I. Antiderivatives:

→ motivation

Q: What is the position function \( s(t) \) of an object with velocity function \( v(t) \)?
A: \( s'(t) = v(t) \), that is \( s(t) \) is an antiderivative of \( v(t) \).

→ Review: table of antiderivatives
→ Remember: general antiderivatives need "+C"

Examples:

1. A piano is dropped from the top of a 10 story (100 ft.) building. At what time does it pass the 2nd floor window at 15 ft.? (\( g = -32 \text{ ft/s}^2 \))

The piano has velocity \( v(t) = -32t \text{ ft/s} \)
Now, the general antiderivative of \( v(t) = -32t \) is:

\[
s(t) = \frac{-32t^2}{2} + C
\]

and we want the antiderivative such that \( s(0) = 100 \):

\[
100 = s(0) = \frac{-32(0)^2}{2} + C \Rightarrow C = 100
\]

and therefore:

\[
s(t) = \frac{-32t^2}{2} + 100 = -16t^2 + 100
\]

Now, we want \( t \) such that \( s(t) = 15 \):

\[
15 = s(t) = -16t^2 + 100
\]

\[
\Rightarrow -85 = -16t^2 \Rightarrow \frac{85}{16} = t^2
\]

\[
\Rightarrow t = \sqrt{\frac{85}{16}} \text{ seconds}
\]
2. An insect flying has a vertical velocity of \( v(t) = -32 \sin(t) \) ft/s with a starting height of 1000 ft.

Where is the insect at 3 seconds?

The general antiderivative of \( v(t) \) is given by: \( s(t) = 32 \cos(t) + C \)
and we are given that \( s(0) = 1000 \), so:

\[
1000 = s(0) = 32 \cos(0) + C \\
= 32 + C
\]

\[ \Rightarrow C = 968 \]

So: \( s(t) = 32 \cos(t) + 968 \)

and thus: \( s(3) = 32 \cos(3) + 968 \) ft.
II. Integration

→ motivation

Q: What is the displacement of an object with given velocity function \( v(t) \) over a time interval \( a \leq t \leq b \)?

A: The area under the graph \( y = v(t) \) over the interval \( a \leq t \leq b \), a quantity denoted \( \int_{a}^{b} v(t) \, dt \) called the integral

→ Review: areas for rectangles, triangles, trapezoids, and circles

Example:

1. Let \( v(t) \) have the graph:

```
\begin{figure}
\centering
\includegraphics[width=\textwidth]{graph.png}
\end{figure}
```
(a) What is the displacement over the time interval $2 \leq t \leq 9$?

We want to know $\int_2^9 v(t)\,dt$.

Consider that part of the graph:

$$\int_2^9 v(t)\,dt = \int_2^3 v(t)\,dt + \int_3^5 v(t)\,dt + \int_5^7 v(t)\,dt + \int_7^9 v(t)\,dt$$

1. $\triangle_{\frac{1}{2}(1)(1)}$
2. $-\Delta_{-\frac{1}{2}(2)(1)}$
3. $\triangle_{\frac{1}{2}(2)(2)}$
4. Complement of $\frac{1}{4}$ circle

$$(2)^2 \left\{ \frac{1}{4}(\pi(2)^2) \right\} = 4 - \pi$$
hence:
\[ \int_2^9 v(t) \, dt = \frac{1}{2} - 1 + 2 + 4 - \pi \]
\[ = \left[ \frac{11}{2} - \pi \right] \]

(b) On what intervals does the area function \( F(x) = \int_0^x v(t) \, dt \) increase and decrease?

See that for \( x \) in the intervals
\[
\begin{align*}
0 \leq x \leq 3, \quad & \text{and} \\
5 < x < 14
\end{align*}
\]
the integral adds positive area, hence it is increasing.

On the other hand, in \( 3 < x < 5 \) and \( 14 < x < 16 \)
the integral counts negative area, hence it is decreasing.
Evaluating integrals: Riemann sums

Recall: \( R_N = \sum_{n=1}^{N} \left( \frac{b-a}{N} \right) f(a + \frac{(b-a)n}{N}) \)

\( L_N = \sum_{n=0}^{N-1} \left( \frac{b-a}{N} \right) f(a + \frac{(b-a)n}{N}) \)

Right & left Riemann sums divide \( N \) rectangles under \( y = f(x) \) over \( a \leq x \leq b \).

Examples:

1. \( f(x) = \frac{3}{x} \) over \( 2 \leq x \leq 7 \); find \( R_5 \)

\[
R_5 = \sum_{n=1}^{5} \left( \frac{7-2}{5} \right) f(2 + \frac{(7-2)n}{5})
= \sum_{n=1}^{5} \left( \frac{5}{5} \right) f(2 + \frac{5n}{5}) = \sum_{n=1}^{5} f(2+n)
= f(3) + f(4) + f(5) + f(6) + f(7)
= \frac{3}{3} + \frac{3}{4} + \frac{3}{5} + \frac{3}{6} + \frac{3}{7}
= \frac{1 + \frac{3}{4} + \frac{3}{5} + \frac{1}{2} + \frac{3}{7}}{1}
\]

must simplify \( \Sigma \),
but adding fractions not necessary
\( L_5 = ? \)

\[
L_5 = \sum_{n=0}^{4} f(2+n)
\]

Same as \( R_5 \), except index changed

\[
= f(2) + f(3) + f(4) + f(5) + f(6)
\]

\[
= \frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \frac{3}{5} + \frac{3}{2}
\]

\[
= \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + \frac{1}{2} = \boxed{3 + \frac{3}{4} + \frac{3}{5}}
\]
IV. Evaluating integrals: FTC2 + substitution

Recall:

**FTC2**
\[ \int_a^b f(t) \, dt = F(b) - F(a) \]
for any \( F'(x) = f(x) \)

**Substitution**
\[ du = \frac{du}{dx} \, dx \]

Examples

1. \( \int_1^3 (2x^3 + \frac{1}{x^2}) \, dx \)

\[ = \int_1^3 2x^3 \, dx + \int_1^3 \frac{1}{x^2} \, dx \]

Now, \( 2x^3 \) has antiderivative \( F(x) = \frac{2}{4}x^4 = \frac{x^4}{2} \)
and \( \frac{1}{x^2} \) has antiderivative \( G(x) = \frac{-1}{x} \)

So we can apply FTC2:

\[ = F(3) - F(1) + G(3) - G(1) \]

\[ = \frac{3^4}{2} - \frac{1^4}{2} + \frac{-1}{3} - \frac{-1}{1} = \frac{81}{2} + \frac{2}{3} = \frac{40}{3} + \frac{2}{3} \]
\( \int_{-2}^{2} |1-x^2| \, dx \)

\( |1-x^2| \) is a piecewise function:

\[
|1-x^2| = \begin{cases} 
1-x^2 & \text{for } 1-x^2 \geq 0 \iff 1 \geq x^2 \\
-(1-x^2) & \text{for } 1-x^2 < 0 \iff 1 < x^2 
\end{cases}
\]

then \( 1 \geq x^2 \iff -1 \leq x \leq 1 \)

and \( 1 < x^2 \iff x < -1 \text{ or } x > 1 \)

So:

\[
|1-x^2| = \begin{cases} 
-(1-x^2) & \text{for } x < -1 \\
(1-x^2) & \text{for } -1 \leq x \leq 1 \\
-(1-x^2) & \text{for } 1 < x 
\end{cases}
\]

Now we divide the integral's interval by the three pieces:

\[
\int_{-2}^{2} |1-x^2| \, dx = \int_{-2}^{-1} (1-x^2) \, dx + \int_{-1}^{1} (1-x^2) \, dx + \int_{1}^{2} (1-x^2) \, dx
\]

\( \stackrel{\text{piecewise definition}}{=} \)

\[
= \int_{-2}^{-1} (1-x^2) \, dx + \int_{-1}^{1} (1-x^2) \, dx + \int_{1}^{2} (1-x^2) \, dx
\]

\( = \int_{-2}^{-1} (x^2-1) \, dx + \int_{-1}^{1} (1-x^2) \, dx + \int_{1}^{2} (x^2-1) \, dx
\]
$x^2 - 1$ has antiderivative $F(x) = \frac{x^3}{3} - x$

$1 - x^2$ has antiderivative $-F(x) = x - \frac{x^3}{3}$

so now apply $\boxed{\text{FTC 2}}$:

$$\int_{-2}^{2} (1 - x^2) \, dx = (F(1) - F(2)) + (-F(1) - (-F(-1)))$$

$$+ (F(2) - F(1))$$

$$= \left( \left( \frac{1}{3} + 1 \right) - \left( \frac{8}{3} + 2 \right) \right) + \left( (1 - \frac{1}{3}) + (\frac{1}{3} + 1) \right)$$

$$+ \left( \left( \frac{8}{3} - 2 \right) - \left( \frac{1}{3} - 1 \right) \right)$$

$$= \left( \frac{2}{3} + \frac{2}{3} \right) + \left( \frac{2}{3} + \frac{2}{3} \right) + \left( \frac{2}{3} + \frac{2}{3} \right)$$

$$= \frac{12}{3} = \boxed{4}$$

$\text{Shortcut: What kind of function is}$ $f(x) = |1 - x^2|$? $\text{Hint: check } f(-x)$ ....
\[ \int_{\pi/2}^{\pi} 3t \sin(5t^2) \, dt \]

Apply \textbf{substitution:} let \( u = 5t^2 \),
then: \( du = 10t \, dt \), hence \( \frac{du}{10} = t \, dt \).

Now:

\[
\int_{\pi/2}^{\pi} 3t \sin(5t^2) \, dt = 3 \int_{\pi/2}^{\pi} t \sin(5t^2) \, dt \]

\[
= 3 \int_{\pi/2}^{\pi} t \sin(u) \, du \]

\[
= 3 \int_{\pi/2}^{\pi} \frac{\sin(u)}{10} \, du \]

\[
= \frac{3}{10} \int_{\pi/2}^{\pi} \sin(u) \, du \]

\[
\Rightarrow \int_{\pi/2}^{\pi} \frac{\sin(u)}{10} \, du = \frac{3}{10} \left[ \cos(u) \right]_{\pi/2}^{\pi} = \frac{3}{10} \left[ \cos(\pi) - \cos(\pi/2) \right] \]

\[
= \frac{3}{10} \left[ \cos(\pi/4) - \cos(5\pi^2) \right] \]
\[ \int_1^e \frac{5\pi \ln(x)^3}{x} \, dx \]

**Apply substitution:** let \( u = \ln(x) \)

Then \( du = \frac{1}{x} \, dx \)

Now:

\[ \int_1^e \frac{5\pi \ln(x)^3}{x} \, dx = 5\pi \int_1^e \frac{\ln(x)^3}{x} \, dx \]

\[ \text{apply FTC2} \]

\[ = \left( \frac{(1)^4}{4} - \frac{(0)^4}{4} \right) \]

\[ = 1 \]
Let \( f(x) \) be continuous; then
\[
F(x) = \int_a^x f(t) \, dt
\]
is an area function
and it is an antiderivative of \( f(x) \) by:

\[
\text{FTC 1}
\]
\[
F'(x) = f(x)
\]

Examples

1. Let \( f(x) \) have the graph
\[
y = f(x)
\]
and let \( F(x) = \int_{-2}^x f(t) \, dt \)
(a) \( F(0) = \int_{-2}^{0} f(t) \, dt = \frac{1}{2} (2)(2) = 2 \) \\
area of triangle

\[
F(0) = \int_{-2}^{0} f(t) \, dt 
\]

\[
= \int_{-2}^{0} f(t) \, dt + \int_{0}^{2} f(t) \, dt + \int_{2}^{4} f(t) \, dt 
\]

by above 

\[
= 2 
\]

cancel: see graph

\[
F(4) = \int_{-2}^{4} f(t) \, dt 
\]

\[
= 0 
\]

(b) Where is \( F(t) \) increasing?

Adding positive area in \(-2 \leq x < 2\) and \(5 < x < 6 \) \( \Rightarrow \) increasing here

Adding negative area in \(2 < x < 5\) \( \Rightarrow \) decreasing here
(c) $F'(3) = \ ?$

by (FTC 1), $F'(x) = f(x)$

hence $F'(3) = f(3) = \boxed{-1}$

by the graph

② Let $G(x) = \int_{-\pi/2}^{x} \sin(x) \, dx$

(a) $G\left(\frac{\pi}{2}\right) = \ ?$, $G(0) = \ ?$

$G\left(\frac{\pi}{2}\right) = \int_{-\pi/2}^{\pi/2} \sin(x) \, dx = \boxed{0}$ by oddness

$G(0) = \int_{-\pi/2}^{0} \sin(x) \, dx = (\cos(0)) - (-\cos(\pi/2))$

$= (-1) + (0) = \boxed{-1}$

(b) $G'(\pi/2) = \ ?$, $G'(0) = \ ?$

$G'(x) = \sin(x) \Rightarrow G'(\pi/2) = \sin(\pi/2) = \boxed{1}$

$G'(0) = \sin(0) = \boxed{0}$

(c) $G(\pi) = \ ?$

$G(\pi) = \int_{-\pi/2}^{\pi} \sin(x) \, dx = \int_{-\pi/2}^{\pi} \sin(x) \, dx = \boxed{1}$

by part (a): $G(\pi/2)$