WEEK 3

1. Integrability

Q: Given a function $f(x)$ and an interval $a \leq x \leq b$, how do we know that

$$\lim_{N \to \infty} R_N = \lim_{n \to \infty} L_N$$

That is, how do we know the limit of the different Riemann sums agree?

A: It isn't always true.... (swept under the rug before)

Defn: functions $f(x)$ where all Riemann sum limits agree on an interval $a \leq x \leq b$ is said to be integrable on $a \leq x \leq b$.
Remark: the all in the above definition includes more than just $\bar{L}_N$ and $\bar{R}_N$.

We can write a general Riemann sum as follows:

→ divide $a \leq x \leq b$ into $N$ equal length sub-intervals:

\[ a = x_0 < x_1 < x_2 < \cdots < x_N = b \]

→ Let $p_n$ be any point in the $n^{th}$ sub-interval:

\[ a = x_0 < x_1 < x_2 < \cdots < x_N = b \]

\[ x_0 < p_1 < p_2 < p_3 < \cdots < p_N < x_N \]

\[ S_N = \sum_{n=1}^{N} \frac{(b-a)}{N} f(p_n) \]

So, if $f(x)$ is integrable on $[a \leq x \leq b]$

Then \( \int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} S_N \)

for any $S_N$
1. Note that we can build
   - $L_N$ by choosing $p_n$ to be left endpoints
   - $R_N$ by choosing $p_n$ to be right endpoints
   - Midpoint rule $M_N$ by choosing points $p_n$ to be the midpoints

The point is, regardless of how we build the sum $S_N$, when we take the limit $N \to \infty$ we should always get the same number!

2. Clearly this is a difficult condition to verify; however, we have the following:

**Theorem**

Continuous functions are integrable
So to find \( \int_{a}^{b} f(x) \, dx \)
we can use whichever Riemann sum is most convenient as long as \( f(x) \) is integrable.

Exercise:
(\[
\lim_{N \to \infty} L_N = \lim_{N \to \infty} R_N
\]
for the function \( f(x) = x^2 \))

\[ N \to \infty \]

\[ N \to \infty \]

we should expect this since polynomials are continuous!
2. Fundamental Theorem of Calculus

We've seen that computing integrals via limits of Riemann sums can be very difficult, even for simple functions.

Now we are ready to introduce a powerful tool for integration:

**Fundamental Theorem of Calculus**

If \( f(x) \) is continuous on \( a \leq x \leq b \) and \( F(x) \) is an antiderivative of \( f(x) \), then:

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

1) Notice that FTC works for any antiderivative \( F(x) \); this is because we lose any constants:

\[
(F(b) + C) - (F(a) + C) = F(b) - F(a) + C - C = F(b) - F(a)
\]
Remarks:

1. The book calls this the "Evaluation Theorem" in §5.3.

2. Notation: the symbol $\int f(x) \, dx$ is used to represent the general antiderivative

   \[(\text{e.g., } f(x) = x^2 \Rightarrow \int f(x) \, dx = \frac{x^3}{3} + C)\]

   and is called an "indefinite integral".

   Both names refer to the same thing.

4. Now we see the importance of the antiderivative!

   See the following table.
# Table of general antiderivatives

<table>
<thead>
<tr>
<th>$f(x) = x^n$</th>
<th>$F(x) = \frac{x^{n+1}}{n+1} + C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^x$</td>
<td>$e^x + C$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$a^x$</td>
<td>$\frac{a^x}{\ln(a)} + C$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$-\cos(x) + C$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sin(x) + C$</td>
</tr>
<tr>
<td>$\sec^2(x)$</td>
<td>$\tan(x) + C$</td>
</tr>
<tr>
<td>$\csc^2(x)$</td>
<td>$-\cot(x) + C$</td>
</tr>
<tr>
<td>$\sec(x) \cdot \tan(x)$</td>
<td>$\sec(x) + C$</td>
</tr>
<tr>
<td>$\csc(x) \cdot \cot(x)$</td>
<td>$-\csc(x) + C$</td>
</tr>
<tr>
<td>$\frac{1}{x^2 + 1}$</td>
<td>$\arctan(x) + C$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\arcsin(x) + C$</td>
</tr>
</tbody>
</table>
Examples:

1. \[ \int_2^4 \sqrt{x} \, dx \]

See that \( F(x) = \frac{2}{3} x^{3/2} \) is an antiderivative.

\[
F'(x) = \left( \frac{2}{3} \right) \cdot \left( \frac{3}{2} \right) x^{1/2} = x^{1/2} = \sqrt{x}
\]

So apply FTC:

\[
\int_2^4 \sqrt{x} \, dx = F(4) - F(2) = \frac{2}{3} (4)^{3/2} - \frac{2}{3} (2)^{3/2} = \frac{16}{3} - 4 = \frac{16 - 12}{3} = \frac{4}{3}
\]
\( \int_{-1}^{2} (x^2 + e^x) \, dx \)

\[
= \int_{-1}^{2} x^2 \, dx + \int_{-1}^{2} e^x \, dx \quad \text{by linearity}
\]

Now, \( F(x) = \frac{x^3}{3} \) and \( G(x) = e^x \) are antiderivatives for \( x^2 \) and \( e^x \), respectively.

So apply \( \text{FTC} \).

\[
\int_{-1}^{2} x^2 \, dx + \int_{-1}^{2} e^x \, dx = \int_{-1}^{2} x^2 \, dx + \int_{-1}^{2} e^x \, dx
\]

\[
= F(2) - F(-1) + G(2) - G(-1)
\]

\[
= \frac{8}{3} - \frac{(-1)^3}{3} + e^2 - e^{-1}
\]

\[
= \left[ \frac{8}{3} + e^2 - \frac{1}{e} \right]
\]
\[ 3 \int_0^{2\pi} 2 \sin(x) \, dx \]

\[ = 2 \int_0^{2\pi} \sin(x) \, dx \]

By linearity; now, \( F(x) = -\cos(x) \)

is an antiderivative of \( \sin(x) \),

so by FTC:

\[ = 2 \left( F(2\pi) - F(\pi) \right) \]

\[ = 2 \left( -\cos(2\pi) - (-\cos(\pi)) \right) \]

\[ = 2 (-1 + (-1)) = 2 (-2) \]

\[ = -4 \]
4. \( \int_{0}^{2\pi} |\cos(x)| \, dx \)

We'll look at \( |\cos(x)| \) as a piecewise function:

\[
|\cos(x)| = \begin{cases} 
\cos(x) & \text{if } \cos(x) \geq 0 \\
-\cos(x) & \text{if } \cos(x) < 0 
\end{cases}
\]

Now, on \( 0 \leq x \leq 2\pi \), we know that:

\[
\cos(x) < 0 \iff \frac{\pi}{2} < x < \frac{3\pi}{2}
\]

and \( \cos(x) \geq 0 \iff x \leq \frac{\pi}{2} \) or \( x \geq \frac{3\pi}{2} \)

check using the graph...

So:

\[
|\cos(x)| = \begin{cases} 
\cos(x) & \text{if } x \leq \frac{\pi}{2} \text{ or } x \geq \frac{3\pi}{2} \\
-\cos(x) & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2}
\end{cases}
\]
So now we'll apply interval splitting using the piecewise intervals:

\[
\int_0^{2\pi} |\cos(x)| \, dx
\]

\[
= \int_0^{\pi} \cos(x) \, dx + \int_{\pi}^{3\pi/2} \cos(x) \, dx + \int_{3\pi/2}^{2\pi} \cos(x) \, dx
\]

\[
= \int_0^{\pi} \cos(x) \, dx + \int_{\pi}^{3\pi/2} \cos(x) \, dx + \int_{3\pi/2}^{2\pi} \cos(x) \, dx
\]

by piecewise definition

⇒ Now, \( F(x) = \sin(x) \) is an antiderivative for \( \cos(x) \), so apply FTC:

\[
= (F(\pi) - F(0)) + (F(\pi) - F(3\pi/2)) + (F(2\pi) - F(\pi))
\]

(remember the negative!)

\[
= (1 - 0) - (1 - 1) + (0 - (-1))
\]

\[
= 1 - (-2) + (1) = 4
\]
3. Finding antiderivatives with FTC

Given a function that is integrable, we can write a new function using its integral; for example

\[ F(x) = \int_0^x f(t) \, dt \]

Consider the graph \( y = f(t) \)

Then \( F(x) = \text{Area of the shaded region} \)

Notice that after \( x = 3 \) the function must be decreasing as negative area is being added.

If \( F(x) \) is increasing before \( x = 3 \) and decreasing after \( x = 3 \), what sort of point is \( x = 3 \) for \( F(x) \)?
Example 1: Consider the graph $y = f(t)$ below:

$\begin{array}{c}
\text{\includegraphics{graph.png}}
\end{array}$

and define $F(x) = \int_0^x f(t) \, dt$
on the interval $0 \leq x \leq 4$

(a) What are the local extrema in the interval?

(b) What is the global max?

(a) Since $F(x) = \text{area under } y = f(t) \text{ over } t$

we see that:

- $F(x)$ is increasing in $0 \leq x < 1$
- $F(x)$ is decreasing in $1 < x \leq 2$
- $F(x)$ is increasing in $2 < x \leq 4$

hence by the first derivative test we have

local max @ $x = 1$

and local min @ $x = 2$
(b) The global max of $F(x)$ is the $x$ value for which the area under $y = f(t)$ over $0 \leq t \leq x$ is maximized.

Therefore we want to pick the $x$ value such that the integral

$$\int_0^x f(t) \, dt$$

passes over the most possible positive area and the least possible negative area.

It is easy to see that $F(2) = 0$ and $F(x) > 0$ for all $x > 2$, hence the global max occurs at the right endpoint: $x = 4$. 
Functions of this kind allow us to compute antiderivatives using integrals by the following version of FTC:

**Fundamental Theorem of Calculus, again**

Let \( f(x) \) be continuous on \( a \leq x \leq b \) and define the function:

\[
F(x) = \int_{a}^{x} f(t) \, dt
\]

Then this is an antiderivative of \( f(x) \):

\[
F'(x) = f(x)
\]

Remark: the book calls the above theorem FTC1 and the previously stated theorem as FTC2. We will use those terms in class when referring to the two versions of the FTC.
Examples:

\( g(x) = \int_3^x \ln(t) \, t^2 \, dt \); \( g'(x) = ? \)

By FTC 1,
\[
 g'(x) = \ln(x) \times x^2
\]

\( p(x) = \int_x^4 \frac{\sin(5t)}{t} \, dt \); \( p'(x) = ? \)

See that FTC 1 does not apply to \( p(x) \) as written. First we must apply interval reversal:
\[
 p(x) = \int_x^4 \frac{\sin(5t)}{t} \, dt = -\int_4^x \frac{\sin(5t)}{t} \, dt
\]

Don't forget the negative.

Now we can apply FTC 1:
\[
 p'(x) = -\frac{\sin(5x)}{x}
\]
Again, we need to be careful:
- FTC 1 only applies to functions with $x$ as the upper bound for the integral.

Hence here we must use the chain rule:

Note that if we let

$$f(x) = \int_1^x \cos^2(t) \, dt$$

and $g(x) = x^3$

Then $g(x) = f(g(x)) = f(g(x))$

and so:

$$g'(x) = f'(g(x)) \cdot g'(x)$$

Now, FTC 1 does apply to $f(x)$, so we have

$$f'(x) = \cos^2(x)$$

and $g'(x) = 3x^2$ by the power rule.

So:

$$g'(x) = \cos^2(0x^3) \cdot 3x^2$$
Substitution Rule:

Suppose we encounter a function which we do not have an obvious antiderivative rule, e.g.:

\[ \int_0^1 x \sqrt{1-x^2} \, dx \]

To solve this problem we can make a change of variables:

Let \( u = 1 - x^2 \) (x)

But what about the \( dx \)? We also need to "tell" the integral that \( u \) is the new variable of integration, i.e. we must also change to \( du \).

We do this by using the following observation:

\[ du = \frac{du}{dx} \, dx \]

Hence in this case: \( du = (-2x) \, dx \)

\[ \Rightarrow \frac{du}{2} = x \, dx \]

\((**)\)
Finally, we must also change the endpoints:

\[ x = 0 \rightarrow u = 1 - x^2 = 1 \]  
\[ x = 1 \rightarrow u = 1 - x^2 = 0 \]  

Now we rewrite \( \int_0^1 x \sqrt{1-x^2} \, dx \)
using the substitutions (x), (**), and (***):

\[
\int_0^1 x \sqrt{1-x^2} \, dx \\
= \int_0^1 \sqrt{1-x^2} \, x \, dx \\
= \int_0^1 \sqrt{u} \left( -\frac{du}{2} \right) \\
\text{(sub)} \\
= -\frac{1}{2} \int_0^1 \sqrt{u} \, du \\
\rightarrow \text{This integral we can solve using FTC2 because we know an antiderivative of } \sqrt{u} \text{ is } F(u) = \frac{2}{3} u^{3/2} \\
= -\frac{1}{2} (F(0) - F(1)) \\
= -\frac{1}{2} \left( \frac{2}{3} \cdot 0^{3/2} - \frac{2}{3} (1)^{3/2} \right) = \frac{1}{3}
Examples:

1. \( \int_0^1 (2x-1)^{99} \, dx \)

Let \( u = 2x - 1 \)

Then \( du = 2 \, dx \Rightarrow \frac{du}{2} = dx \)

Also,
\[
\begin{align*}
  u(0) &= 2(0) - 1 = -1 \\
  u(1) &= 2(1) - 1 = 1
\end{align*}
\]

Now apply these substitutions:

\[
\int_0^1 (2x-1)^{99} \, dx = \int_{-1}^{1} u^{99} \left( \frac{du}{2} \right)
\]

\[
= \frac{1}{2} \int_{-1}^{1} u^{99} \, du
\]

\[
= \frac{1}{2} \left( \frac{1^{100}}{100} - \frac{(-1)^{100}}{100} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{100} - \frac{1}{100} \right) = \boxed{0}
\]
\[ 2 \int_{0}^{\frac{3\pi}{2}} \sin(3\pi t) \, dt \]

Let \( u = 3\pi t \) \( \Rightarrow \) \( du = 3\pi \, dt \)

\[ \Rightarrow \frac{du}{3\pi} = dt \]

And

\[ u(0) = 3\pi(0) = 0 \]
\[ u\left(\frac{3\pi}{2}\right) = 3\pi\left(\frac{3\pi}{2}\right) = \frac{3\pi^2}{2} \]

Now:

\[ \int_{0}^{\frac{3\pi}{2}} \sin(3\pi t) \, dt = \int_{0}^{\frac{3\pi^2}{2}} \sin(u) \, \frac{du}{3\pi} \]

\[ = \frac{1}{3\pi} \int_{0}^{\frac{3\pi^2}{2}} \sin(u) \, du \]

\[ = \frac{1}{3\pi} \left( -\cos\left(\frac{3\pi^2}{2}\right) - (-\cos(0)) \right) \]

\[ = \frac{1}{3\pi} \left( -\cos\left(\frac{3\pi^2}{2}\right) + 1 \right) \]
\[ \int_{1}^{2} \frac{\ln(x)^5}{x} \, dx \]

Let \( u = \ln(x) \Rightarrow du = \frac{1}{x} \, dx \)

and \( u(1) = \ln(1) = 0 \)

\( u(2) = \ln(2) \)

So now:

\[
\int_{1}^{2} \frac{\ln(x)^5}{x} \, dx = \int_{0}^{\ln(2)} u^5 \, du
\]

\[
= \left( \frac{\ln(2)^6}{6} - \frac{0^6}{6} \right) = \frac{\ln(2)^6}{6}
\]
(5) Even and odd functions

Functions with symmetry are often easier to integrate:

Example 1: \( y = \sin(x) \) over \(-\pi \leq x \leq \pi\)

![Graph of \( \sin(x) \)]

Clearly the negative and positive areas are the same size, so:

\[
\int_{-\pi}^{\pi} \sin(x) \, dx = 0
\]

Example 2: \( y = \cos(x) \) over \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\)

Clearly here the area over \(0 \leq x \leq \frac{\pi}{2}\) is the same as the area over \(-\frac{\pi}{2} \leq x \leq 0\)

![Graph of \( \cos(x) \)]
Then we have:

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) \, dx = 2 \int_{0}^{\frac{\pi}{2}} \cos(x) \, dx
\]

\[= 2 \int_{-\frac{\pi}{2}}^{0} \cos(x) \, dx\]

\[\Rightarrow \sinh \text{ and cosh are the main examples of odd and even functions.}\]

**Definition:** Suppose \( f(x) \) is continuous.

- \( f(x) \) is **odd** if \( f(-x) = -f(x) \) for all \( x \)

- \( f(x) \) is **even** if \( f(-x) = f(x) \) for all \( x \)

\[\Rightarrow \text{The integrals of even and odd functions over symmetric intervals, i.e. } -a \leq x \leq a, \text{ have a nice form:}\]

\[\int_{-a}^{a} f(x) \, dx = 0 \quad \text{for odd} \quad f(x) \]

\[\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \quad \text{for even} \quad f(x) \]
example: consider \( \int_{-35}^{35} x^{37} \, dx \)

check first the symmetry of the function \( x^{37} \):

\((-x)^{37} = (-1)^{37} x^{37} = -x^{37}\)

hence \( x^{37} \) is ODD

Therefore \( \int_{-35}^{35} x^{37} \, dx = 0 \)

Since \(-35 \leq x \leq 35\) is symmetric interval around 0.