Recall: \( b > 0 \)
\[
\frac{d}{dx}[b^x] = b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}
\]

by definition

Now see that:
\[
\lim_{h \to 0} \frac{b^h - 1}{h} = \ln(b)
\]

the natural logarithm

Def'n: \( \ln(x) \) is the function with domain \((0, \infty)\) that gives an inverse to \( e^x \):
\[
\begin{aligned}
e^{\ln(x)} &= x \\
\ln(e^x) &= x
\end{aligned}
\]

i.e. it is a logarithm with base \( e \)
\[
\ln(x) = \log_e(x)
\]
With the previous definition we can find the derivative of any exponential function using the chain rule:

Let $f(x) = b^x$ for $b > 0$.

Now:

$$f'(x) = b^x \ln(b) = e^{\ln(b)} x$$

$$f'(x) = e^{\ln(b) x} (\ln(b)) = \ln(b) b^x$$

$$\frac{d}{dx} (b^x) = \ln(b) b^x$$

We can also use implicit differentiation to find the derivative of $\ln(x)$ itself:

$$\frac{d}{dx} (\ln(x)) = \frac{1}{x} \quad (x > 0)$$
Examples (§39)

(18) \( y = 7^{4x-x^2}; \ \frac{dy}{dx} = ? \)

\[
\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[ 7^{4x-x^2} \right] = \frac{d}{dx} \left[ e^{\ln(7)(4x-x^2)} \right] = e^{\ln(7)(4x-x^2)} (\ln(7)(4-2x))
\]

Chain rule + \( e^{\ln(7)} = 7 \)

\[
= 7^{(4x-x^2)} (\ln(7)(4-2x)) \]

\[
= \ln(7)(4-2x) \cdot 7^{(4x-x^2)}
\]

(32) \( f(x) = \ln(x^2) \); find tangent line at \( x = 4 \)

\[
f'(x) = \frac{d}{dx} \left[ \ln(x^2) \right] = \frac{1}{x^2} \cdot \frac{d}{dx} (x^2) = \frac{2x}{x^2} = \frac{2}{x}
\]

Chain rule + \( \frac{d}{dx} (\ln(x)) = \frac{1}{x} \)

\[
\Rightarrow f'(4) = \frac{2}{4} = \frac{1}{2}
\]

So now the tangent line is:

\[
y = f'(4)(x-4) + f(4) = \frac{1}{2} (x-4) + \ln(16)
\]
\( y = \log_2 (1 + 4x^{-1}) \); tangent line \( x = 4 \)?

Let's begin with some algebra:

\[ 2^y = 2 \log_2 (1 + 4x^{-1}) = (1 + 4x^{-1}) \]

\[ e^{\ln(2)}y = (1 + 4x^{-1}) \]

Now we'll differentiate implicitly:

\[ \frac{d}{dx} \left( e^{\ln(2)}y \right) = \frac{d}{dx} (1 + 4x^{-1}) \]

\[ e^{\ln(2)}y \left( \ln(2) \frac{dy}{dx} \right) = 0 + -4x^{-2} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{-4}{\ln(2) x^2 e^{\ln(2)}y} = \frac{-4}{\ln(2) x^2 2^y} \]

\[ e^{\ln(2)} = 2 \]

\[ \text{Sub. } y = \log_2 (1 + 4x^{-1}) \]

\[ \Rightarrow \frac{-4}{\ln(2) x^2 \log_2(1 + 4x^{-1})} = \frac{-4}{\ln(2) x^2 (1 + 4x^{-1})} = \frac{-4}{\ln(2) (x^2 + 4x)} \]
Now look at \( x = 4 \):

\[
\frac{dy}{dx} = \frac{-4}{\ln(2)(4^2 + 4(4))} = \frac{-4}{\ln(2)(32)}
\]

\[= \frac{-1}{\ln(2)8}
\]

and finally:

\[
L(x) = \frac{dy}{dx}(x - 4) + y(4) = \frac{-1}{\ln(2)8}(x - 4) + \log_2(2)
\]

\[= \left[ \frac{-1}{\ln(2)8} (x - 4) + 1 \right]
\]
Review of properties of logarithms:

\[ \log_b(x) \text{ is the inverse of } b^x \]
for \( b > 0, \ x > 0 \)

That is:
\[
\begin{align*}
\log_b (b^a) &= a \\
\log_b (a) &= a
\end{align*}
\]

Important properties:

1. \( \log_b (a) = \frac{\ln(a)}{\ln(b)} \) (\( b > 1 \))

2. \( \ln(ab) = \ln(a) + \ln(b) \) (\( a, b > 0 \))

3. \( \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) \) (\( a, b > 0 \))

4. \( \ln(a^x) = x \cdot \ln(a) \) (\( a > 0, x \in \mathbb{R} \))
Application of logarithmic functions:

**logarithmic differentiation**

\[
\frac{d}{dx} \left[ \ln(f(x)) \right] = \frac{f'(x)}{f(x)}
\]

for \( f(x) > 0 \)

**Proof:**

- Direct result of the chain rule
- \( \frac{d}{dx} (\ln(x)) = \frac{1}{x} \)

---

Example:

**38** \( y = (3x+5)(4x+9) \); find \( \frac{dy}{dx} \)

see that \( \ln(y) = \ln((3x+5)(4x+9)) \)

\[
= \ln(3x+5) + \ln(4x+9)
\]

\[
\Rightarrow \frac{d}{dx} (\ln(y)) = \frac{d}{dx} (\ln(3x+5)) + \frac{d}{dx} (\ln(4x+9))
\]

apply \( \frac{d}{dx} \)
(38) cont'd:

\[ \frac{1}{y} \frac{dy}{dx} = \frac{3}{3x+5} + \frac{4}{4x+9} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{3y}{3x+5} + \frac{4y}{4x+9} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{3(3x+5)(4x+9)}{(3x+5)} + \frac{4(3x+5)(4x+9)}{4x+9} \]

\[ = 3(4x+9) + 4(3x+5) \]

More examples:

(46) \( f(x) = x^3 \); \( f'(x) = \) ?

\[ f(x) = e^{\ln(x)}3^x \]

\[ \Rightarrow f'(x) = e^{\ln(x)}3^x \frac{d}{dx} (\ln(x)3^x) \]

\[ = e^{\ln(x)}3^x \left[ \frac{d}{dx} (\ln(x)) 3^x + \ln(x) \frac{d}{dx} (3^x) \right] \]

\[ = e^{\ln(x)}3^x \left[ \frac{1}{x}3^x + \ln(x) \frac{d}{dx} (e^{\ln(3)x}) \right] \]

\[ = e^{\ln(x)}3^x \left[ \frac{1}{x}3^x + \ln(x) \frac{d}{dx} (e^{\ln(3)x}) \right] \]

\[ = e^{\ln(x)}3^x \left[ \frac{3^x}{x} + \ln(3) \ln(x)3^x \right] \]
Application of derivatives: Related rates

Before: we used the derivative of a function to find a rate of change.

Now: suppose we have two functions that are related or have related rates of change.

Can we use info about one rate of change with data about another?

Examples: “the ladder problem” (§3.10 9-12)

Suppose a 5 m ladder slides down a wall, where the bottom starts 1.5 m away from the wall.

Let:
\[ x(t) = \text{distance} \]
\[ h(t) = \text{height of top from floor} \]

Then: \[ x(0) = 1.5 \text{ m} \]
Suppose the bottom moves away from the wall at 0.8 m/s. How fast will the top fall at 2 seconds?

Translate into math:

Given: \( x'(t) = 0.8 \text{ m/s} \)

Want: \( h'(2) \)

Now, we may use the relationship between \( h(t) \) and \( x(t) \) given by the Pythagorean theorem:

\[
h(t)^2 + x(t)^2 = 25
\]

Apply the derivative:

\[
\frac{d}{dx} \left[ h(t)^2 + x(t)^2 \right] = 0
\]

\[
2h(t)h'(t) + 2x(t)x'(t) = 0
\]

\( t=2 \)

\[
2h(2)h'(2) + 2x(2)x'(2) = 0
\]

\[
h'(2) = \frac{-x(2)x'(2)}{h(2)} = \frac{-0.8 x(2)}{h(2)}
\]
Since we've been given that the derivative of \( x(t) \) is a constant, we have the following:

\[
x(t) = x'(t)t + x(0) = 0.8t + 1.5
\]

hence

\[
x(2) = 0.8(2) + 1.5 = 1.6 + 1.5 = 3.1
\]

and then:

\[
x(2)^2 + h(2)^2 = 25
\]

\[
\Rightarrow h(2) = \sqrt{25 - (3.1)^2} = \sqrt{25 - 9.61}
\]

So now:

\[
h'(2) = \frac{-0.8(3.1)}{\sqrt{25 - 9.61}} \text{ m/s}
\]

\[
\Rightarrow \text{negative: because it is moving down the wall (so } h'(t) \text{ is decreasing)}
\]
10. Suppose that the top slides down the wall at a rate of 1.2 m/s. Find \( \frac{dx}{dt} \) at the point where the top is 3 meters from the floor.

Translate into math:

**Given:** \( h'(t) = -1.2 \) m/s

**Want:** \( x'(a) \) for \( t = a \) such that \( h(a) = 3 \) m

Consider again the Pythagorean relationship:

\[ x(t)^2 + h(t)^2 = 25 \]

Apply at \( t = a \):

\[ \frac{dx}{dt} = 2x(a)x'(a) + 2h(a)h'(a) = 0 \]

\[ x'(a) = -\frac{h(a)h'(a)}{x(a)} = -\frac{3(-1.2)}{x(a)} \]

\[ x'(a) = -\frac{3(-1.2)}{x(a)} \]

To find \( x(a) \), again apply Pythagoras:

\[ x(a) = \sqrt{25 - h(a)^2} = \sqrt{25 - 3^2} = \sqrt{25 - 9} = 4 \]
so now:

\[ x'(a) = \frac{+3(1.2)}{4} = \frac{3.6}{4} = \sqrt{0.9 \text{ m/s}} \]

\[ 3 \text{ positive: moving away from } x=0 \text{ point (the wall)} \]

(11) Suppose \( h(0) = 4 \)
and top slides down the wall at a rate of \( 1.2 \text{ m/s} \)

Find \( x(2) \) and \( x'(2) \)

The setup:
\( h(0) = 4 \text{ m} \)
\( h'(t) = -1.2 \text{ m/s} \)

Use the Pythagorean theorem again:
\[ h(t)^2 + x(t)^2 = 25 \]

and apply the derivative to both sides at \( t=2 \)
\[ \Rightarrow 2h(2)h'(2) + 2x(2)x'(2) = 0 \]
\[ \Rightarrow x'(2) = \frac{-h(2)h'(2)}{x(2)} \]

Now, since \( h'(t) \) is constant, we have:

\[ h(2) = -(1,2)(2) + h(0) \]
\[ = -2.4 + 4 = 1.6 \]

And by Pythagoras:

\[ x(2) = \sqrt{25 - h(2)^2} = \sqrt{25 - 2.56} \text{ meters} \]

and then

\[ x'(2) = \frac{-(1.6)(-1.2)}{\sqrt{25 - 2.56}} = \frac{1.92}{\sqrt{25 - 2.56}} \]
Suppose we are at a time where \( h'(t) \) and \( x'(t) \) are equal. What is the relationship between \( h \) and \( x \)?

Let \( t=a \) such that \( h'(a) = x'(a) = v \)

Recall again the Pythagorean relation:

\[
h(t)^2 + x(t)^2 = 25
\]

\[
\Rightarrow 2 h(a) h'(a) + 2 x(a) x'(a) = 0
\]

Apply \( \frac{dx}{dt} \) at \( t=a \)

\[
\Rightarrow 2 v (h(a) + x(a)) = 0
\]

Divide out 2 and \( v \neq 0 \)

\[
h(a) + x(a) = 0
\]
At a given moment \((t=0)\) a plane passes directly above a radar station at an altitude of 6 km (fig 1)

(a) The plane's speed is 800 km/h.

How fast is the distance between the plane and the radar station changing \(\frac{1}{2}\) a minute later?

Using the fig 2, we translate into math:
(recall \(\frac{1}{2}\) min = \(\frac{1}{2} \cdot \frac{60}{60}\) hour = \(\frac{1}{120}\) hour)

**Given:**
- \(d(0) = 6\) km
- \(s'(t) = 800\) km/h

**Want:**
- \(d'(\frac{1}{120}) = ?\)
Now since altitude remains constant, we may apply Pythagoras: (see Fig 2)

\[ s(t)^2 + d(0)^2 = d(t)^2 \]

and apply \( \frac{d}{dt} \) with \( t = \frac{1}{120} \):

\[ \Rightarrow 2 s\left( \frac{1}{120} \right) s'\left( \frac{1}{120} \right) = 2 d\left( \frac{1}{120} \right) d'\left( \frac{1}{120} \right) \]

\[ \Rightarrow d'\left( \frac{1}{120} \right) = \frac{s\left( \frac{1}{120} \right) s'\left( \frac{1}{120} \right)}{d\left( \frac{1}{120} \right)} = \frac{800 \cdot s\left( \frac{1}{120} \right)}{d\left( \frac{1}{120} \right)} \]

So now we must find \( s\left( \frac{1}{120} \right) \) and \( d\left( \frac{1}{120} \right) \). Since \( s'(t) = 800 \text{ km/h} \), we have:

\[ s\left( \frac{1}{120} \right) = 800 \left( \frac{1}{120} \right) + s(0) = \frac{800}{120} = \frac{20}{3} \]

and by Pythagoras:

\[ d\left( \frac{1}{120} \right) = \sqrt{\left( \frac{20}{3} \right)^2 + 6^2} = \sqrt{\frac{400}{9} + 36} \]

hence:

\[ d'\left( \frac{1}{120} \right) = \frac{800 \left( \frac{20}{3} \sqrt{\frac{400}{9} + 36} \right)}{3} \]
(b) How fast is the distance between the plane and the statue changing when the plane is directly above the statue?

\[ \text{Want: } c'(0) = ? \]

Which we already know to be

\[ c'(0) = \frac{s(0) s'(6)}{c'(0)} = \frac{(0)(800)}{6} = \boxed{0} \]

\[ \text{Why do we know this function had to have a zero?} \]

(IVT!)
Application of derivatives: Linear approximation:

If \( f(x) \) is differentiable at \( x = a \), then the tangent line provides an approximation for \( f(x) \) in some interval around \( a \).

That is:

\[
\begin{align*}
    f(x) & \approx f'(a)(x-a) + f(a) \\
    & = L(x)
\end{align*}
\]

for \( x \) near \( a \).

The approximation error is given by \( |f(x) - L(x)| \) and the \( \% \)-error is:

\[
\left( \frac{|f(x) - L(x)|}{f(x)} \right) \times 100 \%
\]
Examples (§4.1)

17. Estimate \( \sqrt{26} - \sqrt{25} \)

Let \( f(x) = \sqrt{x} \); we will estimate \( \sqrt{26} \) with the tangent line at \( x = 25 \):

\[
f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \implies f'(25) = \frac{1}{2 \sqrt{25}} = \frac{1}{10}
\]

Then \( L(x) = f'(25)(x-25) + f(25) \)

\[
= \frac{1}{10} (x-25) + 5
\]

So now:

\[
\sqrt{26} - \sqrt{25} = f(26) - f(25) \\
\approx L(26) - 5 = \frac{1}{10} (26-25) = \frac{1}{10}
\]
(22) Estimate $\arctan(1.05) - \frac{\pi}{4}$

(Note that $\arctan(1) = \frac{\pi}{4}$)

Let $g(x) = \arctan(x)$; we approx. with $\tan(x) \approx x$ near $x = 1$

$\Rightarrow g'(x) = \frac{1}{1+x^2}$  $\Rightarrow g'(1.05) = \frac{1}{1+1} = \frac{1}{2}$

Now:

$\arctan(1.05) - \frac{\pi}{4} = g(1.05) - g(1)$

$\approx L(1.05) - g(1)$

$= g'(1)(1.05 - 1) + g(1) - g(1)$

$= \frac{1}{2}(0.05) = \boxed{0.025}$
\( y = (1+x)^{-1/2} \); find \( L(x) \) at \( a = 3 \)

\[
\frac{dy}{dx} = -\frac{1}{2} \frac{-3}{2} (1+x)^{-3/2} = -\frac{1}{2(\sqrt{1+x})^3}
\]

\[
\frac{dy}{dx} \bigg|_{x=3} = -\frac{1}{2(\sqrt{4})^3} = -\frac{1}{16}
\]

Then:

\[
L(x) = \frac{dy}{dx} \bigg|_{x=3} (x-3) + y \bigg|_{x=3}
\]

\[
= -\frac{1}{16} (x-3) + \frac{1}{\sqrt{4}}
\]

\[
= -\frac{1}{16} x + \frac{3}{16} + \frac{8}{16}
\]

\[
= \frac{1}{16} (11 - x)
\]
56. \( f(x) = 3x - 4 \) \( \text{and } L(x) \) at \( a = 0 \) and \( a = 2 \)

\[ f'(x) = 3 \]

\[ L_0(x) = 3(x - 0) + f(0) = 3x - 4 \]

\[ L_2(x) = 3(x - 2) + f(2) = 3x - 4 \]

More generally: note that if a function \( f(x) \) is linear, then it coincides with \( L(x) \) everywhere!!

\[ f(x) = mx + b \]

\[ \Rightarrow f'(x) = m \]

\[ \Rightarrow L_a(x) = f'(a)(x - a) + f(a) \]

\[ = m(x - a) + (ma + b) \]

\[ = mx - ma + ma + b = mx + b \]

\[ = f(x) \]
Estimate $\sqrt{16.2}$ using $L(x)$ at $a=16$

For $f(x) = \sqrt{x}$; plot the estimate.

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$$

$$L(x) = f'(16)(x-16) + f(16)$$

$$= \frac{1}{8}(x-16) + 4 = \frac{1}{8}x - 2 + 4$$

$$= \frac{1}{8}x + 2$$

Then

$$\sqrt{16.2} \approx L(16.2)$$

$$= \frac{1}{8}(0.2) + 4 = \left[ \frac{1}{40} + 4 \right]$$

Plot:

![Graph showing the function $f(x) = \sqrt{x}$ and its linear approximation $L(x) = \frac{1}{8}x + 2$.](image)
Approximate \( \ln(1.07) \) and compute the error.

We will use the approximation at \( x = 1 \) for the function \( f(x) = \ln(x) \).

\[
f'(x) = \frac{1}{x} \implies f'(1) = 1
\]

\[
L(x) = f'(1)(x-1) + f(1)
\]

\[
= 1(x-1) + 0 = x - 1
\]

So

\[
\ln(1.07) = f(1.07) \times L(1.07) = 0.07
\]

Then

\[
(\%\text{-error}) = \left| \frac{f(1.07) - L(1.07)}{f(1.07)} \right| \times 100
\]

\[
= \left| \frac{0.06766 - 0.07}{0.06766} \right| \times 100
\]

\[
= \left| \frac{0.00234}{0.06766} \right| \times 100 = 3.4585\%
\]