(x) \, dx \quad \int \tan(x) \sec^2(x) \, dx 

  \int \sin^3(x) \cos^5(x) \, dx \quad \int \tan^2(x) \cos^5(x) \, dx 

- After this section you should be able compute all of the above examples. (Do them as exercises!)

Strategy:

- Identify trig functions
- Use substitution
Old examples:

1. $\int \sin(x) \cos(x) \, dx$

   Apply sub: $u = \sin(x)$
   \[ du = \cos(x) \, dx \]

   Then:
   \[ \int \sin(x) \cos(x) \, dx = \int u \, du \]
   \[ = \frac{u^2}{2} + C \]
   \[ = \frac{\sin^2(x)}{2} + C \]

2. $\int (\sin^2(x) + \sin(x)) \cos(x) \, dx$

   \[ = \int \sin^2(x) \cos(x) \, dx + \int \sin(x) \cos(x) \, dx \]
   \[ = \int u^2 \, du + \int u \, du \]

   Apply sub:
   \[ u = \sin(x) \]
   \[ du = \cos(x) \, dx \]
   \[ = \frac{u^3}{3} + \frac{u^2}{2} + C \]
   \[ = \frac{\sin^3(x)}{3} + \frac{\sin^2(x)}{2} + C \]
1. General idea:
\[ \int f'(\sin(x)) \cos(x) \, dx \]
\[ \overset{u = \sin(x)}{\mapsto} = \int f'(u) \, du = f(u) + C \]
\[ = f(\sin(x)) + C \]

2. What if we have powers of sine and cosine?
   A: We need to use the identity to convert "excess derivatives" into the chosen sub \( u \).

Recall:
\[ \sin^2(x) + \cos^2(x) = 1 \]
(circle identity)
\[ \int \sin^2(x) \cos^3(x) \, dx \]

To use the substitution \( u = \sin(x) \)

\[ (du = \cos(x) \, dx) \]

we need to "save" one factor of \( \cos(x) \) for \( du \), and "convert" the remaining cosine factor to sine using identity \( \sqrt{\sin^2(x) + \cos^2(x)} = 1 \).

See: \( \int \sin^2(x) \cos^3(x) \, dx \)

\[ = \int \sin^2(x) \cdot \cos^2(x) \cdot \cos(x) \, dx \]

\[ = \int \sin^2(x) (1-\sin^2(x)) \cos(x) \, dx \]

\[ = \int u^2 (1-u^2) \, du \]

\[ = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C \]

\[ = \left( \frac{1}{3} \sin^3(x) - \frac{1}{5} \sin^5(x) \right) + C \]
\[ 3 \int \sin^5(x) \cos^7(x) \, dx \]

\[ = \int \sin^5(x) \cos^6(x) \cos(x) \, dx \]

\[ = \int \sin^5(x) (\cos^2(x))^3 \cos(x) \, dx \]

\[ = \int \sin^5(x) (1 - \sin^2(x))^3 \cos(x) \, dx \]

\[ = \int u^5 (1-u^2)^3 \, du \quad \text{sub: } u = \sin(x) \quad du = \cos(x) \, dx \]

\[ = \int u^5 (1-2u^2+u^4)(1-u^2) \, du \]

\[ = \int u^5 (1-2u^2+u^4-u^2+2u^4-u^6) \, du \]

\[ = \int u^5 (1-3u^2+3u^4-u^6) \, du \]

\[ = \int (u^5-3u^7+3u^9-u^{11}) \, du \]

\[ = \frac{u^6}{6} - 3\left( \frac{u^8}{8} \right) + 3\left( \frac{u^{10}}{10} \right) - \frac{u^{12}}{12} + C \quad \text{re-sub} \]

\[ = \frac{\sin^6(x)}{6} - \frac{3}{8} \sin^8(x) + \frac{3}{10} \sin^{10}(x) - \frac{1}{16} \sin^{12}(x) + C \]
\( \int \sin^3(x) \cos^2(x) \, dx \)

\[ \rightarrow \text{This time cosine is the even power, so we need to use } u = \cos(x), \text{ because we can save an single } \sin(x) \text{ from the odd power of sine!} \]

Let \( u = \cos(x) \)

\[ \Rightarrow \begin{cases} 
    du = -\sin(x) \, dx \\
    \int \sin^3(x) \cos^2(x) \, dx 
\end{cases} \]

\[ \int \sin^3(x) \cos^2(x) \, dx \]

\[ = \int \cos^2(x) (\sin^2(x)) \cdot \sin(x) \, dx \]

\[ = \int \cos^2(x) (1-\cos^2(x)) \sin(x) \, dx \]

\[ = \int u^2 (1-u^2) (-du) \]

\[ = -\left( \frac{u^3}{3} - \frac{u^5}{5} \right) + C \]

\[ = \left( -\cos^3(x) + \frac{\cos^5(x)}{5} \right) + C \]
\[ \int \sin^4(x) \cos^2(x) \, dx \]

\[ \to \text{Problem: both are even powers, whichever one we save for } \text{dx will leave an odd power behind} \]

\[ \to \text{our identity only relates even powers! } \quad (\sin^2(x) + \cos^2(x) = 1) \]

\[ \to \text{need the so-called "half-angle" identity to do products of even powers} \]

\[ \text{Try it yourself, but these identities are not required for tests!} \]
We can employ the same strategy as above when tangent and secant are also involved.

Recall that:

\[ \tan^2(x) + 1 = \sec^2(x) \]

(just divide both sides of old identity by \( \cos^2(x) \))

Examples cont'd.

6. \( \int \tan(x) \sec^2(x) \, dx \)

Simple sub:

\[ u = \tan(x) \]
\[ du = \sec^2(x) \, dx \]

\[ \Rightarrow \int \tan(x) \sec^2(x) \, dx = \int u \, du \]
\[ = \frac{u^2}{2} + C = \left[ \frac{\tan^2(x)}{2} \right] + C \]
7. \[ \int \tan^2(x) \sec^2(x) \, dx \]

Simple again:

\[ u = \tan(x) \]
\[ du = \sec^2(x) \, dx \]

\[ \Rightarrow \int \tan^2(x) \sec^2(x) \, dx = \int u^2 \, du \]

\[ = \frac{u^3}{3} + C \]

\[ = \left[ \frac{\tan^3(x)}{3} \right] + C \]

8. \[ \int \tan^8(x) \sec^4(x) \, dx \]

\[ = \int \tan^8(x) \sec^2(x) \sec^2(x) \, dx \]

\[ = \int \tan^8(x) (\tan^2(x) + 1) \sec^2(x) \, dx \]

\[ \text{now use } u = \tan(x) \]
\[ du = \sec^2(x) \, dx \]
So we have

\[ \int \tan^8(x) (\tan^2(x) + 1) \sec^2(x) \, dx \]

= \int u^8 (u^2 + 1) \, du

= \int (u^{10} + u^8) \, du = \frac{u^{11}}{11} + \frac{u^9}{9} + C

= \left[ \frac{\tan^{11}(x)}{11} + \frac{\tan^9(x)}{9} \right] + C

In general, if there is an even power of secant then \( u = \tan(x) \) is a good candidate for the sub
Since letting $u = \sec(x)$ gives $du = \sec(x) \tan(x) \, dx$,
we'll want this sub when we have odd powers of secant.

See that:

\[
\int \tan^5(x) \sec^2(x) \, dx
\]

\[
= \int \tan^4(x) \sec^2(x) \, sec(x) \tan(x) \, dx
\]

\[
= \int (\tan^2(x))^2 \sec^2(x) \, sec(x) \tan(x) \, dx
\]

Apply identity:

\[
= \int (\sec^2(x)-1)^2 \sec^2(x) \, sec(x) \tan(x) \, dx
\]

\[
= \int (u^2-1)^2 u^2 \, du
\]

\[
= \int u^2(u^4-2u^2+1) \, du
\]

\[
= \int (u^6-2u^4+u^2) \, du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C
\]
\[ \int \tan^2(x) \cos^3(x) \, dx \]

Recall \( \tan(x) = \frac{\sin(x)}{\cos(x)} \), so:

\[ \int \tan^2(x) \cos^3(x) \, dx = \int \frac{\sin^2(x)}{\cos^2(x)} \cos(x) \, dx \]

\[ = \int \sin^2(x) \cos(x) \, dx \]

Let \( u = \sin(x) \Rightarrow du = \cos(x) \, dx \)

\[ = \int u^2 \, du \]

\[ = \frac{u^3}{3} + C \]

\[ = \frac{\sin^3(x)}{3} + C \]

\[ \rightarrow \text{(Always check cancellations for } \tan(x) = \frac{\sin(x)}{\cos(x)}\text{)}!! \]
<table>
<thead>
<tr>
<th>Sub</th>
<th>Identity</th>
<th>When?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = \sin(x)$</td>
<td>$\sin^2(x) + \cos^2(x) = 1$</td>
<td>odd power of cosine</td>
</tr>
<tr>
<td>$du = \cos(x)dx$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u = \cos(x)$</td>
<td>$\tan^2(x) + 1 = \sec^2(x)$</td>
<td>even power of secant</td>
</tr>
<tr>
<td>$du = -\sin(x)dx$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u = \tan(x)$</td>
<td></td>
<td>odd power of secant</td>
</tr>
<tr>
<td>$du = \sec^2(x)dx$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u = \sec(x)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$du = \sec(x)\tan(x)dx$</td>
<td></td>
<td>odd power of secant + tangent</td>
</tr>
</tbody>
</table>

→ Need other identities to:

- even/even powers sine/cosine
- even/odd powers tan/secant
2) **Trigonometric Substitution**

Some functions under radicals can be simplified by substituting a trig function in place of the variable, with the goal of simplifying the resulting function using trig identities!

Example:

1. \( \int x^3 \sqrt{1 - x^2} \, dx \)

Let \( x = \sin \theta \)

Note the difference from usual substitution, which we could write as \( \theta = \arcsin(x) \)

Then \( dx = \cos \theta \, d\theta \)

Now we substitute
Now:

\[ \int x^3 \sqrt{1-x^2} \, dx \]

\[ = \int (\sin \theta)^3 \sqrt{1-(\sin \theta)^2} \, \cos \theta \, d\theta \]

\[ = \int \sin^3 \theta \sqrt{\cos^2 \theta} \, \cos \theta \, d\theta \]

apply identity

\[ = \int \sin^3 \theta \cos \theta \, \cos \theta \, d\theta \]

\[ = \int \sin^2 \theta \cos^2 \theta \, \sin \theta \, d\theta \]

\[ = \int (1- \cos^2 \theta) \cos^2 \theta \, \sin \theta \, d\theta \]

\[ u = \cos \theta \]
\[ du = -\sin \theta \, d\theta \]

\[ = \int (1-u^2)u^2 (-du) \]

\[ = \left( \frac{u^3}{3} - \frac{u^5}{5} \right) + C = \left( -\frac{\cos^3 \theta}{3} + \frac{\cos^5 \theta}{5} \right) + C \]

\[ \text{NOT FINISHED!} \]

\[ \text{still need to change } \theta \text{ to } x \]
but there is a problem...

\[ \int x^3 \sqrt{1-x^2} \, dx = \frac{-\cos^3 \theta}{3} + \frac{\cos^5 \theta}{5} + C \]

\[ \sin \theta = x \]

how do we resubstitute \( \sin \theta = x \)?

We must see the relationship between the two quantities \( \sin \theta \) and \( \cos \theta \).

\[ \theta \quad \sin \theta = \frac{x}{1} \]

Recall that \( \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \)

\[ \text{and } \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \]

hence we must find adjacent side using the picture!
Apply the Pythagorean Theorem:

\[ (\text{adj})^2 + x^2 = 1^2 \Rightarrow \text{adj} = \sqrt{1-x^2} \]

Therefore we have:

\[ \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \]

and so:

\[ \int x^3 \sqrt{1-x^2} \, dx = -\frac{\cos^3 \theta}{3} + \frac{\cos^5 \theta}{5} + C \]

Resub: \( \cos \theta = \sqrt{1-x^2} \)
The preceding process is called trig. substitution, and it can be employed to solve integrals with radicals of binomials.

The following chart tells which sub. to use for an integral $\int f(x) \, dx$:

<table>
<thead>
<tr>
<th>$f(x)$ contains</th>
<th>Sub</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a^2-x^2}$</td>
<td>$x = a \sin(t)$</td>
<td>$\sin^2(t) + \cos^2(t) = 1$</td>
</tr>
<tr>
<td>$\sqrt{x^2-a^2}$</td>
<td>$x = a \sec(t)$</td>
<td>$\tan^2(t) + 1 = \sec^2(t)$</td>
</tr>
<tr>
<td>$\sqrt{x^2+a^2}$</td>
<td>$x = a \tan(t)$</td>
<td></td>
</tr>
</tbody>
</table>

Using these subs, we are doing the following:

**Roots of binomials $\rightarrow$ trig sub $\rightarrow$ trig integrals $\rightarrow$ last section**
Examples, cont'd

(2) \( \int x^3 \sqrt{x^2 + 4} \, dx \)

By the chart, we let \( x = 2 \tan(t) \).

Then \( dx = 2 \sec^2(t) \, dt \) and:

\[
\int x^3 \sqrt{x^2 + 4} \, dx = \int 8 \tan^3(t) - 4 \tan^3(t) + 1 \cdot (2 \sec^2(t) \, dt)
\]

\[
= \int 8 \tan^3(t) \cdot 2 \cdot \sqrt{\tan^2(t) + 1} \cdot 2 \cdot \sec^2(t) \, dt
\]

Apply identity: \( \sec^2(t) = \tan^2(t) + 1 \)

\[
= 32 \int \tan^3(t) \sqrt{\sec^2(t)} \cdot \sec^2(t) \, dt
\]

\[
= 32 \int \tan^3(t) \sec(t) \sec^2(t) \, dt \quad \text{trig subst}
\]

\[
\downarrow \quad \text{now we solve the resulting trig integral!}
\]
\[ = 32 \int \tan^2(t) \sec^2(t) \sec(t) \tan(t) \, dt \]

\[ = 32 \int (\sec^2(t) - 1) \sec^2(t) \cdot \sec(t) \tan(t) \, dt \]

\[ \Rightarrow = 32 \int (u^2 - 1) u^2 \, du \]

\[ = 32 \int (u^4 - u^2) \, du \]

\[ = 32 \left( \frac{u^5}{5} - \frac{u^3}{3} \right) + C \]

\[ \Rightarrow = \frac{32}{5} (\sec(t))^5 + \frac{32}{3} (\sec(t))^3 + C \]

\[ \Rightarrow \text{Now we need to re-sub. back to } x, \text{ we know that } x = 2 \tan(t) \]

\[ \text{Now we'll back subst.( } \sec(t)) \]
Method 1: triangle picture

\[
\frac{\text{tangent}}{\text{adj.}} = \frac{\text{opp}}{\text{adj.}}
\]

Now, we know \( \sec(t) = \frac{1}{\cos(t)} = \frac{\text{hyp}}{\text{adj.}} \), hence we must find the hypotenuse.

\( \Rightarrow \) By Pythagorean Theorem:

\[
(hyp)^2 = x^2 + 2^2
\]

\( \Rightarrow \) hyp = \( \sqrt{x^2 + 4} \)

Therefore:

\[
\sec(t) = \frac{\text{hyp}}{\text{adj.}} = \frac{\sqrt{x^2 + 4}}{2}
\]

and therefore:

\[
\text{(ans)} = \frac{32}{5} \left( \frac{\sqrt{x^2 + 4}}{2} \right)^5 + \frac{32}{3} \left( \frac{\sqrt{x^2 + 4}}{2} \right)^3 + C
\]
Method 2: no picture.

We can also find \( \sec(t) \) using our identity. 

Since \( x = 2 \tan(t) \), we know that, using the identity:

\[
\sec^2(t) + 1 = \tan^2(t),
\]

we have:

\[
2 \sec^2(t) + 2^2 = 2 \cdot \tan^2(t) \quad \text{(multiply both sides by } 2^2) \]
\[
\Rightarrow 4 \sec^2(t) + 4 = x^2
\]
\[
\Rightarrow \frac{x^2}{4} = \sec^2(t) + 1
\]
\[
\Rightarrow \sqrt{\frac{x^2}{4} - 1} = \sec(t)
\]
\[
\Rightarrow \sqrt{\frac{x^2 - 4}{4}} = \sec(t)
\]
\[
\Rightarrow \frac{1}{2} \sqrt{x^2 - 4} = \sec(t)
\]

which was the no-subs we found before.
\( \int \frac{x^5 \, dx}{\sqrt{x^2 - 25}} \)

Let \( x = 5 \sec(t) \)

\( dx = 5 \sec(t) \tan(t) \, dt \)

Now:

\[
\int \frac{x^5 \, dx}{\sqrt{x^2 - 25}} = \int \frac{5^5 \sec(t) \left( 5 \sec(t) \tan(t) \, dt \right)}{\sqrt{25 \sec^2(t) - 25}}
\]

\[
= 5^5 \int \frac{\sec^6(t) \tan(t) \, dt}{\sqrt{25(\sec^2(t) - 1)}}
\]

apply identity \( \Rightarrow \)

\[
= \frac{5^6}{5} \int \frac{\sec^6(t) \tan(t) \, dt}{\sqrt{\tan^2(t)}}
\]

\[
= 5^5 \int \sec^6(t) \, dt
\]

save a \( \sec^2(t) \)

\[
= 5^5 \int \sec^4(t) \sec^2(t) \, dt
\]

apply identity \( \Rightarrow \)

\[
\text{sub: } u = \tan(t)
\]

\[
= 5^5 \int (u^2 + 1)^2 \, du
\]
\[ 5^5 \int (u^4 + 2u^2 + 1) \, du \]
\[ = 5^5 \left( \frac{u^5}{5} + \frac{2u^3}{3} + u \right) + C \]
\[ \overset{\text{re-sub}}{=} 5^5 \left( \frac{\tan^5(t)}{5} + \frac{2\tan^3(t)}{3} + \tan(t) \right) + C \]

Now since \( x = 5 \sec(t) \iff \sec(t) = \frac{x}{5} \), we have:

\[
\begin{align*}
\frac{x}{5} & \quad \text{so opposite side is:} \\
\tan(t) & = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2 - 25}}{5}
\end{align*}
\]

and since we have:

\[ \tan(t) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2 - 25}}{5} \]

we know the re-sub for \( \tan(t) \):

\[ 5^5 \left( \left( \frac{\sqrt{x^2 - 25}}{5} \right)^5 + \frac{2}{3} \left( \frac{\sqrt{x^2 - 25}}{5} \right)^3 + \frac{\sqrt{x^2 - 25}}{5} \right) + C \]
\[ = \left( x^2 - 25 \right)^{5/2} + \frac{50}{3} \left( x^2 - 25 \right)^{3/2} + 5^4 \left( x^2 - 25 \right)^{1/2} + C \]
(3) Partial Fractions

Now consider integrals of rational functions:

\[ \frac{x+1}{(2x+1)(x^2+1)} \quad \frac{2x^2}{x^2-2x-3} \]

\[ \frac{dx}{(x+1)(x+2)(x+3)} \quad \frac{(x^5-1)}{(x^3+1)(x^2-1)(x-2)} \]

Using the process of **Partial Fraction Decomposition** (or expansion) we can rewrite most rational functions as sums of simpler functions!

\[\Rightarrow\] The idea comes from "reversing" addition of factors:

\[ \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \]

\[\Rightarrow\] So if we can find \(A\) and \(B\) we may rewrite the complicated rational fn. on the right.
Let's consider this example:

$$\frac{8x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

Now multiply both sides by the common denominator $(x+1)(x+2)$:

$$3x = A \frac{(x+1)(x+2)}{(x+1)} + B \frac{(x+1)(x+2)}{(x+2)}$$

$$\Rightarrow 3x = A(x+2) + B(x+1).$$

Now this is an equality of functions, hence it must be true for every value of $x$. Therefore, we can find $A$ and $B$ by looking at certain values of $x$. 
Consider:

\[ x = -2 \quad \Rightarrow \quad 3(-2) = A(-2+2) + B(-2+1) \]
\[ \Rightarrow \quad -6 = A(0) + B(-1) \]
\[ \Rightarrow \quad -6 = -B \quad \Rightarrow \quad B = 6 \]

\[ x = -1 \quad \Rightarrow \quad 3(-1) = A(-1+2) + B(-1+1) \]
\[ \Rightarrow \quad -3 = A(1) + B(0) \]
\[ \Rightarrow \quad A = -3 \]

\[ \begin{align*}
\text{[we chose our x-values based on what would make each term containing an A, B, etc. to be zero.]} \\
\text{[e.g., to find B, we chose } x = -2 \text{ because that made } A(x+2) = 0] \\
\end{align*} \]

so now we have our decomposition:

\[ \frac{3x}{(x+1)(x+2)} = \frac{-3}{(x+1)} + \frac{6}{(x+2)} \]
and now we can integrate!

\[
\int \frac{3x}{(x+1)(x+2)} \, dx = \int \left( \frac{-3}{x+1} + \frac{5}{x+2} \right) \, dx
\]

\[
= -3 \int \frac{dx}{x+1} + 6 \int \frac{dx}{x+2}
\]

\[
= -3 \ln |x+1| + 6 \ln |x+2| + C
\]

The procedure for finding coefficients \( A, B, C, \) etc. is the same for each problem, however the form of the decompositions may change.

See the follow up.
Writing the PF decompositions

1. One summand for each linear factor:

\[ \frac{2x^2}{(x-3)(2x+1)} = \frac{A}{x-3} + \frac{B}{2x+1} \]

2. An additional summand for repeated linear factors:

\[ \frac{5x-1}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \]

3. An additional x-term for irreducible quadratic factors:

\[ \frac{2x^2+1}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 2} \]

\[ \text{Irreducible! (cannot factor)} \]

\[ \text{need a linear function on top to "offset" the quadratic!} \]
Examples, cont'd

(2) \( \int \frac{(x-1)dx}{(x+2)(x+1)^2} \)

Write:

\[ \frac{x-1}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \]

\[ \Rightarrow x-1 = A(x+1)^2 + B(x+2)(x+1) + C(x+2) \]

Let \( x = -1 \):

\[ -1 - 1 = A(0) + B(0) + C(-1+2) \]

\[ \Rightarrow -2 = C(1) \Rightarrow \boxed{C = -2} \]

Let \( x = -2 \):

\[ -2 - 1 = A(-2+1)^2 + B(0) + C(0) \]

\[ \Rightarrow -3 = A(-1)^2 \Rightarrow \boxed{A = -3} \]

Let \( x = 0 \):

\[ -1 = A(1)^2 + B(2)(1) + C(2) \]

\[ \Rightarrow -1 = A + 3B + 2C \]

\[ \text{Sub.} A + C \Rightarrow -1 = (-3) + 3B + 2(-2) \]

\[ \Rightarrow -1 + 3 + 4 = 3B \Rightarrow 6 = 3B \Rightarrow \boxed{B = 2} \]
Now we can evaluate:

\[ \int \frac{(x-1) \, dx}{(x+2)(x+1)^2} = \int \left( \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right) \, dx \]

\[ = A \int \frac{dx}{x+2} + B \int \frac{dx}{x+1} + C \int \frac{dx}{(x+1)^2} \]

\[ = A \ln|x+2| + B \ln|x+1| + C \left( \frac{-1}{x+1} \right) + \text{Const.} \]

Sub in the constants:

\[ = -3 \ln|x+2| + 2 \ln|x+1| + \frac{2}{x+1} + C \]
\[ \int \frac{2x^2 + 1}{(x-1)(x^2 + 2)} \, dx \]

\[ \frac{2x^2 + 1}{(x-1)(x^2 + 2)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 2} \]

\[ \Rightarrow 2x^2 + 1 = A(x^2 + 2) + (Bx + C)(x-1) \]

\( x = 1 \):
\[ 2 + 1 = A(1 + 2) + 0 \]
\[ \Rightarrow 3 = A(3) \Rightarrow A = 1 \]

\( x = 0 \):
\[ 0 + 1 = A(0 + 2) + (0 + C)(0 - 1) \]
\[ \Rightarrow 1 = 2A - C \Rightarrow 1 = 2 - C \]
\[ \text{Sub } A \]
\[ \Rightarrow C = 1 \]

\( x = 2 \):
\[ 2(2)^2 + 1 = A(2^2 + 2) + (B(2) + C)(2-1) \]
\[ \Rightarrow 9 + 1 = A(6) + (2B + C)(1) \]
\[ \Rightarrow \text{Sub } A, C \]
\[ 9 = 6(1) + 2B + C(1) \]
\[ \Rightarrow 2 = 2B \Rightarrow B = 1 \]
Now we can evaluate:

\[
\int \frac{2x^2 + 1}{(x-1)(x^2+2)} \, dx = \int \left( \frac{1}{(x-1)} + \frac{x+1}{x^2+2} \right) \, dx
\]

\[
= \int \frac{dx}{x-1} + \int \frac{x \, dx}{x^2+2} + \int \frac{dx}{x^2+2}
\]

\[
= \ln|x-1| + \frac{\ln|x^2+2|}{2} + \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}} x\right) + C
\]

\[\text{sub: } u = x-1 \quad \text{sub: } v = x^2+2 \quad \text{rule for } \arctan\]