1. Differential equations \((\S\,7\,1)\)

Algebraic equations have solutions that are numbers:

- \(10 = x^2 + x - 2\) has solns \(x = 1\) and \(2\)
- \(16 = (x - 2)^2\) has solns \(x = 6\) and \(-2\)

Differential equations have solutions that are functions!

The simplest example are antiderivatives:

- **E.g.** 1: What function \(u(x)\) satisfies \(\frac{du}{dx} = 2x\)?

One solution is \(u = x^2 + 1\)

The general family of solutions is given by \(u = x^2 + C\)
(2) What function \( y(x) \) satisfies
\[
\frac{dy}{dx} - \cos(x) = 0
\]?

\( \Rightarrow \) the general solution is \( y = \sin(x) + C \).

→ Differential equations consist of equalities in a "variable function," e.g., \( u \) and \( y \) above, and their derivatives, hence the name.

→ Important first example (from 10A...)
\[
\frac{dy}{dx} = y
\]

→ This is the defining property of the function \( y = e^x \).

→ Since constants factor out of \( \frac{dy}{dx} \), the general family of solutions to this D.E. is \( y = Ae^x \).
2. Important examples

(i) Population: rate of growth of a population is directly proportional to the size of the population; in symbols:

\[ \frac{dP}{dt} = kP \quad \text{\textit{(Exponential Growth D.E.)}} \]

where \( P(t) \) is population at time \( t \) and \( k \) is a constant.

The general family of solutions is:

\[ P(t) = Ae^{kt} \]

\( \Rightarrow \) exercise: check!

(ii) Population with capacity: M carrying capacity of environment. Then:

\[ \frac{dP}{dt} = kP(1 - \frac{P}{M}) \quad \text{\textit{(Logistic D.E.)}} \]

the general family of solutions is:

\[ P(t) = \frac{Ae^{kt}}{(1 + \frac{A}{M}e^{kt})} \]

\( \Rightarrow \) exercise: check!
(iii) Spring:
\[
\frac{d^2x}{dt^2} = -\frac{k}{m} x
\]
\[k = \text{spring constant} \quad m = \text{mass}\]

\[x(t) = A \sin(t/\sqrt{m}) + B \cos(t/\sqrt{m})\]

\[\Rightarrow \text{exercise: check!}\]

3. Initial value problems:

Find specific solution from solution family for a given initial condition.

E.g. Specific antiderivative:

Find \(u(x)\) such that \(\frac{du}{dx} = \cos(x)\)

and \(u(0) = \pi\).

Solution: we know the general solution is \(u(x) = \sin(x) + C\)

so we use the initial condition to solve for \(C\):

\[\pi = u(0) = \sin(0) + C = 0 + C = C\]

\[\Rightarrow C = \pi\]
hence, the specific solution is
\[ u(x) = \sin(x) + \pi \]

that is, \( u(x) \) above is the solution to the given I.V.P.
\[ \begin{cases} u' = \cos(x) \\ u(0) = \pi \end{cases} \]

More examples:

1. **Population growth with \( p(t) = 2 \), \( k = 3 \)**
   
   that is:
   \[ \begin{cases} \frac{dP}{dt} = 3P \\ P(0) = 2 \end{cases} \]

   General solution family is \( P(t) = Ae^{3t} \), so we use the initial data to solve for \( A \):
   
   \[ 2 = P(0) = Ae^{3(0)} = Ae^0 = A \]
   
   \[ \Rightarrow A = 2 \Rightarrow P(t) = 2e^{3t} \]

   is the specific solution for this I.V.P.
Spring with $\frac{k}{m} = 4$ and starting position 1, initial velocity 0,
that is: \[
\begin{cases}
\ddot{u} = -4u \\
\dot{u}(0) = 1, \quad u'(0) = 0
\end{cases}
\]

The general solution family is:
\[ u(t) = A \sin(2t) + B \cos(2t) \]

To use the initial data to solve:
\[ 1 = u(0) = A \sin(0) + B \cos(0) = 0 + B \quad \Rightarrow \quad B = 1 \]

and
\[ u'(t) = 2A \cos(2t) - 2B \sin(2t) \]
so:
\[ 0 = u'(0) = 2A \cos(0) - 2B \sin(0) = 2A - 0 \quad \Rightarrow \quad A = 0 \]

so the particular solution is:
\[ u(t) = \cos(2t) \]
4. More examples of checking solutions of D.E.'s:

(1) Check that \( y = \frac{2}{3}e^x + e^{-2x} \)

is a solution to the diff eq. \( \frac{dy}{dx} + 2y = 2e^x \) 

First, see that \( \frac{dy}{dx} = \frac{2}{3}e^x - 2e^{-2x} \).

Now we plug in \( \frac{dy}{dx} \) and \( y \) into diff equation (x):

\[
\left( \frac{2}{3}e^x - 2e^{-2x} \right) + 2 \left( \frac{2}{3}e^x + e^{-2x} \right) = 2e^x
\]

\[
= \frac{dy}{dx} = y
\]

\[
\Rightarrow \frac{2}{3}e^x - 2e^{-2x} + \frac{4}{3}e^x + 2e^{-2x} = 2e^x
\]

\[
\Rightarrow \frac{6}{3}e^x + 0 = 2e^x \implies 2e^x = 2e^x
\]

Hence the given \( y \) is a solution to D.I.E. (x)
2) Show that \( y = -t \cos(t) - t \) is a solution to the I.V.P.

\[
\begin{align*}
\frac{dy}{dt} &= y + t^2 \sin(t) \quad (\ast) \\
y(\pi) &= 0
\end{align*}
\]

First, we check the initial condition \( y(\pi) = 0 \):

\[
y(\pi) = -\pi \cos(\pi) - \pi = -\pi(-1) - \pi = \pi - \pi = 0 \quad \checkmark
\]

Next, we check the D.E. First compute:

\[
\frac{dy}{dt} = \frac{d}{dt} (-t \cos(t) - t)
\]

\[
= - \frac{d}{dt} (t \cos(t) - t)
\]

\[
= -(\cos(t) + t(-\sin(t))) - 1
\]

\[
= -\cos(t) + t \sin(t) - 1
\]

and then plug in \( \frac{dy}{dt} \) and \( y \) into \((\ast)\):

\[
t(-\cos(t) + t \sin(t) - 1) = (-t \cos(t) - t) + t^2 \sin(t)
\]

\[
\Rightarrow -t \cos(t) + t^2 \sin(t) - t = -t \cos(t) - t + t^2 \sin(t)
\]

\[ \checkmark \text{ hence the given } y \text{ solves the I.V.P.} \]
3) Show that \( y = \frac{\ln(x) + C}{x} \) is a family of solutions to
\[
x^2 y' + xy = 1
\]
Then find particular solutions for the initial conditions
(a) \( y(1) = 2 \)
(b) \( y(2) = 1 \)

See that
\[
y' = \frac{d}{dx} \left( \frac{\ln(x) + C}{x} \right) = \frac{d}{dx} \left( \frac{\ln(x)}{x} \right) + \frac{d}{dx} \left( \frac{C}{x} \right)
\]
\[
= -\frac{\ln(x)}{x^2} + \left( \frac{1}{x} \right) \left( \frac{1}{x} \right) + \frac{-C}{x^2} = \frac{1}{x^2} - \frac{(\ln(x) + C)}{x^2}
\]
Then:
\[
x^2 \left( \frac{1}{x^2} - \frac{(\ln(x) + C)}{x^2} \right) + x \left( \frac{\ln(x) + C}{x} \right) = 1
\]
\[
\Rightarrow 1 - \ln(x) - C + \ln(x) + C = 1 \Rightarrow 1 = 1
\]

The given family is a solution for all values of \( C \).
(a) For \( y(1) = 2 \), the specific solution is:

\[ y = y(1) = \frac{\ln(1) + C}{1} = 0 + C \]

\[ \Rightarrow C = 2 \]

\[ \Rightarrow y(x) = \frac{\ln(x) + 2}{x} \]

(b) For \( y(z) = 1 \), the specific solution is:

\[ 1 = y(z) = \frac{\ln(z) + C}{2z} = \frac{\ln(z)}{2} + \frac{C}{2} \]

\[ \Rightarrow \left(1 - \frac{\ln(z)}{2}\right) = \frac{C}{2} \Rightarrow C = 2 \left(1 - \frac{\ln(z)}{2}\right) \]

\[ \Rightarrow y(x) = \frac{\ln(x) + 2 - \ln(z)}{x} \]
Show that \( y = \frac{1}{x + C} \) is a family of solutions to the D.E.

\[
y' = -y^2
\]

Find the specific solution for \( y(0) = \frac{1}{2} \)

First, compute

\[
y' = \frac{d}{dx} \left( \frac{1}{x + C} \right)
\]

\[
= -\frac{1}{(x+C)^2}
\]

and then

\[
\left( -\frac{1}{(x+C)^2} \right) = -\left( \frac{1}{x+C} \right)^2 = -\frac{1}{(x+C)^2}
\]

\[
y' = y
\]

\[\checkmark\]

The given \( y \) is a solution for all \( C \)

Now, for \( y(0) = \frac{1}{2} \):

\[
\frac{1}{2} = y(0) = \frac{1}{0 + C} = \frac{1}{C} \Rightarrow C = 2
\]

\[
y = \frac{1}{x + 2}
\]

is the specific solution to the I.V.P.
Consider the D.E.

\[ u'' + u' = 2u \]

Find all \( k \) so that the function \( u = e^{kx} \) is a solution.

First, compute the derivatives:

\[ u' = ke^{kx} \]
\[ u'' = k^2e^{kx} \]

and then plug into the D.E.:

\[ (k^2e^{kx}) + (ke^{kx}) = 2(e^{kx}) \]

Cleaning up we see that:

\[ e^{kx}(k^2 + k) = 2e^{kx} \]

\[ \Rightarrow (k^2 + k) = 2 \]

Divide by \( e^{kx} \neq 0 \)

\[ \Rightarrow k^2 + k - 2 = 0 \]

\[ \Rightarrow (k-1)(k+2) = 0 \]

\[ \Rightarrow k = 1 \text{ or } -2 \]
hence the functions
\[ u = e^x \]
and \[ u = e^{-2x} \]
are solutions to the given D.E.

6) Consider the D.E.:
\[ y'' = -\frac{y}{2} \]
Find all \( k \) such that \( y = \sin(kt) \) is a solution.

First:
\[ y' = k\cos(kt) \]
\[ y'' = -k^2\sin(kt) \]

Then:
\[ -k^2\sin(kt) = -\frac{1}{2} \sin(kt) \]
\[ \Rightarrow -k^2 = -\frac{1}{2} \]
\[ \Rightarrow k = \pm \frac{1}{\sqrt{2}} \]

Hence \( y = \sin\left(\frac{\pm 1}{\sqrt{2}}t\right) \) and \( y = \sin\left(\frac{\pm 1}{12}t\right) \) are solutions to the given D.E.
5. **Separable differential equations**

Suppose $y$ is a function of $x$.

Then any differential equation which can be written as:

\[
\frac{dy}{dx} = f(y)g(x)
\]

is called **separable**.

Here $f$ and $g$ are functions of their respective variables, i.e., you can think of:
- $g(x) = \text{expression in } x \text{ only}$
- $f(y) = \text{expression in } y \text{ only}$

We can use integration to find the general solution family for a given separable DE!
By the solution method reveals another link between the \textit{Leibniz notation} for derivatives and the notation for integration:

\[
\frac{dy}{dx} = f(y) \cdot g(x)
\]

\[
\Rightarrow \int \frac{dy}{f(y)} = \int g(x) \, dx
\]

integrate

\[
\Rightarrow (\text{expression in } y) = (\text{expression in } x)
\]

\(\Rightarrow \text{Don't forget } + C\)

algebra

\[
\Rightarrow \text{solve for } y \text{ to get solution family } C
\]

\(\Rightarrow \text{solution for each constant } C!\)
Examples

1. Solve the I.V.P. \( \frac{dy}{dx} = \frac{x}{y} \), \( y(0) = -7 \)

First, we find the general solution since the D.E. is separable:

\[
\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y \, dy = \int x \, dx
\]

\[
\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C
\]

\[
\Rightarrow y^2 = x^2 + C
\]

\[
\Rightarrow y = \pm \sqrt{x^2 + C} \quad \text{gen. soln}
\]

Then we require:

\(-7 = y(0) = \pm \sqrt{0 + C} = -\sqrt{C}\)

\(\Rightarrow 49 = C\) \(\text{choose negative root!}\)

Hence the particular solution is

\[
y = -\sqrt{x^2 + 49}
\]
2. Find general solution to the D.E.

\[ \frac{dz}{dt} + 4e^{t+z} = 0 \]

See that:

\[ \frac{dz}{dt} = -4e^{t+z} = -4e^t e^z \]

\[ e^z = \int \frac{-4e^t dt}{4e^t} = \int -4e^t dt \]

\[ e^z = -4e^t + C \]

\[ e^{-z} = -4e^t + C \]  \( \downarrow \) divide out negative

\[ e^{-z} = 4e^t + C \]  \( \downarrow \) ln both sides

\[ -z = \ln(4e^t + C) \]

\[ z = -\ln(4e^t + C) \]  \( \text{general solution} \)

**Remark:**

would be way different (and wrong) if we left out \( +C \)
3) (e.g. 4) from previous section revisited)

Find general family of solutions to

\[
\frac{dy}{dx} = -y^2
\]

\[
y' = -y^2
\]

Here \( \frac{dy}{y^2} = -y^2 \Rightarrow \int \frac{dy}{y^2} = \int -dx \)

\[
\Rightarrow \frac{-1}{y} = -x + C
\]

\[
\Rightarrow \frac{1}{y} = x + C
\]

\[
\Rightarrow y = \frac{1}{x + C}
\]

\(\leftarrow\) \(\text{Compare with solution family in e.g. 4) in 5.4 of these notes}\)
4) Find general solution for

\[ \frac{dL}{dt} = kL^2 \ln(t) \]

\[ \frac{dL}{dt} = kL^2 \ln(t) \implies \int \frac{dL}{L^2} = \int k \ln(t) \, dt \]

\[ \implies \frac{-1}{L} = \int k \ln(t) \, dt \]

Must find this integral via integration by parts:

\[ u = \ln(t) \quad du = \frac{dt}{t} \]

\[ dv = dt \quad v = t \]

\[ \implies \int \ln(t) \, dt = uv - \int vdu = t \ln(t) - \int t \left( \frac{dt}{t} \right) \]

\[ = t \ln(t) - t + C \]

\[ \frac{-1}{L} = k(\ln(t)-t) + C \]

\[ \implies L = \frac{-1}{k \ln(t)-kt+C} \]
(cont'd): Find the particular solution such that 

\[ L(1) = -1 \]

\[-1 = L(1) = \frac{-1}{k \ln(1) - k(1) + C} \]

\[ \Rightarrow 1 = \frac{1}{-k + C} \]

\[ \Rightarrow C = k + 1 \]

So the solution to the I.V.P. is:

\[ L(t) = \frac{-1}{k(\ln(t) - t) + k + 1} \]
(5) Solve the I.V.P.

\[ \begin{align*}
\frac{dy}{dx} &= \frac{xy \sin(x)}{y+1} \\
\end{align*} \]

\[ y(0) = 1 \]

First we find the general solution:

\[ \frac{dy}{dx} = \frac{xy \sin(x)}{y+1} = \left(\frac{y}{y+1}\right) x \sin(x) \]

\[ \Rightarrow \int \left(\frac{y+1}{y}\right) dy = \int x \sin(x) \, dx \]

\[ \Rightarrow \int \left(1 + \frac{1}{y}\right) dy = \int x \sin(x) \, dx \]

\[ \Rightarrow y + \ln|y| = \int x \sin(x) \, dx \]

\[ \downarrow \text{Solve w/ Int. by parts:} \]

\[ u = x \quad du = dx \]

\[ dv = \sin(x) \, dx \quad v = -\cos(x) \]

\[ \int x \sin(x) \, dx = -x \cos(x) - \int -\cos(x) \, dx \]

\[ = -x \cos(x) + \sin(x) + C \]
\[ y + \ln |y| = -x \cos(x) + \sin(x) + C \]

Then use the initial data to solve for \( C \): \( y(0) = 1 \) so:

\[ (1) + \ln |1| = -(0) \cos(0) + \sin(0) + C \]
\[ \Rightarrow 1 + 0 = 0 + 0 + C \Rightarrow C = 1 \]

Hence the particular solution satisfies:

\[ y + \ln |y| = -x \cos(x) + \sin(x) + 1 \]

Note: we can't solve for \( y \) as an expression of \( x \) in this case;

but it is given implicitly as a function of \( x \).

To check that the above solves our ODE, use implicit differentiation!
OVERVIEW OF DIFFERENTIAL EQUATIONS:

Be able to:

1. Check solutions of D.E.'s and I.V.P.'s

2. Find specific solutions to I.V.P.'s from a general solution family, e.g.:

<table>
<thead>
<tr>
<th>D.E.</th>
<th>Solution family</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponential growth D.E.</td>
<td>( u = Ae^{kx} )</td>
</tr>
<tr>
<td>logistic D.E.</td>
<td>( u = \frac{Ae^{kx}}{(1 + \frac{A}{M}e^{kx})} )</td>
</tr>
<tr>
<td>spring</td>
<td>( u(x) = A \sin(t\sqrt{\frac{k}{m}}) + B \cos(t\sqrt{\frac{k}{m}}) )</td>
</tr>
</tbody>
</table>

\( (k, m, M \text{ are constants}) \)

4. Given initial data \( u(i) = j \) for some \( i \) and \( j \), find the constant \( A \)

3. Find general solutions for separable D.E.'s