Math 10B,
Midterm Exam 2
March 1, 2018

Turn off and put away your cell phone.

No calculators or other electronic devices are allowed during this exam.

You may use one page of notes, but no books or other assistance during this exam. If you violate these instructions or communicate in any way with any other student during this exam, you will receive a zero on the exam, and the zero will not be dropped when calculating your cumulative course average.

Read each question carefully, and answer each question completely.

Show all of your work; no credit will be given for unsupported answers.

Write your solutions clearly and legibly; no credit will be given for illegible solutions.

If any question is not clear, ask for clarification.
1. (4 points) Evaluate the given integral. Please show all of your work.

\[ \int \frac{\ln(t)}{t^5} \, dt \]

Apply integration by parts:

Let \( u = \ln(t) \) and \( du = \frac{dt}{t} \)

\[ dv = \frac{dt}{t^5} \]

So, \( v = \int \frac{dt}{t^5} = \frac{-1}{4t^4} \)

So then:

\[ \int \frac{\ln(t)}{t^5} \, dt = \int u \, dv = uv - \int v \, du \]

\[ = \left( \frac{-1}{4t^4} \right) (\ln(t)) - \int \frac{-1}{4t^4} \left( \frac{dt}{t} \right) \]

\[ = \frac{-\ln(t)}{4t^4} + \int \frac{dt}{4t^5} \]

\[ = \frac{-\ln(t)}{4t^4} + \frac{1}{4} \left( \frac{-1}{4t^4} \right) + C \]

\[ = \left[ \frac{-\ln(t)}{4t^4} - \frac{1}{16t^4} + C \right] \]
2. (4 points) Evaluate the given integral. Please show all of your work.

\[ \int \sin^6(x) \cos^3(x) \, dx \]

Apply trig identities:

We will use the substitution

\[ \begin{align*}
\begin{cases}
  u = \sin(x) \\
  du = \cos(x) \, dx
\end{cases}
\end{align*} \]

hence we use the identity

\[ \sin^2(x) + \cos^2(x) = 1 \]

to set up the integral for this sub.:

\[ \int \sin^6(x) \cos^3(x) \, dx = \int \sin(x) \cos^2(x) \cos(x) \, dx \]

\[ = \int \sin(x) (1 - \sin^2(x)) \cos(x) \, dx \]

\[ = \int u^6 (1 - u^2) \, du \]

\[ = \frac{u^7}{7} - \frac{u^9}{9} + C \]

\[ \downarrow \text{ re-sub} \]

\[ = \frac{\sin^7(x)}{7} - \frac{\sin^9(x)}{9} + C \]
(a) Find the partial fraction expansion (PFE) of the rational function:

\[
\frac{5x^2 + 2x - 1}{(3x - 9)(x + 2)^2} = \frac{A}{3x - 9} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}
\]

\[
5x^2 + 2x - 1 = A(x + 2)^2 + B(3x - 9)(x + 2) + C(3x - 9)
\]

Multiply both sides by common denominator

Consider the following values:

\[x = 3\]: \[5(3)^2 + 2(3) - 1 = A(3+2)^2 + B(0) + C(0)\]

\[
45 + 6 - 1 = 25A \implies 50 = 25A \implies A = 2
\]

\[x = -2\]: \[5(-2)^2 + 2(-2) - 1 = A(0) + B(0) + C(-6-9)\]

\[
20 - 4 - 1 = -15c \implies -15 = -15c \implies C = -1
\]

\[x = 0\]: \[0 + 0 - 1 = A(2)^2 + B(-9)(2) + C(-9)\]

\[
-1 = 4A - 18B - 9C \implies -1 = 4(2) - 18B - 9(-1) \implies -1 - 8 - 9 = -18B \implies B = 1
\]

Problem continued on next page
(b) Evaluate the integral:

\[
\int \frac{5x^2 + 2x - 1}{(3x - 9)(x + 2)^2} \, dx
\]

\[
= \int \frac{A}{3x-9} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \, dx
\]

\[
= A \int \frac{dx}{3x-9} + B \int \frac{2x}{x+2} + C \int \frac{dx}{(x+2)^2}
\]

\[
\begin{align*}
&\quad \quad \text{(sub)} \quad \quad \text{(sub)} \quad \quad \text{(sub)} \\
&u = 3x - 9 \quad \quad \quad v = x + 2 \quad \quad \quad w = x + 2 \\
&du = 3 \, dx \quad \quad \quad dv = dx \quad \quad \quad dw = dx
\end{align*}
\]

\[
= A \int \frac{du}{3u} + B \int \frac{dv}{v} + C \int \frac{dw}{w^2}
\]

\[
= \frac{A}{3} \ln |u| + B \ln |v| + \frac{C}{w} + \text{Const.}
\]

\[
= \frac{A}{3} \ln |3x-9| + B \ln |x+2| + \frac{C}{x+2} + \text{Const.}
\]

\[
= \frac{2}{3} \ln |3x-9| + \ln |x+2| + \frac{1}{x+2} + \text{Const.}
\]
4. (4 points) Determine if the improper integral converges or diverges. If it converges, find its value.

\[
\int_1^\infty x^2 e^{-x^3} \, dx
\]

\[
\int_1^\infty x^2 e^{-x^3} \, dx = \lim_{t \to \infty} \int_1^t x^2 e^{-x^3} \, dx
\]

Now we evaluate the integral inside, with FTC 2

\[
\int x^2 e^{-x^3} \, dx = \int e^{-u} \frac{du}{3} = \frac{-e^{-u}}{3} + C
\]

\[u = x^3, \quad du = 3x^2 \, dx\]

\[
\int_1^t x^2 e^{-x^3} \, dx = \left. \frac{-e^{-u}}{3} \right|_{1}^{t} + \frac{e^{-1}}{3}
\]

\[
= \frac{-e^{-t^3}}{3} + \frac{1}{3e}
\]

hence:

\[
\int_1^\infty x^2 e^{-x^3} \, dx = \lim_{t \to \infty} \int_1^t x^2 e^{-x^3} \, dx
\]

\[
= \lim_{t \to \infty} \left( \frac{-e^{-t^3}}{3} + \frac{1}{3e} \right) = 0 + \frac{1}{3e} = \frac{1}{3e}
\]
5. (6 points)

(a) On the axes provided below, sketch the region enclosed by the graphs of \( y = 2x + 1 \) and \( y = x^2 - 2 \).

Intersections:
\[
\begin{align*}
& x^2 - 2 = 2x + 1 \\
\Rightarrow & x^2 - 2x - 3 = 0 \\
\Rightarrow & (x + 1)(x - 3) = 0 \\
\Rightarrow & x = -1, 3
\end{align*}
\]

(b) Find the area of the region enclosed by the graphs of \( y = 2x + 1 \) and \( y = x^2 - 2 \).

\[
\text{Area} = \int_{-1}^{3} [(2x + 1) - (x^2 - 2)] \, dx
\]

\[
= \int_{-1}^{3} (-x^2 + 2x + 3) \, dx
\]

\[
= \left[ \frac{-x^3}{3} + x^2 + 3x \right]_{-1}^{3}
= \left[ \frac{-3^3}{3} + 3^2 + 3 \cdot 3 \right] - \left[ \frac{-1}{3} + 1 - 3 \right]
= \left[ -9 + 9 + 9 \right] - \left[ \frac{1}{3} - 2 \right]
= \left[ 9 \right] - \left[ -\frac{5}{3} \right]
= 9 + \frac{5}{3}
= \frac{28}{3}
\]

\boxed{\frac{28}{3}}
6. (4 points) Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis:

\[ y = \frac{1}{(5 - x)^2}, \quad x = 0, \quad x = 3, \quad y = 0 \]

In case you find it helpful, the graph of \( y = \frac{1}{(5 - x)^2} \) is provided below.

Volume of one slice = \( \pi \left( \frac{1}{(5 - x)^2} \right)^2 \, dx \)

Volume of figure = \( \int_{0}^{3} \pi \left( \frac{1}{(5 - x)^2} \right)^2 \, dx \)

\[ = \int_{0}^{3} \pi \frac{1}{(5 - x)^4} \, dx = \pi \int_{0}^{3} \frac{1}{u^4} \, du \]

Sub \( u = 5 - x \), \( du = -dx \)

\( u(0) = 5 \quad u(3) = 2 \)

\[ = \pi \left[ \frac{1}{3u^3} \right]_{5}^{2} = \frac{\pi}{3} \left[ \frac{1}{8} - \frac{1}{125} \right] \]