MATH 10B

WEEK 9

Differential equations

1. General and specific solutions:

Recall we have seen function equations, that is, for what numbers \( x \) is the equality

\[ x^2 - 4x + 6 = 4x - x^2 \]

true?

We solve:

\[
\begin{align*}
2x^2 - 8x + 6 &= 0 \\
\Rightarrow 2(x - 1)(x - 3) &= 0 \\
\Rightarrow x &= 1, 3
\end{align*}
\]

→ A differential equation asks for what functions \( u(x) \) does a given equality hold; the first examples are antiderivatives!
Example

1. For which functions \( u(x) \) does the equality
\[
\frac{du}{dx} = \cos(x) + 2x
\]
hold?

This is precisely asking for the antiderivative of \( \cos(x) + 2x \):

\[
u = \int \frac{du}{dx} \, dx = \int (\cos(x) + 2x) \, dx = \sin(x) + x^2 + C
\]

\[\Rightarrow \boxed{u(x) = \sin(x) + x^2 + C}\]

is called the general solution; it gives the form of every possible specific solution, which correspond to a chosen value for \( C \).
we can find a specific solution by specifying some initial data:

\[ u(0) = 2. \]

Then:

\[ Z = u(0) = \sin(0) + 0^2 + C \]

\[ \Rightarrow C = 2 \]

hence the specific solution is:

\[ u(x) = \sin(x) + x^2 + 2 \]

As this initial data:

we say it is the answer to the Initial Value Problem:

\[ \left\{ \begin{array}{l}
\frac{dy}{dx} = \cos(x) + 2x \\
u(0) = 2
\end{array} \right. \]
2. Population exponential growth:

The first (and possibly most important) example of a differential equation (DE) is the population or exponential growth equation:

\[
\frac{dP}{dt} = kP
\]

for some constant \( k \) (growth rate). The general solution is given by:

\[ P(t) = Ae^{kt} \]

and specific solutions correspond to values for \( A \).
Example:

2) Solve the IVP

\[ \frac{dP}{dt} = 3P \]
\[ P(0) = 500 \]

The general solution is \( P(t) = Ae^{3t} \)

and therefore: \( 500 = P(0) = Ae^0 = A \)

\[ \Rightarrow A = 500 \]

\[ \text{hence} \quad P(t) = 500e^{3t} \]
Suppose a bacteria culture has 100 cells and grows exponentially. After 1 hour it has increased to 400 cells.

(a) Find \( P(t) = \) population of culture at time \( t \) hours:

We know \( P(t) = Ae^{kt} \) since \( P(t) \) is exponential (population).

We are given \( \begin{cases} P(0) = 100 \\ P(1) = 400 \end{cases} \)

Therefore:

\[
100 = P(0) = Ae^0 = A \\
400 = P(1) = Ae^k \\
\Rightarrow (A=100) \text{ and } 400 = 100e^k \\
\Rightarrow 4 = e^k \Rightarrow (\ln(4)=k)
\]

Thus:

\[
P(t) = 100e^{\ln(4)t} = 100(4)^t
\]
(b) Find population after 2 hours

\[ P(2) = 100(4)^2 = 100(16) = \frac{1600}{\text{bacteria}} \]

(c) Find rate of growth at 2 hours

i.e., find \( P'(2) \).

\[ P'(t) = \ln(4) 100 e^{\ln(4)t} \]

\[ \Rightarrow P'(2) = \ln(4) 100 e^{\ln(4)2} = \ln(4) \cdot 100 \cdot (16) = \frac{1600 \cdot \ln(4)}{\text{bacteria/hour}} \]

(d) At what time does the population reach 10000 cells?

i.e., for what \( t \) does \( P(t) = 10000 \)?

\[ \Rightarrow 100e^{\ln(4)t} = 10000 \]

\[ \Rightarrow e^{\ln(4)t} = 100 \]

\[ \Rightarrow \ln(4)t = \ln(100) \Rightarrow t = \frac{\ln(100)}{\ln(4)} \text{ hours} \]
3. Checking solutions to D.E.'s:

Given a D.E. and (general or specific) solution, we can check the solution by plugging it in!

Examples:

6. Population: check \( P(t) = Ae^{kt} \)

is a general solution to \( \frac{dP}{dt} = kP \):

\[ \frac{dP}{dt} = A \frac{d}{dt}(e^{kt}) = Ake^{kt} \]

\[ kP = kAe^{kt} = Ake^{kt} \]

Hence \( P(t) \) satisfies the equation (for any \( A \)).
1. Check \( y = \frac{2}{3} e^t + e^{-2t} \)
   is a solution to \( \frac{dy}{dt} + 2y = 2et \)

- \( \frac{dy}{dt} = \frac{d}{dt} \left( \frac{2}{3} e^t + e^{-2t} \right) \)
  \[ = \frac{2}{3} e^t - 2e^{-2t} \]
- \( 2y = 2 \left( \frac{2}{3} e^t + e^{-2t} \right) \)
  \[ = \frac{4}{3} e^t + 2e^{-2t} \]

\[ \Rightarrow \frac{dy}{dt} + 2y = \frac{2}{3} e^t - 2e^{-2t} + \frac{4}{3} e^t + 2e^{-2t} \]
\[ = \frac{6}{3} e^t = 2et \]

\[ \checkmark \]

hence the given \( y \)
 satisfies the D.E.!
2. Find all $A$ so that $y = Ae^t$ is a solution to

$$\frac{dy}{dt} + 2y = 2e^t$$

$$\frac{dy}{dt} = Ae^t$$

$$2y = 2Ae^t$$

$$\frac{dy}{dt} + 2y = Ae^t + 2Ae^t = 2e^t$$

$$e^{t \neq 0} \Rightarrow 3Ae^t = 2e^t$$

$$\Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3}$$

Note: only one value of $A$ works, hence $y = Ae^t$ is not a general solution for the D.E. $y' + 2y = 2e^t$. 

9-10
3. Show that
\[ y = -t\cos(t) - t \]
is a solution to the initial value problem
\[ \begin{cases} ty' = y + t^2\sin(t) \\ y(\pi) = 0 \end{cases} \]

- \[ \frac{dy}{dt} = \frac{d}{dt} (-t\cos(t) - t) \]
  \[ = -[\cos(t) + t(-\sin(t))] - 1 \]
  \[ = -\cos(t) + t\sin(t) - 1 \]
  \[ \Rightarrow t\frac{dy}{dt} = -t\cos(t) + t^2\sin(t) - 1 \quad \text{(LHS)} \]

- \[ y + t^2\sin(t) = -t\cos(t) - t + t^2\sin(t) \quad \text{(RHS)} \]
  
  So \( \text{LHS} = \text{RHS} \checkmark \) so solves the I.V.P.

- \[ y(\pi) = -\pi\cos(\pi) - \pi = -\pi(-1) - \pi \]
  \[ = \pi - \pi = 0 \]
  \[ \Rightarrow y(\pi) = 0 \] so solves the I.V.P.
4) Show that \( y = \frac{\ln(x) + C}{x} \) solves the D.E.
\[ x^2y' + xy = 1 \]

\[
\frac{dy}{dx} = \frac{d}{dx}\left( \frac{\ln(x) + C}{x} \right)
\]
\[
= \frac{d}{dx}\left( \frac{\ln(x)}{x} \right) + C \frac{d}{dx}\left( \frac{1}{x} \right)
\]
\[
= (\frac{-1}{x^2})\ln(x) + \left( \frac{1}{x} \right)\left( \frac{1}{x} \right) + \frac{-C}{x^2}
\]
\[
= -\frac{\ln(x)}{x^2} + \frac{1}{x^2} - \frac{C}{x^2}
\]
\[
\Rightarrow x^2\frac{dy}{dx} = -\ln(x) + 1 - C
\]

and \( xy = \frac{x\ln(x) + Cx}{x} = \ln(x) + C \)

hence \( x^2\frac{dy}{dx} + xy = -\ln(x) + 1 - C + \ln(x) + C \)
\[
= 1
\]
\[
\sqrt{\text{hence } y \text{ solves the D.E. for all } C}
\]
The spring equation is:

\[
\frac{d^2x}{dt^2} = -\frac{k}{m} x
\]

where \( x(t) \) is the position of a mass \( m \) on the end of a spring with constant \( k \).

The general solution is:

\[
x(t) = A \sin(\sqrt{\frac{k}{m}} t) + B \cos(\sqrt{\frac{k}{m}} t)
\]
Notice the general solution has 2 unknowns \( A \) and \( B \).

This is because the spring equation is a 2nd order, i.e., has 2nd derivatives. The general solution to an 2\( n \)th order equation will have \( N \) unknowns.

**Check:**

\[
X'(t) = A \sqrt{\frac{k}{m}} \cos(\sqrt{\frac{k}{m}} t) - B \sqrt{\frac{k}{m}} \sin(\sqrt{\frac{k}{m}} t)
\]

\[
X''(t) = -A \left(\frac{k}{m}\right) \sin\left(\sqrt{\frac{k}{m}} t\right) - B \left(\frac{k}{m}\right)^2 \sin\left(\sqrt{\frac{k}{m}} t\right)
\]

\[
= -\frac{Ak}{m} \sin\left(\frac{k}{m} t\right) - \frac{Bk}{m} \sin\left(\frac{k}{m} t\right)
\]

And:

\[
-\frac{k}{m} X(t) = -\frac{Ak}{m} \sin\left(\frac{k}{m} t\right) - \frac{Bk}{m} \sin\left(\frac{k}{m} t\right)
\]

Hence \( X(t) \) is a solution for all \( A, B \).
Examples:

0. Solve I.V.P.:
\[
\begin{align*}
\frac{d^2x}{dt^2} &= -4x \\
x(0) &= 0 \\
x'(0) &= 2
\end{align*}
\]

The general solution is:
\[x(t) = A\sin(2t) + B\cos(2t)\]

Then:
\[0 = x(0) = A\sin(0) + B\cos(0) = 0 + B \Rightarrow B = 0\]

and:
\[x'(t) = 2A\cos(2t) - 2B\sin(2t)\]

\[= 2 = x'(0) = 2A\cos(0) - 2B\sin(0) = 2A - 0 \Rightarrow A = 1\]

(specific)

Hence the solution is:
\[x(t) = \sin(2t)\]
5) Separable D.E.'s

Now we'll see how to build general solutions to a nice type of differential equation.

Recall the antiderivative problem, e.g.

\[
\frac{du}{dx} = \frac{1}{x} + 2
\]

We can use the notation to see the solution; multiply both sides by \(dx\) and then integrate!

\[
\Rightarrow \quad du = \left(\frac{1}{x} + 2\right)\,dx
\]

\[
\Rightarrow \quad u = \int \left(\frac{1}{x} + 2\right)\,dx
\]

\[
= \ln|x| + 2x + C
\]

Hence the general solution is

\[
U(x) = \ln|x| + 2x + C
\]
Notice that a D.E. is an antiderivative problem & it can be written:

\[ \frac{dy}{dx} = (\text{expression in } x) \]

A separable D.E. is a D.E. which can be written:

\[ \frac{du}{dx} = (\text{expression in } x) \left( \text{expression in } u \right) \]

Which we then re-write as:

\[ \frac{du}{\left( \text{expression in } u \right)} = (\text{expression in } x) \, dx \]

and integrate to solve for \( u \):

\[ \int \frac{du}{\left( \text{exp in } u \right)} = \int (\text{exp in } x) \, dx \]
Examples:

1. \( \frac{du}{dx} = (\frac{1}{x} + 2)(u) \)

\[ \Rightarrow \quad \frac{du}{u} = (\frac{1}{x} + 2) \, dx \]

\[ \Rightarrow \quad \int \frac{du}{u} = \int (\frac{1}{x} + 2) \, dx \]

\[ \Rightarrow \quad \ln |u| + C_1 = \ln |x| + 2x + C_2 \]

\[ \Rightarrow \quad \ln |u| = \ln |x| + 2x + C_2 - C_1 \]

\[ = \ln |x| + 2x + C \]

\[ \text{exponentialize both sides by } e \]

\[ e^{\ln |u|} = e^{\ln |x| + 2x + C} \]

\[ \Rightarrow \quad u = e^{\ln |x| + 2x + C} \]

\[ = xe^{2x}e^C \]

\[ = xe^{2x}A \]

\[ A = e^C \]

\[ \Rightarrow \quad u(x) = A \cdot e^{2x} \]

is the general solution
(2) \[ u' = 1 + u + 2t + 2tu \]

\[ \Rightarrow u' = 1 + 2t + u(1+2t) \]

\[ = (1+2t)(1+u) \]

\[ \Rightarrow \frac{du}{dt} = (1+2t)(1+u) \]

\[ \Rightarrow \frac{du}{1+u} = (1+2t)dt \]

\[ \Rightarrow \int \frac{du}{1+u} = \int (1+2t)dt \]

\[ \Rightarrow \ln|1+u| = t + t^2 + C \]

\[ \Rightarrow e^{\ln|1+u|} = e^{t + t^2 + C} \]

\[ \Rightarrow 1+u = Ae^{t}e^{t^2} \]

\[ \Rightarrow u = Ae^{t}e^{t^2} - 1 \]  

General solution
\[ \frac{dy}{dx} = \frac{x}{2y}, \quad \text{solve IVP.} \]

\[ y(0) = -3 \]

\[ \Rightarrow 2y \, dy = x \, dx \]

\[ \Rightarrow \int 2y \, dy = \int x \, dx \]

\[ \Rightarrow y^2 = \frac{x^2}{2} + C \]

\[ \Rightarrow y = \pm \sqrt{\frac{x^2}{2} + C} \]

Don't forget the general solution.

Now \( y(0) = -3 \), hence:

\[ -3 = y(0) = \pm \sqrt{\frac{0}{2} + C} = \pm \sqrt{C} \]

\[ \Rightarrow C = 9 \]

\[ \Rightarrow y(x) = -\sqrt{\frac{x^2}{2} + 9} \]

Specific solution (to the IVP)
Logistic equation:

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)
\]

- models population with carrying capacity \( M \):
  note \( \frac{dP}{dt} = 0 \) at \( P = 0, M \)
  (no growth/change at population 0 or the carrying capacity)

- separable:

\[
\int \frac{dP}{(1 - \frac{P}{M})P} = \int k \, dt
\]

\[
k \, t + C = \int \frac{dP}{(1 - \frac{P}{M})P} \quad \text{need partial fractions!}
\]

\[
\frac{1}{(1 - \frac{P}{M})P} = \frac{A}{(1 - \frac{P}{M})} + \frac{B}{P}
\]

\[
\Rightarrow 1 = AP + B(1 - \frac{P}{M}) \Rightarrow P = 0: 1 = B
\]

\[
P = M: \frac{1}{M} = A
\]
\[\begin{align*}
\text{hence} & \quad kt + C = \int \frac{dP}{(1 - \frac{P}{M})^\gamma} = \int \left( \frac{1}{m(1 - \frac{P}{M})} + \frac{1}{P} \right) dP \\
& = \int \frac{dP}{M - P} + \int \frac{dP}{P} \\
& = -\ln|\frac{P}{M-P}| + \ln|P| \\
\Rightarrow \quad e^{kt+C} &= e^{-\ln(\frac{P}{M-P}) + \ln|P|} \\
& = e^{\ln|P|} \\
& = \frac{P}{e^{\ln|M-P|}} \\
& = \frac{P}{e^{\ln(M-P)}} = \frac{P}{M-P} \\
\Rightarrow \quad Ae^{kt} &= \frac{P}{M-P} \\
\Rightarrow \quad \frac{M-P}{P} &= A e^{-kt} \Rightarrow \frac{M}{P} - 1 = A e^{-kt} \\
& \Rightarrow \frac{M}{P} = A e^{-kt} + 1 \\
& \Rightarrow \frac{1}{P} = \frac{A e^{-kt} + 1}{M} \\
\Rightarrow \quad P &= \frac{M}{A e^{-kt} + 1} \quad \text{general solution}
\end{align*}\]