(1.) Approximating integrals (§5.9)

The chance that any given continuous function has a "closed form" antiderivative (i.e., that we can write $F(x)$ as a function of $x$ explicitly) is very small!

\[ \text{e.g., } \int_0^1 e^{-x^2} \, dx \text{ cannot be evaluated using FTC2 because we cannot write a function } F(x) \text{ such that } F'(x) = e^{-x^2} \]

In such cases we must approximate $\int_0^1 f(x) \, dx$ using the definition of integration via Riemann sums.
Recall:

Let $f(x)$ be continuous on the closed interval $a \leq x \leq b$.

Let the interval $a \leq x \leq b$ be subdivided into $N$ sub-intervals of equal length $\frac{b-a}{N}$.

Then we define the \underline{left} and \underline{right} Riemann sums given by areas of the $N$ rectangles over each subinterval:

$L_N = \sum_{n=0}^{N-1} \left( \frac{b-a}{N} \right) f\left( a + \left( \frac{b-a}{N} \right) n \right)$

- width of sub-interval = width of $n^{th}$ rectangle
- height of $n^{th}$ rectangle from left endpoint of the $n^{th}$ subinterval

$R_N = \sum_{n=0}^{N-1} \left( \frac{b-a}{N} \right) f\left( a + \left( \frac{b-a}{N} \right)(n+1) \right)$

(same) - $\text{right endpt.}$
Remark: we also can write \( R_N \) using a different index of \( n \):

\[
R_N = \left( \sum_{n=1}^{N} \frac{b-a}{N} f\left(a + \frac{(b-a)}{N} n\right) \right)
\]

Note the change.

However, we use the \( n = 0, \ldots, N-1 \) index so our other formulas use the same index.

Now we can define two new Riemann sums that are given by the areas of the \( N \) trapezoids over each endpoint:

\[
M_N = \sum_{n=0}^{N-1} \left( \frac{b-a}{N} \right) f\left(a + \frac{(b-a)}{N}(\frac{2n+1}{2})\right)
\]

- width of \( n^{th} \) subinterval
- height of rectangle from midpoint of \( n^{th} \) subinterval
- "average height" of tangent line to trapezoid at the midpoint of \( n^{th} \) subinterval

This is the Midpoint Sum, the Riemann sum with rectangles whose heights are given
the midpoints of each sub-interval

However, these rectangles have the same area as the trapezoids associated with the tangent line to $f(x)$ at the midpoints; see the following picture:

\[
\text{(e.g.) } M_1 = \frac{a+b}{2} \quad \text{heights} \quad f(\frac{a+b}{2}) \quad f(\frac{a+b}{2}) \quad f(\frac{a+b}{2}) - L \quad f(\frac{a+b}{2}) + L
\]

\[
\text{area of rectangle: } f(\frac{a+b}{2})(b-a) \quad \text{area of trapezoid: } \frac{1}{2}(f(\frac{a+b}{2})-L+f(\frac{a+b}{2}+L)(b-a) \quad \Rightarrow \quad \text{equal areas}
\]
We also define the *Trapezoid Sum* which is given by the trapezoids over each subinterval associated with the secant line for $f(x)$ over the subinterval; see the picture:

$$\text{e.g. } T_1 = \sum_{n=0}^{N-1} \left( \frac{b-a}{N} \right) \left[ \frac{f(a + \left( \frac{b-a}{N} \right) n) + f\left(a + \left( \frac{b-a}{N} \right) (n+1)\right)}{2} \right]$$

The general formula is:

- **width of $n$th trapezoid**
- **height of left side of $n$th trapezoid**
- **height of right side of $n$th trapezoid**
All four Riemann sums $R_N, L_N, M_N, T_N$ all increase in accuracy as $N$ approaches $\infty$.

Letting $I = \int_a^b f(x)\,dx$ the actual value of the integral, we can see in advance which sums are over/under-estimaters and which are most accurate; see the following:

**If Increasing**

$f_1 < T_1 < I < M_1 < R_1$

**If Decreasing**

$R_1 < T_1 < I < M_1 < L_1$
(2) Improper integrals (S5.10)

\[ \int_a^b f(x) \, dx \] is improper if any of the points in the interval \( a \leq x \leq b \) are outside the domain of \( f \)

\[ \Rightarrow \text{i.e., where } f(x) \text{ includes dividing by } 0, \text{ or if } a \text{ or } b \text{ is } \pm \infty \]

We can still evaluate such integrals using limits; for example, suppose \( b \) is not in the domain of \( f \). Then:

\[ \int_a^b f(x) \, dx := \lim_{t \to b} \int_a^t f(x) \, dx \]

Now we may evaluate the integral since working inside the limit ensures we stay away from the "bad point" at \( b \).
Examples

(a) Is \( \int_0^1 \frac{dx}{3x-1} \) an improper integral?

See that \( f(x) = \frac{1}{3x-1} \) has domain all real numbers except \( x = \frac{1}{3} \).

This is in the interval \( 0 \leq x \leq 1 \), hence

\[ \text{yes} \] this integral is improper.

How do we set it up?

\[ \Rightarrow \text{Split the integral @ } x = \frac{1}{3} : \]

\[ \int_0^1 \frac{dx}{3x-1} = \int_0^{1/3} \frac{dx}{3x-1} + \int_{1/3}^1 \frac{dx}{3x-1} \]

Now we can write it using limits:

\[ \int_0^1 \frac{dx}{3x-1} = \lim_{t \to 1/3^-} \left( \int_0^t \frac{dx}{3x-1} \right) + \lim_{t \to 1/3^+} \left( \int_t^1 \frac{dx}{3x-1} \right) \]

\[ \text{Exercise: Show the integral diverges to } \infty \]
1. Evaluate $\int_0^\infty \frac{x \, dx}{(x^2 + 1)^2}$

\[ \lim_{t \to \infty} \left( \int_0^t \frac{x \, dx}{(x^2 + 1)^2} \right) \]

Substitute:

\[ u = x^2 + 1 \]
\[ du = 2x \, dx \]
\[ u(t) = t^2 + 1 \]
\[ u(0) = 1 \]

\[ \lim_{t \to \infty} \left( \int_1^{t^2 + 1} \frac{1}{u^2} \, du \right) \]

\[ = \frac{1}{2} \lim_{t \to \infty} \left( \int_1^{t^2 + 1} \frac{du}{u^2} \right) \]

Solve the integral:

\[ = \frac{1}{2} \lim_{t \to \infty} \left( \frac{1}{(t^2 + 1)} - \frac{1}{1} \right) \]

Using FTC 2:

\[ F(u) = \frac{-1}{u} \]

\[ = \frac{1}{2} \lim_{t \to \infty} \left( 1 - \frac{1}{t^2 + 1} \right) \]

Solve the resulting limit:

\[ = \frac{1}{2} - \frac{1}{2} \lim_{t \to \infty} \frac{1}{t^2 + 1} \]

Recall $\lim_{x \to \infty} \frac{1}{x} = 0$

\[ = \frac{1}{2} \]

\[ \boxed{\frac{1}{2}} \]
\[ \int_{-\infty}^{0} \frac{dw}{\sqrt{1-w}} \]

\[ \lim_{t \to -\infty} \int_{t}^{0} \frac{dw}{\sqrt{1-w}} = \lim_{t \to -\infty} \int_{1-t}^{1} \frac{-du}{\sqrt{u}} \]

\[ \text{setup} \]

\[ \text{Substitute:} \]
\[ u = 1-w \]
\[ du = -dw \]
\[ u(0) = 1 \]
\[ u(t) = 1-t \]

\[ \text{solve integral:} \]
\[ \text{with FTC2} \]
\[ \text{and antiderivative} \]
\[ F(u) = 2\sqrt{u} \]

\[ \lim_{t \to -\infty} \left[ 2(1 - \sqrt{1-t}) \right] \]

\[ = \lim_{t \to -\infty} \left[ 2(1 - \sqrt{1+t}) \right] \]

\[ \text{now it is clear that the limit diverges to } -\infty \]

\[ \text{so this integral diverges} \]
\[ \int_{-\infty}^{-1} e^{-2x} \, dx \]

\[ = \lim_{t \to -\infty} \int_{0}^{t} e^{-2x} \, dx \]

\[ = \lim_{t \to \infty} \int_{-2t}^{2} e^{u} \left( \frac{du}{-2} \right) \]

Sub:
\[ u = -2x \]
\[ du = -2 \, dx \]
\[ u(-1) = 2 \]
\[ u(t) = -2t \]

\[ \quad \Rightarrow \quad = \lim_{t \to \infty} \int_{-2t}^{2} e^{u} \, du \]

\[ = \frac{1}{2} \lim_{t \to \infty} \left[ e^{2} - e^{-2t} \right] \]

Evaluate the integral inside:
\[ = \frac{1}{2} \left( e^{2} - \lim_{t \to \infty} e^{-2t} \right) \]

Evaluate limit:
\[ = \frac{1}{2} \left( e^{2} - \infty \right) \]

\[ = \frac{-e^{2}}{2} \]
\[ \int_2^3 \frac{dx}{\sqrt{3-x}} = \lim_{t \to 3} \left( \int_2^t \frac{dx}{\sqrt{3-x}} \right) \\
= \lim_{t \to 3} \left( \left[ \frac{2\sqrt{u}}{u} \right]_1^{3-t} \right) \\
= \lim_{t \to 3} \left[ -(2\sqrt{3-t}) - 2 \right] \\
= \left( \lim_{t \to 3} \sqrt{2(3-t)} - \lim_{t \to 3} 2 \right) \\
= -(-2) = 2 \]
\( \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \)

Remarks: Since the endpoints are \( \pm \infty \) and 0, we must split the integral.
Since our definition is for one undefined endpoint.

\[
= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}
\]

\[
= \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{1+x^2} + \lim_{s \to \infty} \int_{0}^{s} \frac{dx}{1+x^2}
\]

\[
= \lim_{t \to -\infty} \left[ \arctan(x) \right]_{t}^{0} + \lim_{s \to \infty} \left[ \arctan(x) \right]_{0}^{s}
\]

\[
= \lim_{t \to -\infty} \left[ \arctan(0) - \arctan(t) \right] + \lim_{s \to \infty} \left[ \arctan(s) - \arctan(0) \right]
\]

\[
= \lim_{t \to -\infty} (-\arctan(t)) + \lim_{s \to \infty} (\arctan(s))
\]

\[
= -\left( -\frac{\pi}{2} \right) + \frac{\pi}{2} = \frac{\pi}{2}
\]
Remark on $\lim \arctan(x)$:

Recall the graph of $\tan(x)$:

\[
\begin{align*}
\lim_{x \to \frac{\pi}{2}} \tan(x) &= +\infty \\
\lim_{x \to -\frac{\pi}{2}} \tan(x) &= -\infty
\end{align*}
\]

So the graph of $\arctan(x)$, the inverse of $\tan(x)$, is given by:

\[
\begin{align*}
\arctan(x) &= \frac{\pi}{2} \quad (x \to \infty) \\
\arctan(x) &= -\frac{\pi}{2} \quad (x \to -\infty)
\end{align*}
\]

and now it is easy to see that:

\[
\begin{align*}
\lim_{x \to \infty} \arctan(x) &= \frac{\pi}{2} \\
\lim_{x \to -\infty} \arctan(x) &= -\frac{\pi}{2}
\end{align*}
\]
\[
\int_1^\infty \frac{dx}{x^7} = \lim_{t \to \infty} \int_1^t \frac{dx}{x^7} = \lim_{t \to \infty} \left[ \frac{-1}{6x^6} \right]_1^t
\]
\[
= \lim_{t \to \infty} \left( \frac{-1}{6t^6} + \frac{1}{6(1)^6} \right)
= \lim_{t \to \infty} \left( \frac{-1}{6t^6} \right) + \lim_{t \to \infty} \left( \frac{1}{6} \right)
= 0 + \frac{1}{6} = \frac{1}{6}
\]

\rightarrow \text{There is a general rule which tells us, in general, when an integral like the above is convergent:}

**INTEGRAL P-TEST**

\[
\int_1^\infty \frac{dx}{x^p} \text{ converges if } p > 1 \\
\text{and diverges if } p \leq 1
\]
\[ \int_1^\infty \frac{x \, dx}{1 + x^2} = \lim_{t \to \infty} \left( \int_1^t \frac{x \, dx}{1 + x^2} \right) = \lim_{t \to \infty} \left( \int_{u(1)}^{u(t)} \frac{1}{u} \, (2 \, dx) \right) \]

Substitution: 
\[ u = 1 + x^2 \quad u(1) = 1 + 1^2 = 2 \]
\[ du = 2 \, x \, dx \quad u(t) = 1 + t^2 \]

\[ = \lim_{t \to \infty} \left( \frac{1}{2} \int_2^{1 + t^2} \frac{du}{u} \right) \]
\[ = \frac{1}{2} \lim_{t \to \infty} \left( \ln |1 + t^2| - \ln |2| \right) = + \infty \]

(since \( \ln |1 + t^2| \) is increasing without bound for all \( t \))
→ Sometimes we just want to know whether a given improper integral converges, i.e. is less than oo.

→ For this purpose, we have:

\[ \text{COMPARISON TEST} \]

Suppose \( f \) and \( g \) are continuous and \( 0 \leq g(x) \leq f(x) \) for all \( x \geq \alpha \).

Then:

\[ \int_{\alpha}^{\infty} g(x) \, dx \leq \int_{\alpha}^{\infty} f(x) \, dx \]

and therefore:

1. \( \int_{\alpha}^{\infty} f(x) \, dx \) convergent \( \Rightarrow \) \( \int_{\alpha}^{\infty} g(x) \, dx \) convergent

2. \( \int_{\alpha}^{\infty} g(x) \, dx \) divergent \( \Rightarrow \) \( \int_{\alpha}^{\infty} f(x) \, dx \) divergent
Now let's revisit example 7:

7' Is \( \int_1^\infty \frac{\ln x}{x^2} \) convergent?

Since \( x > 1 \), we have \( x^2 > 1 \).

\( x^2 \) is also positive, so we'll add it to both sides:

\[
x^2 + x^2 \geq 1 + x^2
\]

Now we invert:

\[
\frac{1}{2x^2} \leq \frac{1}{1 + x^2}
\]

And multiply both sides by \( x \):

\[
\frac{1}{2x} \leq \frac{x}{1 + x^2}
\]

Now apply comparison:

\[
\int_1^\infty \frac{1}{2x} \, dx \leq \int_1^\infty \frac{x}{1 + x^2} \, dx
\]

\[
\frac{1}{2} \left( \int_1^\infty \frac{dx}{x} \right) \quad \text{this is divergent by the } \frac{1}{x^p} \text{-test!}
\]
therefore, $\int_1^\infty \frac{x}{1+x^2} \, dx$ must also be divergent by comparison!

more examples:

8. Is $\int_0^\infty e^{-x^2} \, dx$ convergent?

To make our comparison we'll split the integral:

\[
\int_0^\infty e^{-x^2} \, dx = \int_0^1 e^{-x^2} \, dx + \int_1^\infty e^{-x^2} \, dx
\]

\[\rightarrow \text{this is a usual integral of a continuous function, hence we know it is finite.}\]

Now consider the function $e^{-x^2}$ for $x > 1$
since $x > 1$, we know $x^2 \geq x$,
 hence $e^{x^2} \geq e^x$, hence:

\[
e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{e^x} = e^{-x}
\]
Now apply comparison:
\[\int_1^\infty e^{-x^2} \, dx \leq \int_1^\infty e^{-x} \, dx\]
\[= \lim_{t \to \infty} \int_1^t e^{-x} \, dx\]
\[= \lim_{t \to \infty} (-e^{-t} - e^{-1})\]
\[= -\lim_{t \to \infty} e^{-t} + \lim_{t \to \infty} e^{-1}\]
\[= e^{-1}\]

hence \(\int_1^\infty e^{-x^2} \, dx\) is convergent

and thus \(\int_1^\infty e^{-x} \, dx\) is convergent by comparison

Hence
\[\int_0^\infty e^{-x^2} \, dx = \int_1^\infty e^{-x^2} \, dx + \int_0^1 e^{-x^2} \, dx\]
\[
\text{convergent, convergent}\]

Therefore \(\int_0^\infty e^{-x^2} \, dx\) is convergent
Important

Remark:

we needed to split the integral because we only get our companion inequality $e^{-x^2} \leq e^x$ for $x \geq 1$.