Math 10C.
Practice Midterm Exam 2
February 26, 2019

Exam rules:

1. No electronic devices of any kind are allowed during this exam.

2. You may use one 2-sided US Letter sized page of notes, but no books or other assistance are allowed during this exam.

3. If you violate these instructions or communicate in any way with any other student during this exam, you will receive a zero on the exam, and the zero will not be dropped when calculating your cumulative course average.

Instructions:

1. Write your first and last name at the top of every page of this exam.

2. Write your solutions clearly and legibly. No credit will be given for illegible solutions.

3. Show the work leading to your solutions. Partial credit can only be given based on the work you show.

4. Please put a box around your final answer to each part.

→ If any question is not clear, ask for clarification.
1. (10 points) Find a vector (parametric) equation for the tangent line to the parameterized curve \( r(t) \) in \( \mathbb{R}^3 \) given below at the point \( r(\pi) \).

\[ r(t) = (t \sin(t), t \cos(t), t^2) \]

The tangent line to \( r(t) \) at \( r(\pi) \) is the line in the \( r'(\pi) \) direction (passing through \( r(\pi) \)).

\[ r'(\pi) = \langle \pi \sin(\pi), \pi \cos(\pi), \pi^2 \rangle = \langle 0, -\pi, \pi^2 \rangle \]

\[ r'(t) = \langle \sin(t) + t \cos(t), \cos(t) - t \sin(t), 2t \rangle \]

\[ r'(\pi) = \langle \sin(\pi) + \pi \cos(\pi), \cos(\pi) - \pi \sin(\pi), 2\pi \rangle = \langle 0 - \pi, -1 - 0, 2\pi \rangle = \langle -\pi, -1, 2\pi \rangle \]

So the tangent line is

\[ \ell(t) = t \langle -\pi, -1, 2\pi \rangle + \langle 0, -\pi, \pi^2 \rangle = \langle -\pi t, -(t+\pi), 2\pi t + \pi^2 \rangle \]
2. (15 points) Consider the 2-variable function \( f(x, y) = ye^{x^2+1} + x \ln(y - 1) \).

(a) (5 points) Find the gradient vector for \( f(x, y) \) at the point \((1, 2)\).
That is, find the vector \( \nabla f(1, 2) \).

\[
\nabla f(1, 2) = \langle f_x(1, 2), f_y(1, 2) \rangle
\]

So \( f_x(x, y) = (2x)y e^{x^2+1} + \ln(y - 1) \)

\[= f_x(1, 2) = 2(1)(2)e^{1+1} + \ln(2-1)\]

\[= 4e^2 + \ln(1) = 4e^2\]

\[\text{and} \quad f_y(x, y) = e^{x^2+1} + \frac{x}{y-1} \Rightarrow f_y(1, 2) = e^{1+1} + \frac{1}{2-1} = e^2 + 1\]

So \( \nabla f(1, 2) = \langle 4e^2, e^2+1 \rangle \)

(b) (5 points) Find the directional derivative of \( f(x, y) \) at the point \((1, 2)\) in the direction of the vector \((-1, 1)\).

First, we normalize \((-1, 1)\)

\[1 < -1, 1 > = \sqrt{2} \Rightarrow \text{let} \quad \vec{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle\]

Then \( D_u f(1, 2) = \nabla f(1, 2) \cdot \vec{u} \)

\[= \langle 4e^2, e^2+1 \rangle \cdot \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)\]

\[= \frac{-4e^2}{\sqrt{2}} + \frac{e^2+1}{\sqrt{2}} = \boxed{\frac{1-3e^2}{\sqrt{2}}}\]
(c) (5 points) Find an equation for the plane in $\mathbb{R}^3$ tangent to the graph $z = f(x, y)$ at the point $(1, 2, f(1, 2))$.

The tangent plane at $P$ is given by

$$z - f(1, 2) = \nabla f(1, 2) \cdot (x - 1, y - 2)$$

$$f(1, 2) = 2e^{1+1} + 1 \ln(2-1)$$

$$= 2e^2 + \ln(1) = 2e^2$$

and so

$$z - 2e^2 = \langle 4e^2, e^2 + 1 \rangle \cdot (x - 1, y - 2)$$

$$\Rightarrow \quad z - 2e^2 = 4e^2(x - 1) + (e^2 + 1)(y - 2)$$
3. (15 points) Consider the 2-variable function \( f(x, y) = xy^2 + xy - 2x + 3y \).

(a) (5 points) Find the critical points of \( f(x, y) \).

where \( \nabla f(x, y) = 0 \)

\[ \nabla f(x, y) = \langle y^2 + y - 2, 2xy + x + 3 \rangle = \langle 0, 0 \rangle \]

\[ \Rightarrow \begin{cases} y^2 + y - 2 = 0 & \Rightarrow (y+2)(y-1) = 0 \\ 2xy + x + 3 = 0 & \Rightarrow y = -2, 1 \end{cases} \]

\[ \begin{align*}
\text{for } y = -2: & \ 2x(-2) + x + 3 = 0 \Rightarrow -4x + x = -3 \\
& \Rightarrow x = 1
\end{align*} \]

\[ \begin{align*}
\text{for } y = 1: & \ 2x(1) + x + 3 = 0 \Rightarrow 2x + x + 3 = 0 \\
& \Rightarrow x = -1
\end{align*} \]

\[ \text{hence } (1, 1) \text{ and } (1, -2) \text{ are the critical points.} \]

(b) (5 points) Tell whether each critical point is a local minimum, local maximum, or saddle point.

we apply 2nd derivative test:

\[ f_x(x, y) = y^2 + y - 2, \ f_y(x, y) = 2xy + x + 3 \]

\[ \Rightarrow f_{xx}(x, y) = 0, \ f_{xy}(x, y) = 2y + 1, \ f_{yy}(x, y) = 2x \]

\[ \begin{align*}
\text{now:} & \\
\text{for } (1,1): & \ f_{xx}(1,1) = 0, \ f_{xy}(1,1) = 3, \ f_{yy}(1,1) = -2 \\
& \Rightarrow D = \begin{vmatrix} 0 & 3 \\ 3 & -2 \end{vmatrix} = 0 - 9 = -9 < 0 \Rightarrow (1,1) \text{ is a saddle point}
\end{align*} \]

\[ \begin{align*}
\text{for } (1, -2): & \ f_{xx} = 0, \ f_{xy}(1, -2) = 3, \ f_{yy}(1, -2) = 2 \\
& \Rightarrow D = \begin{vmatrix} 0 & -3 \\ -3 & 2 \end{vmatrix} = 0 - (3)^2 = -9 < 0
\end{align*} \]

\[ \text{for } (1, -2) \text{ is a saddle point.} \]
\[ f(x, y) = xy^2 + xy - 2x + 3y \]

(c) (5 points) Find the global maximum values for \( f(x, y) \) over the triangular region in \( \mathbb{R}^2 \) with vertices: \( P(0, 0), \ Q(-2, 0), \ R(-2, 4) \)

Critical points on boundary:

1. \( y = -2x \) : 
   \[ f(x, -2x) = x(-2x)^2 + x(-2x) - 2x + 3(-2x) \]
   \[ = 4x^3 - 2x^2 - 2x - 6x \]
   \[ \frac{\partial}{\partial x}(f(x, -2x)) = 12x^2 - 4x - 8 = 0 \iff x = \frac{4 \pm \sqrt{16 - 4(-6)(2)}}{2(2)} = \frac{4 \pm 4\sqrt{2}}{2} = 2 \pm 2\sqrt{2} \]
   So check: \((1, -2), \ (-\frac{2}{3}, 4)\)

2. \( x = -2 \) : 
   \[ f(-2, y) = -2y^2 + 2y + 4 + 3y \]
   \[ = -2y^2 + y + 4 \]
   \[ \frac{\partial}{\partial y}(-2y^2 + y + 4) = -4y + 1 = 0 \iff y = \frac{1}{4} \]
   So check: \((-2, \frac{1}{4})\)

3. \( y = 0 \) : 
   \[ f(x, 0) = -2x \Rightarrow \frac{\partial}{\partial x}(-2x) = -2 \neq 0 \], so no critical points

End points of boundary lines: \((0, 0), \ (-2, 0), \ (-2, 4)\)

Now check:

- \( f(1, -2) = 4 - 2 - 2 - 6 = -6 \)
- \( f\left(-\frac{2}{3}, \frac{4}{3}\right) = -\frac{32}{27} - \frac{8}{9} + \frac{4}{3} + 1 = \frac{4 - 20}{27} \)
- \( f(-2, 1) = -2\)
- \( f(0, 0) = 0 \)
- \( f(-2) = 4 + \frac{1}{8} = 4 + \frac{1}{8} \)

So:

- \( \min = -6 \)
- \( \max = 0 \)
- \( \max = 4 + \frac{1}{8} \)