(1) Area, revisited (§60.1)

Recall the definition of $\int_a^b f(x) \, dx$ via Riemann sums:

$$\int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{n=0}^{N-1} \left( \frac{(b-a)}{N} \right) f(a + n \cdot \frac{(b-a)}{N})$$

and adopt the following notation:

\[
\begin{align*}
\Delta x &:= \frac{b-a}{N} & \text{(width of each rectangle)} \\
X_n &= a + n\Delta x & \text{(endpoints of each subinterval)}
\end{align*}
\]

see: $X_0 = a$  $X_N = b$

then:

$$\int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{n=0}^{N-1} \Delta x \cdot f(X_n)$$
In pictures this is:

\[
\begin{align*}
\text{area under } y = f(x) \text{ over } a \leq x \leq b & \quad = \quad \lim_{N \to \infty} \sum_{n=0}^{N-1} \left( \frac{b-a}{N} \cdot f(x_n) \right) \\
\text{area of } n\text{th rectangle out of } N & \quad \text{sum of the areas of } N \text{ rectangles}
\end{align*}
\]

\[\text{limit as } N \to \infty\]

The notation for integration can be interpreted as shorthand for elements of the Riemann sum when \( N \) is close to infinity:

\[
\left\{ \begin{array}{l}
\lim_{N \to \infty} \Delta x \quad \lim_{n} \Delta x \\
\lim_{N \to \infty} \sum f(x_n) \quad \lim_{n} \sum f(x_n)
\end{array} \right. = \int_a^b f(x) \, dx
\]
then the left picture can be rewritten:

\[ \int_{a}^{b} f(x) \, dx = \text{area of infinitely thin rectangle of height } f(x) \]

infinite sum of infinitely thin rectangles over interval \( a \leq x \leq b \)

( so we change \( \Sigma \) to \( \int \) )
using this formalism we can setup integrals for more complicated areas, such as:

→ Area between curves:

Consider the following picture:

\[
y = f(x) \\
y = g(x)
\]

\[a \quad b \quad x\]

Suppose we want to know the area of the shaded region; we can proceed as before by considering an infinite sum of infinitely thin rectangles:

\[
y = f(x) \\
y = g(x)
\]

\[a \quad dx \quad b \quad x\]
Notice that the height of such a rectangle is different:

\[ f(x) = y \]

\[ g(x) = y \]

So the area of one such rectangle is:

\[ (f(x) - g(x)) \cdot dx \]

and summing (integrating) them we have:

\[ \int_{a}^{b} (f(x) - g(x)) \, dx = \text{area between } f(x) \text{ and } g(x) \text{ over } a \leq x \leq b \]

Make sure you know which function is "top" and which is "bottom".
Examples:

1. Find the area enclosed by
   \[ y = e^x, y = x, x=0, x=1 \]

   **Step 1:** Sketch
   \[ y = e^x \]
   \[ y = x \]
   \[ y = 1 \]
   \[ x = 0 \]
   \[ x = 1 \]

   **Step 2:** Set up integral
   See that \( x < e^x \) on the interval, so:
   \[
   \text{area} = \int_0^1 (e^x - x) \, dx
   \]
   \[
   = \int_0^1 e^x \, dx - \int_0^1 x \, dx
   \]
   \[
   = e - 1 - \frac{1}{2} = \left[ e - \frac{3}{2} \right]
   \]
(2) Find the area enclosed by
\[ y = x^3, \ y = x^2, \ x = 1, \ x = 2 \]

Step 1: sketch

\[ y = x^3, \ y = x^2 \]

\[ x=1 \quad x=2 \]

Step 2: setup integral:

\[ \int_1^2 (x^3 - x^2) \, dx = \int_1^2 x^3 \, dx - \int_1^2 x^2 \, dx \]

\[ = \left( \frac{2^4}{4} - \frac{1^4}{4} \right) - \left( \frac{(2)^3}{3} - \frac{(1)^3}{3} \right) \]

\[ = 2^2 - \frac{1}{4} - \frac{8}{3} + \frac{1}{3} \]

\[ = 4 - \frac{1}{4} - \frac{7}{3} = \sqrt{\frac{15}{4} - \frac{7}{3}} \]
(3) Find the area enclosed by
\[ y = 4x - x^2 \] and \[ y = x \]

Remark: we need to find the intersection points for these two curves to determine the region and bounds of integration.

Step 1: Sketch:

\[ y = x \]
\[ y = 4x - x^2 \]

Step 2: Find intersection points; i.e. \( x \) such that both functions have the same \( y \):

\[ 4x - x^2 = x \implies x = 0 \]

Suppose \( x \neq 0 \):
\[ x(4-x) = x \]
\[ (4-x) = 1 \implies x = 3 \]
Step 3: set up integral:

\[
\int_0^3 [(4x - x^2) - (x)] \, dx
\]

Intersection points: top curve - bottom curve

\[
= \int_0^3 (3x - x^2) \, dx
\]

\[
= \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3
\]

\[
= \frac{3(3)^2}{2} - \frac{3^3}{3} = \frac{27}{2} - \frac{27}{3}
\]

\[
= \frac{27}{6} = \frac{9}{2}
\]
4) Find the area enclosed by 

\[ y = 2x^2 - 5 \quad \text{and} \quad y = -3x^2 + 5 \]

- **Step 1**: sketch

- **Step 2**: find intersection points:
  \[ 2x^2 - 5 = -3x^2 + 5 \]
  \[ \Rightarrow 2x^2 + 3x^2 = 10 \]
  \[ \Rightarrow 5x^2 = 10 \quad \Rightarrow \quad x^2 = 2 \]
  \[ \Rightarrow x = \pm \sqrt{2} \]

- **Step 3**: set up integral:

\[
\int_{-\sqrt{2}}^{\sqrt{2}} \left( -3x^2 + 5 \right) - \left( 2x^2 - 5 \right) \, dx = \int_{-\sqrt{2}}^{\sqrt{2}} (5x^2 + 10) \, dx
\]

\[
= \left[ \frac{-5x^3}{3} + 10x \right]_{-\sqrt{2}}^{\sqrt{2}} = \left( \frac{-5(\sqrt{2})^3}{3} + 10\sqrt{2} \right) - \left( \frac{5(\sqrt{2})^3}{3} - 10\sqrt{2} \right)
\]

\[
= \frac{-5(2)^{3/2}}{3} + 20\sqrt{2} = \frac{-5(2)^{3/2} + 60\sqrt{2}}{3}
\]
2. Volume: Solids of revolution (5.6.2)

We can also use integration to find volumes; consider solid regions obtained by rotating an area about the x-axis.

1) The most basic example is a cylinder!

We obtain a cylinder of radius $R$ and height $(b-a)$ by rotating the shaded region around the x-axis.
Recall the area of such a cylinder:

\[ \pi R^2 (b-a) \]

Notice that the shaded region is precisely \( \int_a^b f(x) \, dx \) for \( f(x) = R \).

What is \( f(x) \) is not constant?

We apply our formalism for area and find the volume of each cylinder associated to each \( dx \)-width rectangle!

E.g.:

\[ \text{Full cylinder is volume:} \]
\[ \int_a^b \pi f(x)^2 \, dx \]
\[ \leq \int_a^b \pi R^2 \, dx \]
\[ \leq \pi R^2 (b-a) \]
More generally:

\[ y = f(x) \]

rotation about \( x \)-axis

infinitely thin cylindrical slice of volume

\[ \pi f(x)^2 \, dx \]

radius \( \text{thicknes}\)

volume of solid of revolution

\[ \int_{a}^{b} \pi f(x)^2 \, dx \]

(= infinite sum of volumes of infinitely thin cylinders)
Examples:

1. \( y = \sqrt{x}, \ 0 \leq x \leq 1 \); volume of solid of rev.?

Step 1: sketch

Step 2: set up integral

- Slices are radius \( f(x) \), height \( dx \)

\[ \Rightarrow \text{Volume} = \pi \int f(x)^2 \, dx \]

\( \Rightarrow \) then the solid is:

\[ \text{volume} = \int_0^1 \pi (\sqrt{x})^2 \, dx \]

\[ = \pi \int_0^1 x \, dx \]

\[ = \pi \left[ \frac{1}{2} x \right]_0^1 = \frac{\pi}{2} \]

\[ \boxed{\frac{\pi}{2}} \]
② $y = e^x$, $0 \leq x \leq 1$, volume of solid of revolution?

**Step 1:** Sketch

![Sketch of the function $y = e^x$ with a line from $y = 1$ at $x = 0$ and $x = 1$]

**Step 2:** Set up integral

Let the cylindrical slice be volume $\pi (e^x)^2 \, dx$.

Let the region be volume:

$$
\int_0^1 \pi (e^x)^2 \, dx = \pi \int_0^1 e^{2x} \, dx
$$

$$
= \pi \left[ \frac{e^{2x}}{2} \right]_0^1
$$

$$
= \pi \left( \frac{e^2}{2} - \frac{1}{2} \right)
$$

$$
= \frac{\pi (e^2 - 1)}{2}
$$
(3) \( y = x^2, \ y = x^3, \ 1 \leq x \leq 2 \)

Volume of solid of revolution?

Remark: This solid will have a hole through the middle!
Hence the slices of thickness \( dx \) will be washers rather than cylinders
and we must adjust the volumes accordingly.

Step 1: sketch
Step 2: Set up integral.

Now consider the volume of a washer:

\[
\text{outer radius } = R \\
\text{inner radius } = r \\
\text{height/thickness } = H
\]

\[
\text{volume } = \pi R^2 H - \pi r^2 H = \pi H (R^2 - r^2)
\]

In our example see that the outer radius is \( x^3 \) and the inner radius is \( x^2 \), so the volume of a slice is
\[
\pi \int (x^3^2 - x^2^2) \, dx
\]
So the total volume is:

\[ \int_1^2 \pi (x^6 - x^4) \, dx \]

\[ = \left[ \pi \left( \frac{x^7}{7} - \frac{x^5}{5} \right) \right]_1^2 \]

\[ = \pi \left[ \left( \frac{2^7}{7} - \frac{2^5}{5} \right) - \left( \frac{1^7}{7} - \frac{1^5}{5} \right) \right] \]

\[ = \pi \left[ \frac{2^7 - 1}{7} \cdot \frac{1 - 2^5}{5} \right] \]
\(y = e^x, \ y = 3 - x^2, \ 0 \leq x \leq 1\)

Find volume of solid of revolution

Step 1: Sketch:

\[y = e^x, \ y = 3 - x^2\]

Step 2: Set up integral:

The washes associated to the rectangle of thickness \(dx\) has outer radius \(3 - x^2\) and inner radius \(e^x\).

Hence the volume of a slice is:

\[\pi \left( (3-x^2)^2 - (e^x)^2 \right) \, dx\]

And the total volume is:

\[\int_0^1 \pi \left( (3-x^2)^2 - (e^x)^2 \right) \, dx\]
\[ \int_0^1 \left( (9 - 6x^2 + x^4) - e^{2x} \right) \, dx \]

\[ = \pi \left[ 9x - \frac{6x^3}{3} + \frac{x^5}{5} - \frac{e^{2x}}{2} \right]_0^1 \]

\[ = \pi \left[ 9 - \frac{6}{3} + \frac{1}{5} - \frac{e^2}{2} + \frac{e^0}{2} \right] \]

\[ = \pi \left[ 9 - 2 + \frac{1}{5} - \frac{e^2}{2} + \frac{1}{2} \right] = \pi \left[ 7 + \frac{6}{5} - \frac{e^2}{2} \right] \]
Sometimes we need to integrate with respect to y. The formalism is the same:

\[
\text{Area of region} = \int_a^b f(y) \, dy
\]

**Examples:**

1. Find the area enclosed by \( x = y^2 - 4y \) and \( x = y \).

   **Step 1:** sketch
Step 2: set up the integral

\[ (y - (y^2 - 4y)) \, dy \]

The area of the rectangle is

\[ (y - (y^2 - 4y)) \, dy \]

and the intersection points:

\[ y = y^2 - 4y \quad \text{if} \quad y = 0 \]

and if \( y \neq 0 \) then \( 1 = y - 4 \Rightarrow y = 5 \)

so then the area of the region is:

\[
\int_0^5 (y - (y^2 - 4y)) \, dy = \int_0^5 (5y - y^2) \, dy = \left[ \frac{5y^2}{2} - \frac{y^3}{3} \right]_0^5 = \frac{5^3}{2} - \frac{5^3}{3}
\]
\[ x = 1 - y^2, \quad x = y^2 - 1 \]

Find area enclosed by these curves

**Step 1: Sketch**

![Sketch of curves](image)

**Step 2: Set up the integral:**

The area of the rectangle is \[ \int [(1-y^2) - (y^2-1)] \, dy \]

and the intersection points:

\[ 1 - y^2 = y^2 - 1 \implies 2 = 2y^2 \]

\[ \implies 1 = y^2 \implies \pm 1 = y \]

so the area of the region:

\[ \int_{-1}^{1} [(1-y^2) - (y^2-1)] \, dy = \int_{-1}^{1} (y^2 + 2) \, dy \]

\[ = \left[ -\frac{y^3}{3} + 2y \right]_{-1}^{1} = \left( -\frac{1}{3} + 2 \right) - \left( \frac{1}{3} + 2 \right) = \frac{2}{3} + 4 \]
3) \(x = y^2, \quad x = 4 - y^2\)

Find the volume of the enclosed area’s associated solid of revolution about the y-axis.

**Step 1: Sketch**

\[x = 4 - y^2 \quad x = y^2 \quad (4 - y^2) - y^2\]

[Diagram showing the curves and the region of interest]

**Step 2: Setup**

Each slice is a washer with outer radius \(x = 4 - y^2\) and inner radius \(x = y^2\) and thickness (height) \(dy\).

The intersection points are:

\[4 - y^2 = y^2 \Rightarrow 4 = 2y^2 \Rightarrow \pm \sqrt{2} = y\]
The volume of one washer is then
\[ \pi \left( (4-y^2)^2 - (y^2)^2 \right) \, dy \]

and the volume of the solid is
\[ \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left( (4-y^2)^2 - (y^2)^2 \right) \, dy \]

\[ = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left( (16 - 8y^2 + y^4) - y^4 \right) \, dy \]

\[ = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (16 - 8y^2) \, dy \]

\[ = \pi \left[ 16y - \frac{8y^3}{3} \right]_{-\sqrt{2}}^{\sqrt{2}} \]

\[ = \pi \left[ (16\sqrt{2} - \frac{8\sqrt{2}^3}{3}) - (-16\sqrt{2} + \frac{16\sqrt{2}}{3}) \right] \]

\[ = \pi \left[ 32\sqrt{2} - \frac{32\sqrt{2}}{3} \right] = \pi \left[ \frac{64\sqrt{2}}{3} \right] \]
4. Volume: more complicated regions (§6.2)

We can consider solids with other symmetries that allow us to compute the volumes:

2) Square cross sections:

- Consider the solid whose base is the region given by $y = \sqrt{x}$, $x=0$, $x=1$ and whose cross-sections perpendicular to the x-axis are all squares.

Sketch:

![Sketch of a solid with square cross-sections]

Area is $f(x)^2$
then the volume of a square prism slice can be found as follows:

\begin{align*}
\text{volume} &= f(x) f(x) \, dx \\
&= f(x)^2 \, dx
\end{align*}

and then the volume of the whole figure is:

\begin{align*}
\int_0^1 f(x)^2 \, dx &= \int_0^1 (\sqrt{x})^2 \, dx = \int_0^1 x \, dx \\
&= \left[ \frac{1}{2} \right]
\end{align*}
Isosceles triangle sections

Consider the solid whose base is the region enclosed by $y = e^x$, $x = 0$, $x = 1$ and $y = 0$ and whose cross-sections perpendicular to the $x$-axis are isosceles triangles with equal base and height.

Sketch:

See that the slice has volume

$$\frac{1}{2} (e^x)(e^x) \, dx$$

height, base, thickness
then the total volume is:

\[ \int_0^1 \frac{1}{2} (e^x)^2 \, dx = \int_0^1 \frac{1}{2} e^{2x} \, dx \]

\[ = \frac{1}{2} \left[ \frac{e^{2x}}{2} \right]_0^1 \]

\[ = \frac{1}{2} \left[ \frac{e^2}{2} - \frac{1}{2} \right] \]

(W #15) Solid with base enclosed by \( y = 1-x^2 \) and the x-axis and isosceles cross-sections with height = base volume?

Sketch:

(\text{Isosceles slices have area} \left( \frac{1}{2} \right) (1-x^2)(1-x^2))
Set up:

The volume of a slice is

\[ \frac{1}{2} (1-x^2)(1-x^2) \, dx \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

base height thickness

So the volume of the solid is

\[ \int_{-1}^{1} \frac{1}{2} (1-x^2)^2 \, dx = \int_{-1}^{1} \frac{1}{2} (1-2x^2+x^4) \, dx \]

\[ = \frac{1}{2} \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1} \]

\[ = \frac{1}{2} \left[ 1 - \frac{2}{3} + \frac{1}{5} \right] - \left[ -1 + \frac{2}{3} - \frac{1}{5} \right] \]

\[ = \frac{1}{2} \left[ \frac{1}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} \right] = \frac{1}{2} \left[ \frac{1}{3} + \frac{2}{5} \right] \]

\[ = \frac{1}{2} - \frac{1}{6} + \frac{1}{5} \]