Find a line that intersects the point $P(0,3,1)$ and intersects the line $\mathbf{l}(t) = \langle 2 + t, 3 - t, 4 + 4t \rangle$ at a right angle.

Solution method: use geometry.

The line we're looking for.
We want the line in the direction of the vector $\overrightarrow{RP}$ and passing through $P$ and $R$.

To find $R$, we consider the following:

- $\vec{l}(t)$ is vector $\overrightarrow{OP}$ in the picture.
- $\text{proj}_{\vec{v}}(\overrightarrow{QP})$ is vector $\overrightarrow{QR}$.

Therefore, $\overrightarrow{QR} = \vec{l}(t) + \text{proj}_{\vec{v}}(\overrightarrow{QP})$

So we compute:

- $\vec{l}(t) = \langle 2, 3, 4 \rangle + t \langle 1, -1, 4 \rangle$
  
  So $\overrightarrow{v} = \langle 1, -1, 4 \rangle$ and $|\overrightarrow{v}| = \sqrt{18}$

- $\overrightarrow{QP} = \langle -2, 0, -3 \rangle$ and $\overrightarrow{QP} \cdot \overrightarrow{v} = -14$

$\Rightarrow \text{proj}_{\vec{v}}(\overrightarrow{QP}) = \langle -\frac{7}{9}, \frac{7}{9}, -\frac{28}{9} \rangle$
\[ \vec{OR} = \left\langle -\frac{7}{9}, \frac{7}{9}, -\frac{28}{9} \right\rangle \]

hence \( \vec{R} = \left\langle -\frac{7}{9}, \frac{7}{9}, -\frac{28}{9} \right\rangle \).

\[ \vec{PR} = \left\langle \frac{11}{9}, \frac{7}{9}, \frac{1}{9} \right\rangle \]

which is the direction vector we wanted.

\[ r(t) = \left\langle 0, 3, 1 \right\rangle + t \left\langle \frac{11}{9}, \frac{7}{9}, \frac{1}{9} \right\rangle \]

So our line is
Solution method 2: use system of equations.

The online homework gives a hint to help with this:

\[ \vec{r}(s) = \langle -19s, ?, ?, ? \rangle \]

By the hint, not passing through \( P(0, 3, 1) \), we have:

\[ \vec{r}(s) = \langle -19s, bs + 3, cs + 1 \rangle \]

\( \Rightarrow \) so we only need to find \( b \) and \( c \) via some equations.

0. There must be \( s \) and \( t \) such that

\[ \vec{r}(s) = \vec{r}(t) \]

\[ \Rightarrow \begin{cases} -19s = 2 + t \\ bs + 3 = 3 - t \\ cs + 1 = 4 + 4t \end{cases} \]
Furthermore, they are 1, so:

\[ \langle -19, b, c \rangle \cdot <1, -1, 47> = 0 \]

\[ \Rightarrow -19 + b + 4c = 0 \]

Now, solving these equations we can solve:

\[
\begin{align*}
(1) + (2) & : (-19 + b) s = 2 \\
(1) + (3) & : (76 + c) s = -5
\end{align*}
\]

\[ \Rightarrow -19 + b = \frac{1}{s} = \frac{76 + c}{-5} \]

\[ c = \frac{19}{11} \quad \text{and} \quad b = -\frac{132}{11} \]

(using these two values we can also get \( s = \frac{-11}{919} \) and \( t = -\frac{7}{9} \) for the intersection.)
therefore the direction vector

\[ \langle -19, -\frac{133}{11}, \frac{19}{11} \rangle \]

\[ = -19 \langle 1, \frac{7}{11}, \frac{1}{11} \rangle \]

\[ = -\frac{19 \cdot 9}{11} \langle \frac{11}{9}, \frac{7}{9}, \frac{1}{9} \rangle \]

\[ \Rightarrow \text{same direction (parallel) as the direction vector in previous method.} \]

\[ \Rightarrow \vec{F}(s) = \langle 0, 3, 1 \rangle + s \langle -19, -\frac{133}{11}, \frac{19}{11} \rangle \]

\[ \text{Note that } \vec{F}(\frac{19 \cdot 9}{11}) = \vec{F}(1) \]

\[ \text{second method} \]

\[ \text{first method} \]