1. (6 points) Evaluate each of the following limits, or state that it does not exist.

(a) \( \lim_{x \to 2} \frac{x^2 - 4}{x^2 + 3x - 10} \)

\[
= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-5)(x+2)}
\]

\[
= \lim_{x \to 2} \frac{2+2}{x-5} = \frac{4}{-3} = \frac{-4}{3}
\]

(b) \( \lim_{x \to 0} x^2 \sin \left( \frac{2}{x} \right) \)

See that \( -1 \leq \sin \left( \frac{2}{x} \right) \leq 1 \) for all \( x \neq 0 \).

Since \( x^2 > 0 \), we have: \( -x^2 \leq x^2 \sin \left( \frac{2}{x} \right) \leq x^2 \) for all \( x \neq 0 \).

Recall that \( \lim_{x \to 0} x^2 = 0 = \lim_{x \to 0} x^2 \)

Hence, by the Squeeze Theorem, \( 0 \leq \lim_{x \to 0} x^2 \sin \left( \frac{2}{x} \right) \leq 0 \), hence \( \lim_{x \to 0} x^2 \sin \left( \frac{2}{x} \right) = 0 \)

(c) \( \lim_{x \to 0} f(x) \) where \( f(x) = \begin{cases} \frac{2|x|}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \)

Recall that we have:

\[
1 \times 1 = \begin{cases} -x & \text{for } x \in (-\infty, 0) \\ x & \text{for } x \in [0, \infty) \end{cases}
\]

Hence we may rewrite

\[
f(x) = \begin{cases} -2 & \text{for } x \in (-\infty, 0) \\ 0 & \text{for } x = 0 \\ 2 & \text{for } x \in (0, \infty) \end{cases}
\]

\[
\Rightarrow \lim_{x \to 0^-} f(x) = -2, \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 2,
\]

hence \( \lim_{x \to 0} f(x) \) does not exist.
2. (6 points) Show that the equation \( e^x = \frac{1}{4}x^2 + 2 \) has at least one solution in the interval \([0, 1]\). (Note: \( e \approx 2.7 \))

The statement is equivalent to the function \( f(x) = e^x - \frac{1}{4}x^2 - 2 \) having a zero on the interval \([0, 1]\).

Note that \( f(x) \) is a continuous function.

Now:

\[
f(0) = e^0 - \frac{1}{4}0^2 - 2 = 1 - \frac{1}{4}(0) - 2 = -1 < 0
\]

and

\[
f(1) = e^1 - \frac{1}{4}(1)^2 - 2 = e - 2.25 > 0
\]

hence by the Intermediate Value Theorem, we have that there must be some \( c \in (0, 1) \) such that

\[
f(c) = 0,
\]

i.e. such that \( e^c = \frac{1}{4}c^2 + 2 \)
3. (6 points) Let \( f(x) = \sqrt{2x+1} \). Compute \( f'(x) \) using the limit definition of the derivative.

We will use the second definition:

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}
= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{h} \left( \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} \right)
= \lim_{h \to 0} \frac{2x+2h+1 - (2x+1)}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})}
= \lim_{h \to 0} \frac{2h}{\sqrt{2x+2h+1} + \sqrt{2x+1}}
= \lim_{h \to 0} \frac{2h}{h} \left( \frac{1}{\sqrt{2x+2h+1}} - \frac{1}{\sqrt{2x+1}} \right)
= \lim_{h \to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}
\]
4. (6 points) Let \( g(x) = \sqrt{5-x} + 3. \)

(a) Determine the domain and range of \( g. \)

\[
\text{dom}(g) = \{ x \in \mathbb{R} \mid 5-x > 0 \} = \{ x \in \mathbb{R} \mid 5 > x \} = (-\infty, 5]
\]

\( g(x) = \sqrt{5-x} + 3 \) is defined only when \( 5-x \) is positive.

(b) Find a formula for the inverse \( g^{-1}(x) \) and state its domain and range.

Let \( y = g(x) = \sqrt{5-x} + 3 \)

\[ (y-3)^2 = 5-x \Rightarrow 5-(y-3)^2 = x \]

Therefore \( 5-(g(x)-3)^2 = x \), hence

\[
g^{-1}(x) = 5 - (x-3)^2
\]

\( g^{-1} \) is a polynomial, hence \( \text{dom}(g^{-1}) = \mathbb{R} \).

\( g^{-1} \) is an inverted parabola with vertex at \((3,5)\), hence \( \text{range}(g^{-1}) = (-\infty, 5] \).

\[ \exists \]

Can see this from the graph of the parabola.
5. (4 points) Let \( f \) be a function such that \( f(2) = 2 \) and \( f'(2) = -5 \).

(a) Find an equation for the line tangent to the graph of \( f \) at the point \( (2, 2) \).

The tangent line is given by

\[
y = f'(2)(x - 2) + f(2) \\
= -5(x - 2) + 2 \\
= -5x + 10 + 2 \\
= \boxed{-5x + 12}
\]

(b) Find the value of \( \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \) and justify your answer.

by definition,

\[
\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = f'(2) = \boxed{-5}
\]