(1) [3 points] Find the equation of the line tangent to \( x \cos(y) + 2x^2 + 2xy = 3 \) at \((1,0)\).

(2) [4 points] Calculate the following limits or state that they do not exist (DNE).

(a) \( \lim_{x \to 0^+} xe^x \) 
(b) \( \lim_{x \to \infty} x^{1/2} \)

Hospitals rule
ln properties

(3) [6 points] Consider the function \( f(x) = 2^x \) on \([0,6]\).

(a) Compute the right endpoint approximation \( R_3 \) to \( \int_0^6 f(x)\,dx \).

\[ R_3 = \frac{6-0}{3} \sum_{j=1}^{3} f\left(\frac{j \cdot 6}{3}\right) \]

(b) Write down the most accurate phrase in your blue book:

"The correct answer to Part (a) is less than \( \int_0^6 f(x)\,dx \)."

"The correct answer to Part (a) is greater than \( \int_0^6 f(x)\,dx \)."

"The correct answer to Part (a) is equal to \( \int_0^6 f(x)\,dx \)."

(4) [6 points] A farmer has 24 feet of fencing and wishes to fence off a rectangular field that boarders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

(5) [6 points] (a) Use linear approximation and the fact that \( \sqrt{64} = 8 \) to estimate \( \sqrt{65} \).

(b) Write down the most accurate phrase in your blue book:

"The correct answer to Part (a) is less than \( \sqrt{65} \)."

"The correct answer to Part (a) is greater than \( \sqrt{65} \)."

"The correct answer to Part (a) is equal to \( \sqrt{65} \)."

(6) [6 points] Calculate the following integrals.

(a) \( \int \frac{x^{2/3} + x^{1/2}}{x^{3/2}}\,dx \)

(b) \( \int \frac{1}{x^2} + \sec(2x)\tan(2x)\,dx \)

(c) \( \int_0^{\pi/2} \sin(2x)\,dx \)

L power rule
L build your own antiderivative...
L FTC I

(7) [6 points] Calculate the derivatives of the following functions.

(a) \( f(x) = \sin(x)\cos(x) \)

(b) \( g(x) = \cos(\ln(x^2 + 1)) \)

(c) \( h(x) = \int_1^{2/x} \tan(t^2)\,dt \)

Properties of ln
chain rule
 FTC II +

(8) [10 points] Let \( f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3 \).

(a) Find the \( x \)-coordinates of all critical points of \( f(x) \).

(b) Find the intervals of increase and decrease of \( f(x) \).

(c) Classify all critical points of \( f(x) \) as local maxima, local minima, or neither.

(d) Find the \( x \)-coordinates of all points of inflection of \( f(x) \).

(e) Find the intervals of concavity of \( f(x) \).
1. Find equation of the line to \( x \cos(y) + 2x^2 + 2xy = 5 \) at the point \((1, 0)\).

Recall the tangent line is the line through \((1, 0)\) with slope \( \frac{dy}{dx} \bigg|_{x=1, y=0} \), i.e., \( y = \frac{dy}{dx} \bigg|_{x=1, y=0} (x - 1) + 0 \).

So we must find \( \frac{dy}{dx} \); apply implicit differentiation:

\[
\frac{d}{dx} (x \cos(y) + 2x^2 + 2xy) = \frac{d}{dx} (5)
\]

\[
\Rightarrow \frac{d}{dx} (x \cos(y)) + \frac{d}{dx} (2x^2) + \frac{d}{dx} (2xy) = 0
\]

\[
\Rightarrow (x \cos(y)) + x \frac{d}{dx} (\cos(y)) + 4x + (2y + 2x \frac{dy}{dx}) = 0
\]

\[
\Rightarrow \cos(y) - x \sin(y) \frac{dy}{dx} + 4x + 2y + 2x \frac{dy}{dx} = 0
\]

\[
\Rightarrow \cos(y) + 4x + 2y = x \sin(y) \frac{dy}{dx} - 2x \frac{dy}{dx}
\]

\[
\Rightarrow \frac{\cos(y) + 4x + 2y}{x \sin(y) - 2x} = \frac{dy}{dx}
\]

Then we evaluate at \((1, 0)\):

\[
\frac{dy}{dx} \bigg|_{x=1, y=0} = \frac{\cos(0) + 4(1) + 2(0)}{(1) \sin(0) - 2(1)} = \frac{1 + 4 + 0}{-2} = -\frac{5}{2}
\]
hence; the tangent line is given by:

\[ y = -\frac{5}{2} (x-1) \]

2. Calculate the limits:

(a) \( \lim_{x \to \infty} x e^x = \lim_{x \to \infty} \frac{x}{e^{-x}} \)

- Algebra
- Limit approaches \( \infty \) in the numerator and denominator
- Indeterminate form
- Apply L'Hôpital's rule

\[
\left[ \lim_{x \to -\infty} \frac{1}{-e^{-x}} \right] = 0
\]

- Facts about exponential functions (recall limits at infinity chapters!)
(b) \( \lim_{x \to \infty} x^{1/x} \):

Set \( L = \lim_{x \to \infty} x^{1/x} \)

and apply \( \ln \) to both sides \( \text{by}\)

Continuity we get:

\[
\ln(L) = \lim_{x \to \infty} \ln(x^{1/x})
\]

\[
\ln(L) = \lim_{x \to \infty} \frac{1}{x} \ln(x)
\]

\[
\ln(L) = \lim_{x \to \infty} \frac{1}{x} \ln(x) = \lim_{x \to \infty} \frac{\ln(x)}{x} = 0
\]

\[\Rightarrow \ln(L) = 0\]

\[\Rightarrow e^{\ln(L)} = e^0\]

\[
L = 1
\]
(3) Consider \( f(x) = 2^x \) on \([0,6]\).

(a) Find \( R_3 \) for an approx. of \( \int_0^6 f(x) \, dx \)

\[
R_N = \frac{(b-a)}{N} \sum_{j=1}^{N} f(a + \frac{(b-a)}{N} j)
\]

\[
\Rightarrow R_3 = \frac{6}{3} \sum_{j=1}^{3} f(0 + \frac{6}{3} j)
\]

\[
= 2(f(2) + f(4) + f(6))
\]

\[
= \left[ 2 \left( 2^2 + 2^4 + 2^6 \right) \right]
\]

\[
= 2(4 + 16 + 64) = 2.84
\]

\[
= 1.68
\]

(b) \( R_3 \geq \int_0^6 f(x) \, dx \)

because \( f \) is increasing on \([0,6]\).

(can check \( f'(x) \) to verify if not clear!)
(a) We linearly approximate and $\sqrt{64} = 8$ to estimate $\sqrt{65}$.

We want to use $f(x) = \sqrt{x}$; here $f'(64) = \sqrt{64} = 8$ and we want to estimate $f(65)$. Now we approximate at $x=64$.

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

and

$$f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{16}$$

and then

$$L(x) = f'(64)(x-64) + f(64)$$

$$= \frac{1}{16}(x-64) + 8$$

$$= \frac{1}{16}$$

Then:

$$\sqrt{65} = f(65) \approx L(65) = \frac{1}{16}(65-64) + 8$$

$$= \frac{1}{16} + 8$$
(b) Consider the graph of \( f(x) = y \)

- Tangent line is above.

\[ y = 5x \]

\[ \text{over-estimate} \]

---

(a) Find the general antiderivative for

\[ f(x) = \frac{x^{2/3} + x^{1/2}}{x^{3/2}} \]

See first that

\[ f(x) = \frac{x^{2/3}}{x^{3/2}} + \frac{x^{1/2}}{x^{3/2}} = x^{-1/2} + x^{-1} \]

\[ = x^{-1/3} - 6 + x^{-1} \]

Then

\[ F(x) = \frac{x^{1/6}}{1/6} + \ln|x| + C \]

\[ = 6x^{1/6} + \ln|x| + C \]
(b) Find the general antiderivative of
\[ f(x) = \frac{1}{x^2} + \sec(2x) \tan(2x) \]

First, see that
\[ \frac{d}{dx} \left[ \frac{-1}{x} \right] = \frac{1}{x^2} \]
by the power rule.

Furthermore, see that
\[ \frac{d}{dx} [\tan(2x)] = 2 \sec(2x) \tan(2x) \]
by the chain rule.

Hence:
\[ \frac{1}{2} \frac{d}{dx} [\tan(2x)] = \sec(2x) \tan(2x) \]

and thus:
\[ \Rightarrow \frac{d}{dx} \left[ \frac{1}{2} \tan(2x) \right] = \sec(2x) \tan(2x) \]

So now
\[ F(x) = \frac{-1}{x} + \frac{1}{2} \tan(2x) + C \]
(c) \[ \int_{0}^{1/2} \sin(2x) \, dx \]

See first that \[ F(x) = \frac{-\cos(2x)}{2} \]
is an antiderivative of \( \sin(2x) \). 
So by FTC I:

\[ \int_{0}^{1/2} \sin(2x) \, dx = F(\frac{1}{2}) - F(0) \]

\[ = \frac{-\cos(2(\frac{1}{2}))}{2} + \frac{\cos(0)}{2} \]

\[ = \frac{-\cos(\pi)}{2} + \frac{1}{2} \]

\[ = \frac{1}{2} + \frac{1}{2} = 1 \]
(a) \( f(x) = \sin(x) \cos(x) \)

\[
f'(x) = ?
\]

First, take \( \ln \) of both sides:

\[
\ln(f(x)) = \ln(\sin(x) \cos(x))
\]

Using the properties of \( \ln \):

\[
\ln(f(x)) = \cos(x) \ln(\sin(x))
\]

Then:

\[
\frac{f'(x)}{f(x)} = \frac{-\sin(x) \ln(\sin(x)) + \cos(x) \frac{d}{dx} \ln(\sin(x))}{\sin(x) \cos(x)}
\]

Apply the chain rule:

\[
\frac{f'(x)}{f(x)} = -\sin(x) \ln(\sin(x)) + \frac{\cos(x)}{\sin(x)} \cdot \cos(x)
\]

Multiply by \( f(x) \):

\[
f'(x) = \left( -\sin(x) \ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)} \right) \frac{\cos(x) \ln(x) + \sin(x) \cos(x) - \cos^2(x)}{f(x)}
\]

\[
f'(x) = \frac{\cos(x) + 1 - \sin(x) \ln(\sin(x)) + \sin(x) \cos(x) - \cos^2(x)}{\sin(x) \cos(x) - \cos^2(x)}
\]
(b) \[ g(x) = \cos(\ln(x^2+1)) \]

\[ g'(x) = ? \]

\[ g'(x) = \frac{d}{dx} \left[ \cos(\ln(x^2+1)) \right] \]

\[ = -\sin(\ln(x^2+1)) \frac{d}{dx} \left[ \ln(x^2+1) \right] \]

\[ \text{chain rule} \]

\[ = -\sin(\ln(x^2+1)) \left( \frac{1}{x^2+1} \right) \frac{d}{dx} (x^2+1) \]

\[ = -\sin(\ln(x^2+1)) \frac{1}{x^2+1} (2x) \]

\[ = \boxed{\frac{-2x \sin(\ln(x^2+1))}{x^2+1}} \]
(c) \( h(x) = \int_1^{2/x} \tan(t^2) \, dt \)

\[ h'(x) = ? \]

Apply FTC II and the chain rule:

\[ h'(x) = \frac{d}{dx} \int_1^{2/x} \tan(t^2) \, dt \]

outside by FTC II

chain rule

\[ = \tan\left(\frac{2}{x^2}\right) \frac{d}{dx} \left(\frac{2}{x}\right) \]

chain rule with power rule

\[ = 2 \tan\left(\frac{4}{x^2}\right) \frac{d}{dx} \left(\frac{1}{x}\right) \]

\[ = -2 \tan\left(\frac{4}{x^2}\right) \frac{1}{x^2} \]
8. \( f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3 \)

(a) Critical points:
\[ f'(x) = x^2 + x - 2 \quad \text{(clearly defined everywhere)} \]
\[ = (x-1)(x+2) = 0 \iff x = 1, -2 \]

(b) Intervals of increase/decrease

Check the signs of \( f' \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f'(x) )</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>(-)(-)</td>
<td>+</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>CP</td>
</tr>
<tr>
<td>0</td>
<td>(-)(2)</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>EP</td>
</tr>
<tr>
<td>2</td>
<td>(+)(+)</td>
<td>+</td>
</tr>
</tbody>
</table>

So \( f \) increases on \( (-\infty, -2) \cup (1, \infty) \) and decreases on \( (-2, 1) \)
(c) determine local max/min

By the sign table, there is a local max at $x = -2$
local min at $x = 1$

(d) Inflection points

$f''(x) = 2x + 1 = 0 \iff x = -\frac{1}{2}$

Then we check the signs:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\phi$</td>
<td>$-2\phi + 1 = -1$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$0$</td>
<td>1</td>
<td>$\bigcirc$</td>
</tr>
</tbody>
</table>

There is an inflection point at $x = -\frac{1}{2}$
(From concave $\downarrow$ to $\uparrow$)

(e) Intervals of concavity

By the sign table,
- Concave down on $(-\infty, -\frac{1}{2})$
- Concave up on $(-\frac{1}{2}, \infty)$