1. (6 points) Find the each of the following limits, or state that it does not exist.

(a) \( \lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} \); this is approaching indeterminate \( \frac{0}{0} \).

Multiply top/bottom by conjugate \( \sqrt{x} + 2 \):

\[
\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{(x - 4)(\sqrt{x} + 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)}
\]

\[
= \lim_{x \to 4} \frac{(x - 4)(\sqrt{x} + 2)}{(x - 4)} = \lim_{x \to 4} \sqrt{x} + 2 = 2
\]

(b) \( \lim_{x \to 0} \frac{\sin x - x}{x^3} \); this is approaching indeterminate \( \frac{0}{0} \).

Apply L'Hospital's rule:

\[
\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos(x) - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin(x)}{6x}
\]

\[
= \lim_{x \to 0} \frac{-\cos(x)}{6} = -\frac{1}{6}
\]

(c) \( \lim_{x \to \infty} \frac{5x^4 - 6x^2 + 3}{3x^4 - x} \); apply asymptotes of rational function (or L'Hospital)

To see that

\[
\lim_{x \to \infty} \frac{5x^4 - 6x^2 + 3}{3x^4 - x} = \frac{5}{3}
\]
2. (6 points) Find the derivative of the following functions. You need not simplify the resulting expressions.

(a) \( f(x) = \sin^2(\cos(5x)) \)

\[
\frac{d}{dx} f(x) = 2 \sin(\cos(5x)) \frac{d}{dx} (\sin(\cos(5x)))
\]

\[
= 2 \sin(\cos(5x)) \cos(\cos(5x)) (-5 \sin(5x))
\]

\[
= [-10 \sin(5x) \sin(\cos(5x)) \cos(\cos(5x))]
\]

(b) \( F(x) = \int_{-1}^{2x^2} \sin^2(5\theta) \, d\theta \)

Apply the chain rule and FTC II:

\[
F'(x) = \frac{d}{dx} \int_{-1}^{2x^2} \sin^2(5\theta) \, d\theta = \frac{d}{dx} \sin^2(10x^2) \cdot 4x
\]

\[
= 4x \sin^2(10x^2)
\]

(c) \( g(x) = (x^2 - 1)^2(2x^3 - 5x) \)

Apply the product and chain rules:

\[
g'(x) = \frac{d}{dx} ((x^2 - 1)^2) \cdot (2x^3 - 5x) + (x^2 - 1)^2 \frac{d}{dx} (2x^3 - 5x)
\]

\[
= 2(x^2 - 1)(2x)(2x^3 - 5x) + (x^2 - 1)^2 (6x^2 - 5)
\]

\[
= (x^2 - 1)(4x(2x^3 - 5x) + (x^2 - 1)(6x^2 - 5))
\]
3. (8 points) If \( \int_0^1 f(x) \, dx = 5 \), \( \int_0^2 f(x) \, dx = 2 \), and \( \int_0^2 g(x) \, dx = -3 \), find

(a) \( \int_1^2 f(x) \, dx = \int_1^0 f(x) \, dx + \int_0^2 f(x) \, dx \)
\[ = -\int_0^1 f(x) \, dx + \int_0^2 f(x) \, dx \]
\[ = -5 + 2 = \boxed{-3} \]

(b) \( \int_0^2 3f(u) \, du = 3\int_0^2 f(u) \, du = 3(2) = \boxed{6} \)

(c) \( \int_1^0 f(x) \, dx = -\int_0^1 f(x) \, dx = \boxed{-5} \)

(d) \( \int_0^2 \{2g(x) - 3f(x)\} \, dx = \int_0^2 2g(x) \, dx - \int_0^2 3f(x) \, dx \)
\[ = 2\int_0^2 g(x) \, dx - 3\int_0^2 f(x) \, dx \]
\[ = 2(-3) - 3(2) = -6 - 6 = \boxed{-12} \]
4. (6 points) A metal water trough with equal semicircular ends and open top needs to have a capacity of $64\pi$ cubic feet. Determine its radius $r$ and length $h$ if the trough is to require the least material for its construction.
5. (6 points) A spherical balloon is being inflated at the rate of 12 cubic inches per minute. What is the radius of the balloon when the rate of change of its surface area is 3 square inches per minute? (Note: the volume $V$ and surface area $A$ of a sphere of radius $r$ are given by the the formulas $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$)

Given:
- Spherical balloon: $V(t) = \frac{4}{3}\pi r(t)^3$
  \hspace{1cm} $A(t) = 4\pi r(t)^2$

Inflated: $V'(t) = 12$ in$^3$/min

Wanted: $r(a)$ such that $A'(a) = 3$ in$^2$/min

1. Find a formula for $r(a)$:

   \[ V(t) = \frac{4}{3}\pi r(t)^3 \Rightarrow V'(t) = 4\pi r(t)^2 r'(t) \]
   \[ \Rightarrow 12 = 4\pi r(t)^2 r'(t) \Rightarrow r(a)^2 = \frac{3}{\pi r'(a)} \]

   \[ \Rightarrow r(a) = \sqrt{\frac{3}{\pi r'(a)}} \]

2. Now find $r'(a)$:

   \[ A(t) = 4\pi r(t)^2 \Rightarrow A'(t) = 8\pi r(t) r'(t) \]

   Now, we know $A'(a) = 3$, so:

   \[ 3 = 8\pi r(a) r'(a) \Rightarrow \frac{3}{8\pi r(a)} = r'(a) \]

3. Solve the resulting system

   \[ r(a) = \sqrt{\frac{3}{\pi r'(a)}} = \sqrt{\frac{3}{8\pi r(a)}} = \sqrt{\frac{8\pi r'(a)}{4}} \]
6. (6 points) Calculate the following definite integrals.

(a) \( \int_{-3}^{4} |x^2 - 4| \, dx \) \; we \; must \; treat \; \( (x^2-4) \) \; as \; a

piecewise function!

\[ |x^2 - 4| = \begin{cases} 
(x^2 - 4) & \text{when } x^2 - 4 \geq 0 \Rightarrow x \geq 2 \\
-(x^2 - 4) & \text{when } x^2 - 4 < 0 \Rightarrow x < 2
\end{cases} \]

\[ = \begin{cases} 
(x^2 - 4) & \text{for } x \in (-\infty, -2] \cup [2, \infty) \\
-(x^2 - 4) & \text{for } x \in (-2, 2)
\end{cases} \]

now we integrate:

\[ \int_{-3}^{4} |x^2 - 4| \, dx = \int_{-3}^{-2} (x^2 - 4) \, dx + \int_{-2}^{2} -(x^2 - 4) \, dx + \int_{2}^{4} (x^2 - 4) \, dx \]

\[ = (F(-2) - F(-3)) + (\frac{x^3}{3} - 12x + 16) \bigg|_{2}^{4} \]

(b) \( \int_{-3}^{1} \frac{3 + 2x^2}{x} \, dx \)

\[ = 3 \left( \ln 1 - \ln |x| \right) + (1)^2 - (-3)^2 \]

\[ = 3(\ln(1) - \ln(3)) + (1 - 9) \]

\[ = 3 \ln(3) - 8 \]
7. (8 points) Consider the graph of \( f(x) = \frac{1}{1-x^2} \).

(a) Determine the vertical asymptote(s), if any.

\[ \text{dom}(f) = \{ x \in \mathbb{R} \mid x \neq \pm 1 \} \]

hence there are vertical asymptotes at \( x = -1 \) and \( x = 1 \).

(b) Determine the horizontal asymptote(s), if any.

See that \( \lim_{{x \to \pm \infty}} \frac{1}{1-x^2} = 0 \) by asymptote of rational function.

hence a horizontal asymptote at \( y = 0 \).

(c) Determine the interval(s) of increase and the interval(s) of decrease.

First, we find the critical points. \( f'(x) = -\frac{{(1-x^2)}^2 - 2x}{(1-x^2)^3} \)

hence the \( f \) and \( f' \) have same domain \( = \frac{2x}{(1-x^2)^2} \)

and \( f'(x) = 0 \) \( \implies \) \( x = 0 \).

Now we'll check the signs:

\[ \begin{array}{c|c|c|c}
\hline
x & f'(x) & \text{sign} \\
\hline
-1 & - & - \\
0 & 0 & 0 \\
\frac{1}{3} & + & + \\
1 & - & - \\
\hline
\end{array} \]

(d) Determine the intervals of concavity.

we check possible inflection:

\[ f''(x) = \frac{(2)(1-x^2)(-2x)(1-x^2)+2x(2)(1-x^2)(1-x^2)}{(1-x^2)^3} \]

\[ = \frac{2(1-x^2)(8x^2)}{(1-x^2)^3} = \frac{2-2x^2+8x^2}{1-x^4} = \frac{2-6x^2}{(1-x^2)^2} \]

\[ \frac{2-6x^2}{(1-x^2)^2} \quad \text{at} \quad x = \pm 1 \]

and \( f''(x) = 0 \) \( \implies \) \( x = \pm \frac{\sqrt{3}}{3} \)

Now we'll check the signs:
\[
\begin{array}{|c|c|c|}
\hline
x & f''(x) & \text{sign} \\
\hline
-2 & - & + \\
-1 & \text{undef} & \text{pt} \\
-\frac{2}{3} & + & - \\
-\frac{1}{3} & 0 & \text{pt} \\
0 & + & - \\
\frac{1}{3} & 0 & \text{pt} \\
\frac{2}{3} & + & - \\
1 & \text{undef} & \text{pt} \\
2 & - & + \\
\hline
\end{array}
\]

Hence we are

Concave up on \((-\infty, -1), (-\frac{1}{3}, \frac{1}{3}), (1, \infty)\)

Concave down on \((-1, -\frac{1}{3}), (\frac{1}{3}, 1)\)
8. (6 points) Find the linear approximation to $f(x) = x^{\frac{3}{2}}$ at $x = 25$ and use it to estimate $(25.06)^{\frac{3}{2}}$. (Note: $25^{\frac{3}{2}} = 125$.)

First we'll find the linear approx of $f$ at $x = 25$:

$$f'(x) = \frac{3}{2} x^{\frac{1}{2}}$$

$$f'(25) = \frac{3}{2} \sqrt{25} = \frac{15}{2}$$

$$L(x) = f'(25)(x-25) + f(25)$$

$$= \frac{15}{2} (x-25) + 125$$

$$\Rightarrow L(x) = \frac{15}{2} x - \frac{15 \cdot 25}{2} + 125$$

Hence now

$$(25.06)^{\frac{3}{2}} = f(25.06) \approx L(25.06)$$

$$= \left( \frac{15}{2} (25.06) - \frac{15 \cdot 25 + 250}{2} \right)$$