Computing derivatives

Given a real function \( f(x) \), we want to understand the function \( f'(x) \) given by its derivative.

Previously:

1. We call the domain of \( f' \) the set of points where \( f \) is differentiable:
   \[
   \text{dom}(f') = \{ x \in \mathbb{R} \mid f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ defined} \}
   \]

2. Power rule: \( g(x) = x^n \Rightarrow g'(x) = nx^{n-1} \)

3. Exponent rule: \( h(x) = e^x \Rightarrow h'(x) = e^x \)

4. Linearity:
   \[
   (f+g)'(x) = f'(x) + g'(x) \\
   (cf)'(x) = cf'(x)
   \]
Now: we will deal with products and quotients.

**Product rule:** \( f, g \) are differentiable at \( x = a \)

Then:

\[
(fg)'(a) = f'(a)g(a) + f(a)g'(a)
\]

**Quotient rule:** \( f, g \) are differentiable at \( x = a \) and \( g(a) \neq 0 \)

Then:

\[
\left( \frac{f}{g} \right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}
\]
Examples:

1. \( f(x) = x^2(3+x^{-1}) \)

   Let \( g(x)' = x^2 \) and \( h(x) = (3+x^{-1}) \)

   Then \( g'(x) = 2x \) and \( h'(x) = -x^{-2} \)

   So now:

   \[
   f'(x) = g'(x)h(x) + g(x)h'(x) \\
   = (2x)(3+x^{-1}) + (x^2)(-x^{-2}) \\
   = 6x + 2 - 1 = 6x + 1
   \]

2. \( g(x) = \frac{x^4 + e^x}{x^4 + 1} \)

   Let \( t(x) = x^4 + e^x \) and \( b(x) = x + 1 \)

   Then \( t'(x) = 4x^3 + e^x \) and \( b'(x) = 1 \)

   So now:

   \[
   g'(x) = \frac{b(x)t'(x) - b'(x)t(x)}{b(x)^2} \\
   = \frac{(x+1)(4x^3 + e^x) - (x^4 + e^x)}{(x+1)^2} = \frac{3x^4 + 4x^3 + xe^x}{(x+1)^2}
   \]
Using the derivative

1. Find the rate of change of \( f(x) \) at \( x = a \)

2. Estimate \( f(x) \) near \( x = a \) using the tangent line at \( x = a \):

   \[ f(x) \text{ is differentiable at } x = a \Leftrightarrow f(x) \approx f'(a)(x-a) + f(a) \text{ for } x \text{ near } a \]

3. Show \( f \) is continuous:

   \[ \text{[Thm]} \ f \text{ differentiable at } x = a \Rightarrow f \text{ continuous at } x = a \]

   **Caution:** \( \neq \) not true
Examples: (from §3.4)

14. An object has temperature \( T(t) \) degrees Celsius at time \( t \) minutes given by:

\[
T(t) = \frac{3}{8}t^2 - 15t + 180
\]

for \( t \in [0, 20] \)

What is the rate of cooling at time \( t = 10 \) minutes?

The rate of change in temperature at \( t = 10 \) min. is given by \( T'(10) \):

\[
T'(t) = \frac{6}{8}t - 15
\]

\( (\text{derivative rules}) \) \Rightarrow \( T'(10) = \frac{6}{8}(10) - 15 \)

\[
= \frac{-15}{2} \text{ °C/minute}
\]

Remark: the rate of change is negative because the object is cooling.
It costs \( C(x) \) dollars to produce \( x \) bagels, given as follows:

\[
C(x) = 300 + (0.25)x + (0.5)\left(\frac{x}{1000}\right)^3
\]

Estimate the marginal cost of making the 2001st donut, and compare with the actual value.

\[
C'(x) = \text{manufacturing cost in dollars per donut at donut } x.
\]

hence

\[
C'(2000) = \text{approximate cost of producing 1 donut} = \text{approximate cost of the 2001st donut}.
\]
Let now we compute:

\[ C'(x) = 0.25 + \frac{3(0.5)}{(1000)^3} x^2 \]

\[ \Rightarrow C'(2000) = 0.25 + \frac{3}{2} \cdot \frac{1}{(1000)^3} \cdot (2000)^2 \]

\[ = 0.25 + \frac{6}{1000} = 0.256 \]

What was the actual cost of donut #2001?

\[ C(2001) - C(2000) \]

\[ \approx 804,256,003,005 - 804 \]

Use calculator

\[ = 0.256003005 \]

Hence our estimation error was only $0.000003005$.
An object falls under the influence of gravity near Earth's surface.

**Setting Up:**

- \( S(t) = \text{position function (in meters)} \)
  - \( S(0) = h \text{ meters} \leftarrow \text{starting height} \)
- \( V(t) = \text{velocity function (in meters/second)} \)
  - Galileo: acceleration due to gravity near Earth's surface is \(-g \text{ m/s}^2\)

\[ V(t) = -gt \]

**(a)** T/F: distance travelled increases by equal amounts per equal time intervals.

Consider the interval \([1,2]\). If the statement were true we would have

\[ \frac{s(1+h)-s(1)}{h} = \frac{s(2+h)-s(2)}{h} \]

for all \( h \neq 0 \). Hence we may apply \( \lim_{h \to 0} \) to both sides.
then:
\[ \lim_{h \to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \]

\[ \Rightarrow \quad s'(1) = s'(2) \]

\[ \Rightarrow \quad -g = v(1) = v(2) = -2g \]

but \(-g \neq -2g\), hence our assumption must have been \textbf{False}.

Exercise: try to solve the preceding using the properties of average rates of change and decreasing function

(b) \( T/F \): velocity increases by equal amount per equal time intervals.

Consider interval \([a, b] \)

Then \[ \frac{v(b) - v(a)}{b-a} = -g \frac{b-a}{b-a} = -g \]

Hence \( \Delta v \) is same regardless of length of \([a, b] \), hence statement is \textbf{TRUE}.
(c) \( \text{T/F: the derivative of velocity increases as } t \text{ increases.} \)

Since \( v(t) = -gt \) we have

\( v'(t) = -g \), hence the derivative

is constant and not increasing

\[ \text{False} \]

Remark:

\[ \text{acceleration } \dot{v} = \text{ derivative of velocity} \]

hence constant acceleration

\[ \Rightarrow \text{ velocity function is linear} \]
Higher derivatives

Recall:
\[ f(x) \text{ real function} \implies f'(x) \text{ real function} \]

What if we iterate?
\[ f'(x) \implies f''(x) \implies \ldots \implies f^{(n)}(x) \implies \ldots \]
\[ \left( \frac{df}{dx} \implies \frac{d^2f}{dx^2} \implies \ldots \implies \frac{d^nf}{dx^n} \implies \ldots \right) \]

Examples: (from §3.5)

(14) \[ f(x) = \frac{1}{1-x} \]

\[ f'(x) = \frac{1}{(1-x)^2} \quad (\text{quotient rule!}) \]

\[ f''(x) = \frac{2}{(1-x)^3} \]

\[ f'''(x) = \frac{6}{(1-x)^4} \]

Question:
Is there some \( n \) such that \( f^{(n)}(x) = 0 \)?
Polynomials \[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]

Show: \[ p^{(n)}(x) = (n!) \ a_n \]
\[ p^{(n+1)}(x) = 0 \quad (easy!) \]

24. \[ f(t) = \frac{t}{t+1} ; \text{find } f^{(1)}(1) \]

\[ f'(t) = \frac{t+1-t}{(t+1)^2} = \frac{1}{(t+1)^2} \]

\[ f''(t) = \frac{-2}{(t+1)^3} \]

\[ \Rightarrow f''(1) = \frac{-2}{2^3} = \frac{-1}{2^2} = \frac{-1}{4} \]
Geometry of \( f''(x) \):

1. \( f''(x) \) tells how fast the slope of the tangent line to \( y = f(x) \) is changing.

2. This property tells how fast the graph is "bending" — this property is called concavity.

\[ \text{concave up} \quad f''(t) > 0 \]

\[ \text{concave down} \quad f''(t) < 0 \]

\( \text{graph of } \quad f(t) = \frac{t}{t+1} \)
Examples cont'd:

(36) Find a formula for \( f^{(n)}(x) \) when \( f(x) = x^2e^x \)

\[
f'(x) = 2xe^x + x^2e^x
\]

\[
f''(x) = 2e^x + 2xe^x + x^2e^x + 2xe^x = 2e^x + 4xe^x + x^2e^x
\]

\[
f'''(x) = 2e^x + 4e^x + 4xe^x + 2xe^x + x^2e^x = 6e^x + 6xe^x + x^2e^x
\]

\[
f''''(x) = 6e^x + 6e^x + 6xe^x + 2xe^x + x^2e^x = 12e^x + 8xe^x + x^2e^x
\]

\[
f'''''(x) = 12e^x + 8e^x + 8xe^x + 2xe^x + x^2e^x = 20e^x + 10xe^x + x^2e^x
\]

\[
f^{(n)}(x) = (n-1)nxe^x + 2nxe^x + x^2e^x
\]
The two main derivatives:

\[
\frac{d}{dx} \sin(x) = \cos(x) \\
\frac{d}{dx} \cos(x) = -\sin(x)
\]

Now recall that:

\[
\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \sec(x) = \frac{1}{\cos(x)}
\]

\[
\cot(x) = \frac{\cos(x)}{\sin(x)} \quad \csc(x) = \frac{1}{\sin(x)}
\]

And use the quotient rule to obtain the other basic derivatives:

\[
\frac{d}{dx} \tan(x) = \sec^2(x) \quad \frac{d}{dx} \sec(x) = \sec(x) \tan(x)
\]

\[
\frac{d}{dx} \cot(x) = -\csc^2(x) \quad \frac{d}{dx} \csc(x) = -\csc(x) \cot(x)
\]
Examples: \( f(x) \) (from § 5.6)

18. \[ f(x) = \frac{x}{\sin(x) + 2} \]

\[ f'(x) = \frac{(x)'(\sin(x) + 2) - (x)(-\sin(x) + 2)}{(\sin(x) + 2)^2} \]

\[ = \frac{(\sin(x) + 2) - x \cos(x)}{\sin^2(x) + 2 \sin(x) + 4} \]

\[ = \frac{\sin(x) - x \cos(x) + 2}{\sin^2(x) + 2 \sin(x) + 4} \]


26. \( \frac{d}{dx} \sec(x) = ? \)

\[ \sec(x) = \frac{1}{\cos(x)} \]

\( \Rightarrow \frac{d}{dx} \sec(x) = \frac{\frac{d}{dx} (1) \cos(x) - (\frac{d}{dx} \cos(x))(1)}{\cos(x)^2} \]

\[ = \frac{0 - (-\sin(x))}{\cos(x)^2} = \frac{\sin(x)}{\cos(x)^2} \]

\[ = \tan(x) \sec(x) \]
43. \( f(x) = \cos(x) \); \( f^{(8)}(x) = ? \), \( f^{(37)}(x) = ? \)

Consider the following:

\[
\begin{align*}
 f'(x) &= -\sin(x) \\
 f''(x) &= -\cos(x) \\
 f'''(x) &= \sin(x) \\
 f^{(iv)}(x) &= \cos(x)
\end{align*}
\]

\(
\Rightarrow \quad \text{derivative of cosine has period } 2\pi \)

\[
\begin{align*}
 f^{(4k)}(x) &= \cos(x) \\
 f^{(4k+1)}(x) &= -\sin(x) \\
 f^{(4k+2)}(x) &= -\cos(x) \\
 f^{(4k+3)}(x) &= \sin(x)
\end{align*}
\]

\[
\begin{align*}
 f^{(8)}(x) &= f^{(2\cdot4)}(x) = \cos(x) \\
 f^{(37)}(x) &= f^{(4\cdot9 + 1)}(x) = -\sin(x)
\end{align*}
\]