Name: Solution

Math 10B. 10c
Practice Midterm Exam 1
October 22, 2018

Exam rules:

1. No electronic devices of any kind are allowed during this exam.
2. You may use one 2-sided US Letter sized page of notes, but no books or other assistance are allowed during this exam.
3. If you violate these instructions or communicate in any way with any other student during this exam, you will receive a zero on the exam, and the zero will not be dropped when calculating your cumulative course average.

Instructions:

1. Write your first and last name at the top of every page of this exam.
2. Write your solutions clearly and legibly. No credit will be given for illegible solutions.
3. Show the work leading to your solutions. Partial credit can only be given based on the work you show.
4. Please put a box around your final answer to each part.

→ If any question is not clear, ask for clarification.
1. Consider the following points in $\mathbb{R}^3$:

$A(1,0,3) \quad B(5,-3,1) \quad C(-2,-1,3)$

(a) Let $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{BC}$. Find $|\vec{u} + \vec{v}|$.

\[
\vec{u} = \langle 5-1, -3-0, 1-3 \rangle = \langle 4, -3, -2 \rangle \\
\vec{v} = \langle -2-5, -1+3, 3-1 \rangle = \langle -7, 2, 2 \rangle \\
\vec{u} + \vec{v} = \langle -3, -1, 0 \rangle \\
|\vec{u} + \vec{v}| = \sqrt{9 + 1 + 0} = \sqrt{10}
\]

(b) Are $\vec{u}$ and $\vec{v}$ orthogonal? If not, find the angle between them. (Your answer can be left in terms of an inverse trigonometric function.)

\[
\vec{u} \cdot \vec{v} = -28 + -6 + -4 = -38 \neq 0
\]

Then

\[
\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos \theta
\]

\[
\Rightarrow \quad \frac{-38}{|\vec{u}| \cdot |\vec{v}|} = \cos \theta
\]

\[
\Rightarrow \quad \cos \theta = \frac{-38}{\sqrt{16+9+4} \cdot \sqrt{49+8}}
\]

\[
\theta = \arccos \left( \frac{-38}{\sqrt{29.57}} \right)
\]
(c) Find a vector (parametric) equation for the line containing the points $A$ and $C$.

$$\vec{AC} = \langle -3, -1, 0 \rangle$$

and so let $\vec{v} = \langle -3, -1, 0 \rangle$

and $\vec{v}_0 = \vec{OA} = \langle 1, 0, 3 \rangle$

and thus

$$r(t) = \vec{v}_0 + t \vec{v}$$

$$= \langle 1, 0, 3 \rangle + t \langle -3, -1, 0 \rangle$$

$$\Rightarrow r(t) = \langle 1 - 3t, -t, 3 \rangle$$
(d) Find an equation for the plane containing the point \( C \) and the line segment given by vector \( \vec{u} \).

\[ \vec{u} = \langle 4, -3, -2 \rangle \]

Let \( \vec{v} = \vec{AC} = \langle -3, -1, 0 \rangle \).

Now the normal vector is \( \vec{u} \times \vec{v} \):

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-3 & -2 & 1 \\
-1 & 0 & -3
\end{vmatrix} = (0-2)\hat{i} - (4-0)\hat{j} + (3-1)\hat{k} = \langle -2, 6, -2 \rangle
\]

So, using \( \vec{r}_0 = \vec{OC} = \langle -2, -1, 3 \rangle \),

we have

\[ 0 = (\vec{u} \times \vec{v}) \cdot (\vec{r} - \vec{r}_0) = \langle -2, 6, -13 \rangle \cdot \langle x+2, y+1, z-3 \rangle \]

\[ 0 = -2(x+2) + 6(y+1) - 13(z-3) \]
2. Consider the following lines in $\mathbb{R}^3$:

$$\vec{l}(t) = (-1 + t, 5 - t, -3 + 2t) \quad \vec{r}(t) = (9 + 5t, 13 + 4t, -9 - 3t)$$

(a) Do the lines $\vec{l}$ and $\vec{r}$ intersect? If so, give the intersection point in $\mathbb{R}^3$.

Let $\vec{x}(t) = \vec{r}(s)$

$$\Rightarrow \begin{pmatrix} -1 + t \\ 5 - t \\ -3 + 2t \end{pmatrix} = \begin{pmatrix} 9 + 5s \\ 13 + 4s \\ -9 - 3s \end{pmatrix}$$

\[\begin{cases} -1 + t = 9 + 5s \\ 5 - t = 13 + 4s \\ -3 + 2t = -9 - 3s \end{cases}\]

\[\begin{cases} 4 = 22 + 9s \\
\Rightarrow -18 = 9s \\
\Rightarrow s = -2 \]

and then

$$t = 10 + 5(-2)$$

\[\Rightarrow t = 0\]

check:

$$\vec{l}(0) = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$$

and $\vec{r}(-2) = \begin{pmatrix} 9 - 10 \\ 13 - 8 \\ -9 + 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$
(b) Find an equation for the plane containing the two lines.

The plane containing the two lines must be orthogonal to both and contain a point on both.

Let \( \mathbf{r}_0 = \langle -1, 5, -3 \rangle \) give the point (see previous problem.)

Now, to be \( \perp \) to both \( \mathbf{l} \) and \( \mathbf{r} \) we need \( \langle 1, -1, 2 \rangle \times \langle 5, 4, -3 \rangle \)

\[
\begin{vmatrix}
1 & -1 & 2 \\
4 & -3 & -5 \\
1 & 5 & 4
\end{vmatrix}
\]

\[
= \mathbf{i} (3 - 8) - \mathbf{j} (3 - 10) + \mathbf{k} (4 + 5)
\]

\[
= \langle -5, 13, 9 \rangle = \mathbf{n}
\]

So our plane is \( 0 = \langle -5, 13, 9 \rangle \cdot \langle x+1, y-5, z+3 \rangle \)

\[
\Rightarrow 0 = -5(x+1) + 13(y-5) + 9(z+3)
\]